Accelerator Physics
Particle Acceleration

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Lecture 8

Statistical Treatments of Beams

- Distribution Functions Defined
  - Statistical Averaging
  - Examples
- Kinetic Equations
  - Liouville Theorem
  - Vlasov Theory
  - Collision Corrections
- Self-consistent Fields
- Collective Effects
  - KV Equation
  - Landau Damping
- Beam-Beam Effect

Beam *rms* Emittance

Treat the beam as a statistical ensemble as in Statistical Mechanics. Define the distribution of particles within the beam statistically. Define single particle distribution function

$$\psi\left(x, x'\right),$$

where $\psi(x,x')dxdx'$ is the number of particles in $[x,x+dx]$ and $[x',x'+dx']$, and statistical averaging as in Statistical Mechanics, e. g.

$$\left\langle q\right\rangle \equiv \int q\left(x, x'\right)\psi\left(x, x'\right)dxdx' / N$$

$$\left\langle q^2\right\rangle \equiv \int q^2\left(x, x'\right)\psi\left(x, x'\right)dxdx' / N$$

$$\vdots$$

Closest *rms* Fit Ellipses

For zero-centered distributions, i.e., distributions that have zero average value for x and x'

$$\varepsilon_{rms} \equiv \sqrt{\left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle - \left\langle xx' \right\rangle^2}$$

$$\beta = \frac{\left\langle x^2 \right\rangle}{\varepsilon_{rms}} = \frac{\sigma_x^2}{\varepsilon_{rms}}$$

$$\alpha = -\frac{\left\langle xx' \right\rangle}{\varepsilon_{rms}}$$

$$\gamma = \frac{\left\langle x'^2 \right\rangle}{\varepsilon_{rms}} = \frac{\sigma_{x'}^2}{\varepsilon_{rms}}$$

Case: Uniformly Filled Ellipse

$$\psi(x,x') = \frac{1}{\pi\varepsilon}\Theta\left(1-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{\varepsilon}\right)$$

Θ here is the Heavyside step function, 1 for positive values of its argument and zero for negative values of its argument

$$\sigma_x^2 = \langle x^2 \rangle = \frac{\varepsilon\beta}{4}$$

$$\langle xx' \rangle = -\frac{\varepsilon\alpha}{4}$$

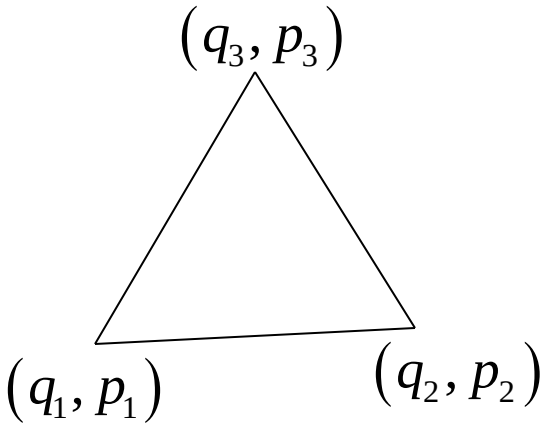
$$\sigma_{x'}^2 = \langle x'^2 \rangle = \frac{\varepsilon}{4\beta}\left(1+\alpha^2\right)$$

$$\therefore \varepsilon_{rms} = \frac{\varepsilon}{4}$$

Gaussian models (HW) are good, especially for lepton machines

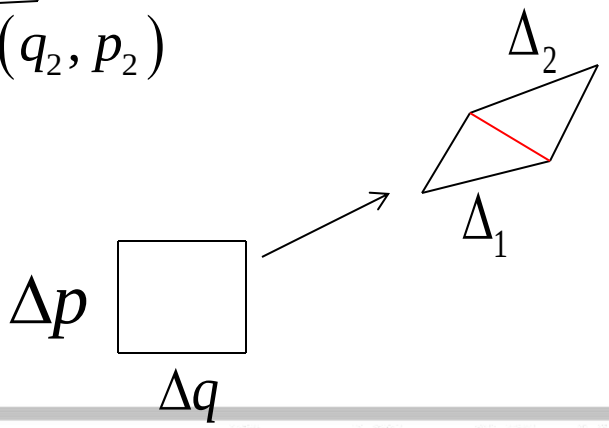
Dynamics? Start with Liouville Thm.

Generalization of the Area Theorem of Linear Optics. Simple Statement: For a dynamical system that may be described by a conserved energy function (Hamiltonian), the relevant phase space volume is conserved by the flow, *even if the forces are non-linear.* Start with some simple geometry!



$$\text{Area } \Delta = \frac{(q_2 - q_1)(p_3 - p_1)}{2} - \frac{(q_3 - q_1)(p_2 - p_1)}{2}$$

$$\left(\text{acute angle has line } 1-2 \text{ clockwise wrt line } 1-3\right)$$



In phase space
Area Before=Area After

Case: Uniformly Filled Ellipse

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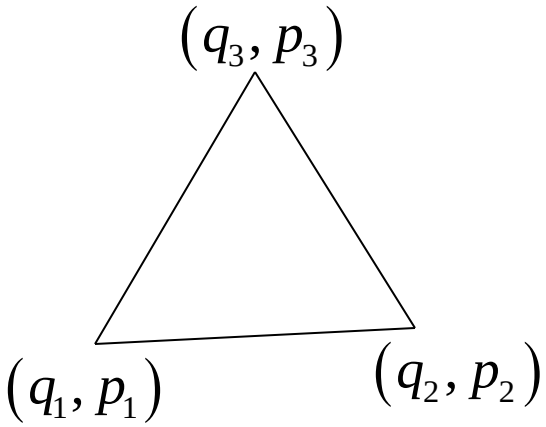

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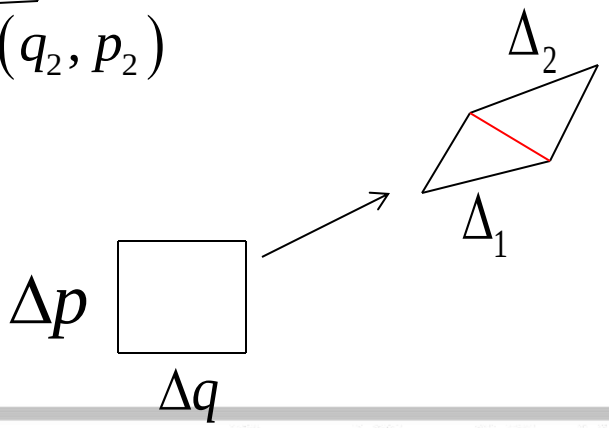


In phase space
Area Before=Area After

Liouville Theorem

$$(q_0, p_0) \rightarrow \left(q_0 + \frac{\partial H}{\partial p}(q_0, p_0)\Delta t + \cdots,\ p_0 - \frac{\partial H}{\partial q}(q_0, p_0)\Delta t + \cdots \right)$$

$$(q_0 + \Delta q, p_0) \rightarrow \left(q_0 + \Delta q + \frac{\partial H}{\partial p}(q_0 + \Delta q, p_0)\Delta t + \cdots,\ p_0 - \frac{\partial H}{\partial q}(q_0 + \Delta q, p_0)\Delta t + \cdots \right)$$

$$(q_0, p_0 + \Delta p) \rightarrow \left(q_0 + \frac{\partial H}{\partial p}(q_0, p_0 + \Delta p)\Delta t + \cdots,\ p_0 + \Delta p - \frac{\partial H}{\partial q}(q_0, p_0 + \Delta p)\Delta t + \cdots \right)$$

$$(q_0 + \Delta q, p_0 + \Delta p) \rightarrow \begin{pmatrix} q_0 + \Delta q + \dfrac{\partial H}{\partial p}(q_0 + \Delta q, p_0 + \Delta p)\Delta t + \cdots, \\ p_0 + \Delta p - \dfrac{\partial H}{\partial q}(q_0 + \Delta q, p_0 + \Delta p)\Delta t + \cdots \end{pmatrix}$$

$$\therefore \text{Area } \Delta_1 = \frac{1}{2}\det \begin{vmatrix} \Delta q + \left[\dfrac{\partial H}{\partial p}(q_0 + \Delta q, p_0) - \dfrac{\partial H}{\partial p}(q_0, p_0) \right]\Delta t & -\left[\dfrac{\partial H}{\partial q}(q_0 + \Delta q, p_0) - \dfrac{\partial H}{\partial q}(q_0, p_0) \right]\Delta t \\ \left[\dfrac{\partial H}{\partial p}(q_0, p_0 + \Delta p) - \dfrac{\partial H}{\partial p}(q_0, p_0) \right]\Delta t & \Delta p - \left[\dfrac{\partial H}{\partial q}(q_0, p_0 + \Delta p) - \dfrac{\partial H}{\partial q}(q_0, p_0) \right]\Delta t \end{vmatrix}$$

$$\underset{\Delta t \to 0}{\rightarrow} \frac{1}{2}(\Delta q \Delta p)\left[1 + \left\{ \frac{\partial^2 H}{\partial q \partial p}(q_0, p_0) - \frac{\partial^2 H}{\partial p \partial q}(q_0, p_0) \right\}\Delta t \right] = \frac{(\Delta q \Delta p)}{2}$$

Likewise

$$\text{Area } \Delta_2 = \frac{1}{2}\det\begin{vmatrix} \Delta q + \left[\frac{\partial H}{\partial p}(q_0+\Delta q, p_0) - \frac{\partial H}{\partial p}(q_0+\Delta q, p_0+\Delta p)\right]\Delta t & -\left[\frac{\partial H}{\partial q}(q_0+\Delta q, p_0) - \frac{\partial H}{\partial q}(q_0+\Delta q, p_0+\Delta p)\right]\Delta t \\ \left[\frac{\partial H}{\partial p}(q_0, p_0+\Delta p) - \frac{\partial H}{\partial p}(q_0+\Delta q, p_0+\Delta p)\right]\Delta t & \Delta p - \left[\frac{\partial H}{\partial q}(q_0, p_0+\Delta p) - \frac{\partial H}{\partial q}(q_0+\Delta q, p_0+\Delta p)\right]\Delta t \end{vmatrix}$$

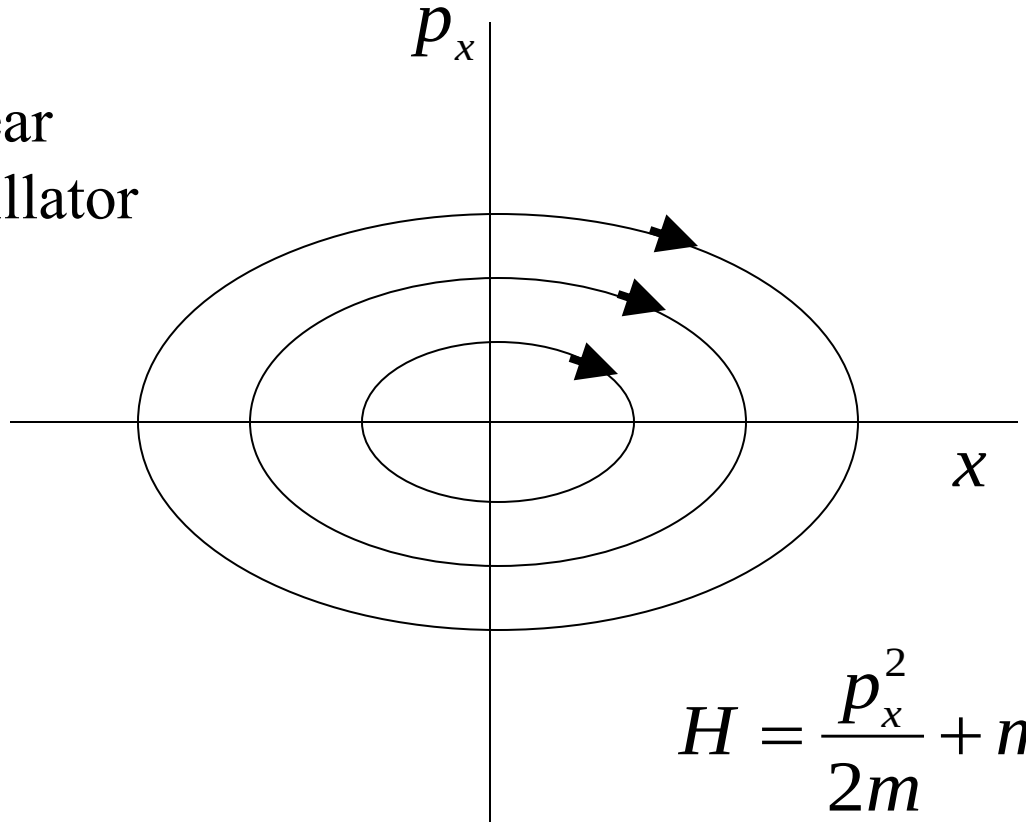
$$\underset{\Delta t\to 0}{\to} \frac{1}{2}(\Delta q \Delta p)\left[1 + \left\{\frac{\partial^2 H}{\partial q \partial p}(q_0+\Delta p, p_0+\Delta p) - \frac{\partial^2 H}{\partial p \partial q}(q_0+\Delta p, p_0+\Delta p)\right\}\Delta t\right] = \frac{(\Delta q \Delta p)}{2}$$

Because the starting point is arbitrary, phase space area is conserved at each location in phase space. In three dimensions, the full 6-D phase volume is conserved by essentially the same argument, as is the sum of the projected areas in each individual projected phase spaces (the so-called third Poincare and first Poincare invariants, respectively). Defeat it by adding non-Hamiltonian (dissipative!) terms later.

Phase Space

- Plot of dynamical system "state" with coordinate along abscissa and momentum along the ordinate

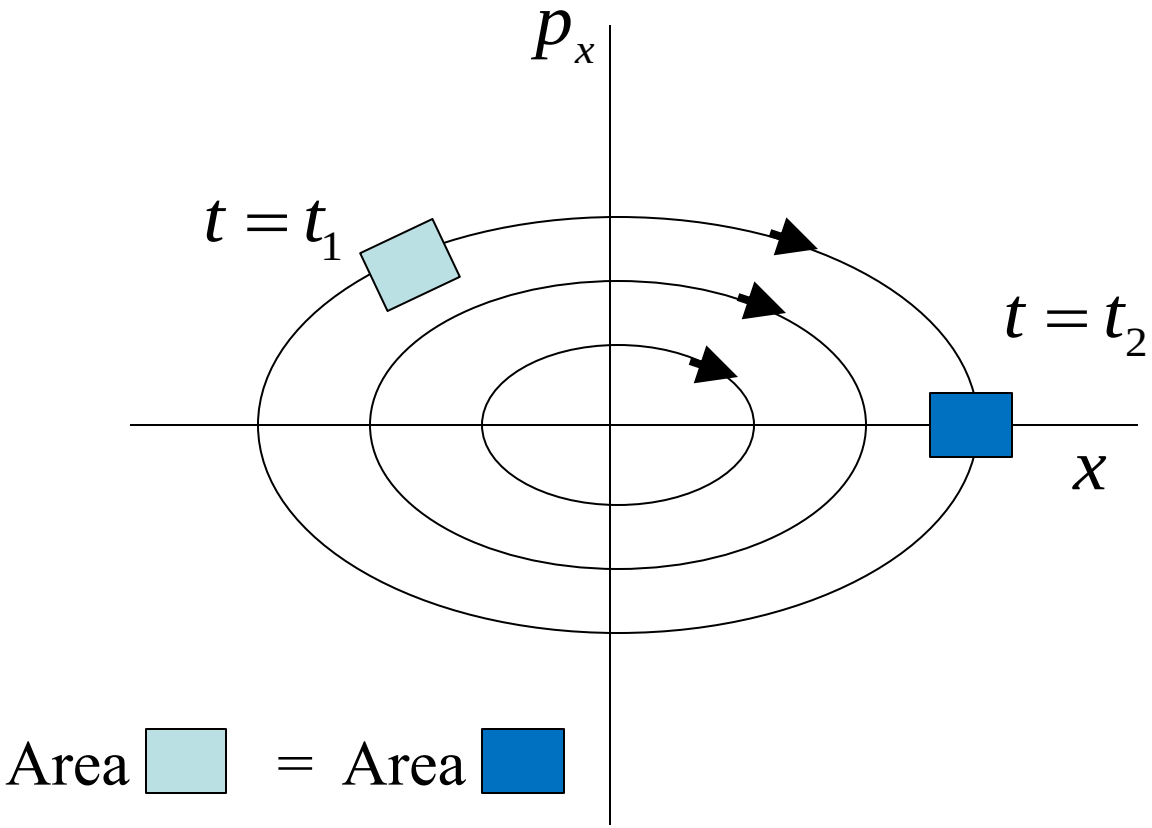
Linear Oscillator



$$H = \frac{p_x^2}{2m} + m\omega^2 \frac{x^2}{2}$$

Liouville Theorem

- Area in phase space is preserved when the dynamics is Hamiltonian



1D Proof

$$\frac{dV}{dt} = \lim_{\Delta t \to \infty} \frac{V(t+\Delta t) - V(t)}{\Delta t}$$

$$x(s, t+\Delta t) \doteq x(s) + \left.\frac{dx}{dt}\right|_t \Delta t + \cdots \doteq x(s) + \frac{\partial H}{\partial p_x}\left(x(s), p_x(s)\right)\Delta t + \cdots$$

$$p_x(s, t+\Delta t) \doteq p_x(s) + \left.\frac{dp_x}{dt}\right|_t \Delta t + \cdots \doteq p_x(s) - \frac{\partial H}{\partial x}\left(x(s), p_x(s)\right)\Delta t + \cdots$$

$$V(t) = \oint_{c(s,t)} p_x dx = \int_0^L p_x(s,t)\frac{dx}{ds}(s,t)\,ds$$

$$V(t+\Delta t) = \oint_{c(s,t+\Delta t)} p_x dx = \int_0^L p_x(s,t+\Delta t)\frac{dx}{ds}(s,t+\Delta t)\,ds$$

$$\doteq \int_0^L \left[p_x(s) + \frac{\partial H}{\partial x}\left(x(s), p_x(s)\right)\Delta t \right] \frac{d}{ds}\left[x(s) - \frac{\partial H}{\partial p_x}\left(x(s), p_x(s)\right)\Delta t \right] ds$$

$$\frac{dV}{dt} = \int_0^L \left[-p_x(s) \frac{d}{ds} \frac{\partial H}{\partial p_x}\left(x(s), p_x(s)\right) + \frac{\partial H}{\partial x}\left(x(s), p_x(s)\right) \frac{dx(s)}{ds} \right] ds$$

$$= \int_0^L \left[\frac{dp_x(s)}{ds} \frac{\partial H}{\partial p_x}\left(x(s), p_x(s)\right) + \frac{\partial H}{\partial x}\left(x(s), p_x(s)\right) \frac{dx(s)}{ds} \right] ds$$

(why is boundary term of integration by parts zero?)

$$= \oint_{c(s,t)} \left[\frac{\partial H}{\partial p_x} dp_x + \frac{\partial H}{\partial x} dx \right]$$

By Green's Thm.

=0 when the differential is an exact differential

i.e., $\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial p_x}\right) = \frac{\partial}{\partial p_x}\left(\frac{\partial H}{\partial x}\right)$, in other words always

(note the integrand above is really dH, so H is a "potential" for phase space!!!)

3D Poincare Invariants

- In a three dimensional Hamiltonian motion, the 6D phase space volume is conserved (also called Liouville's Thm.)

$$V_{6D} = \int_{V_6} dp_x dp_y dp_z dx dy dz$$

- Additionally, the sum of the projected volumes (Poincare invariants) are conserved

$$\int_{\text{proj}(V_2)} dp_x dx + \int_{\text{proj}(V_2)} dp_y dy + \int_{\text{proj}(V_2)} dp_z dz$$

$$\int_{\text{proj}(V_4)} dp_y dp_z dy dz + \int_{\text{proj}(V_4)} dp_z dp_x dz dx + \int_{\text{proj}(V_4)} dp_x dp_y dx dy$$

Emittance (phase space area) exchange based on this idea

- More complicated to prove, but are true because, as in 1D

$$\frac{\partial^2 H}{\partial q_i \partial p_i} = \frac{\partial^2 H}{\partial p_i \partial q_i}$$

γ is a loop in 6D phase space

$$\gamma(t) = \left(\vec{p}(s,t), \vec{q}(s,t)\right)$$

$$\frac{d}{dt}\left[\oint_{\gamma(t)} \sum_{i=1}^{3} p_i dx_i\right] = \int_0^L \sum_{i=1}^{3}\left[-p_i(s)\frac{d}{ds}\frac{\partial H}{\partial p_i}\left(\vec{x}(s),\vec{p}(s)\right) + \frac{\partial H}{\partial x_i}\left(\vec{x}(s),\vec{p}(s)\right)\frac{dx_i(s)}{ds}\right]$$

$$= \int_0^L \sum_{i=1}^{3}\left[\frac{dp_i(s)}{ds}\frac{\partial H}{\partial p_i}\left(\vec{x}(s),\vec{p}(s)\right) + \frac{\partial H}{\partial x_i}\left(\vec{x}(s),\vec{p}(s)\right)\frac{dx_i(s)}{ds}\right] = \oint_{\gamma(t)} dH = 0$$

for any surface in 6D phase space V_2, with $\gamma = \partial V_2$

$$\oint_{\partial V_2} \sum_{i=1}^{3} p_i dx_i = \int_{V_2} \sum_{i=1}^{3} dp_i dx_i = \sum_{i=1}^{3} \int_{\text{proj}(V_2)} dp_i dx_i$$

$$\left(\sum_{i=1}^{3} dp_i dx_i\right)^2 = dp_y dp_z dy dz + dp_z dp_x dz dx + dp_x dp_y dx dy$$

$$\left(\sum_{i=1}^{3} dp_i dx_i\right)^3 = dp_x dp_y dp_z dx dy dz$$

Vlasov Equation

By interpretation of ψ as the single particle distribution function, and because the individual particles in the distribution are assumed to not cross the boundaries of the phase space volumes (collisions neglected!), ψ must evolve so that

$$\frac{d\psi}{dt} = 0 \qquad \text{as the distibution evolves}$$

$$\frac{d\psi}{dt} = \lim_{\delta t \to 0} \frac{\psi\left(t+\delta t, \vec{q}\left(t+\delta t\right), \vec{p}\left(t+\delta t\right)\right) - \psi\left(t, \vec{q}\left(t\right), \vec{p}\left(t\right)\right)}{\delta t} = 0$$

where the equation for ANY (this is what makes it hard to solve in general!) individual orbits through phase space is given by $\vec{q}\left(t\right), \vec{p}\left(t\right)$

$$\therefore \frac{\partial \psi}{\partial t} + \frac{d\vec{q}}{dt}\frac{\partial \psi}{\partial \vec{q}} + \frac{d\vec{p}}{dt}\frac{\partial \psi}{\partial \vec{p}} = 0$$

Conservation of Probability

$N(t) = \int \psi(t;\vec{q},\vec{p}) d^3\vec{x} d^3\vec{p}$ is a conserved quantity

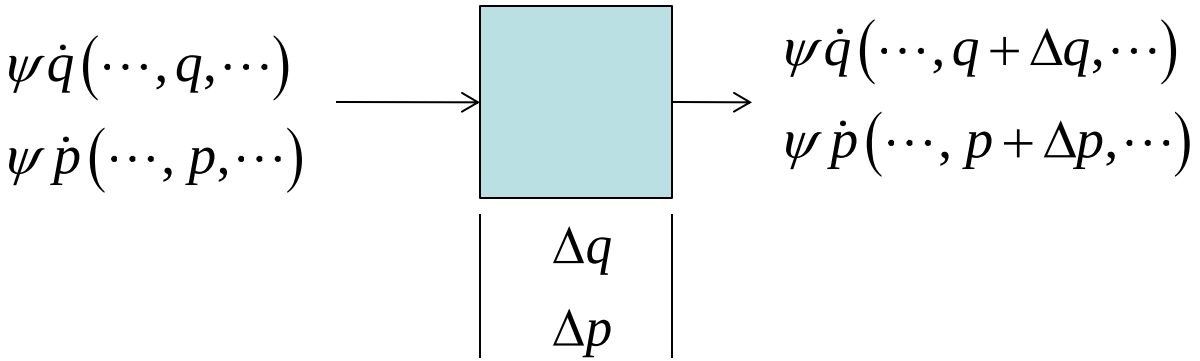
continuity equation for ψ is

$$\frac{\partial \psi}{\partial t} + \nabla_{\vec{q}}\left(\dot{\vec{q}}\psi\right) + \nabla_{\vec{p}}\left(\dot{\vec{p}}\psi\right) = 0$$

$$\therefore \frac{\partial \psi}{\partial t} + \frac{d\vec{q}}{dt}\frac{\partial \psi}{\partial \vec{q}} + \frac{d\vec{p}}{dt}\frac{\partial \psi}{\partial \vec{p}} + \psi\left[\nabla_{\vec{q}}\frac{\partial H}{\partial \vec{p}} - \nabla_{\vec{p}}\frac{\partial H}{\partial \vec{q}}\right] = 0$$

for the Hamiltonian system

$$\therefore \frac{\partial \psi}{\partial t} + \frac{d\vec{q}}{dt}\frac{\partial \psi}{\partial \vec{q}} + \frac{d\vec{p}}{dt}\frac{\partial \psi}{\partial \vec{p}} = 0$$



Jean's Theorem

The independent variable in the Vlasov equation is often changed to the variable *s*. In this case the Vlasov equation is

$$\frac{\partial \psi}{\partial s}+\frac{d\vec{q}}{ds}\frac{\partial \psi}{\partial \vec{q}}+\frac{d\vec{p}}{ds}\frac{\partial \psi}{\partial \vec{p}}=0$$

The equilibrium Vlasov problem, $\partial\psi/\partial t = 0$, is solved by any function of the constants of the motion. This result is called Jean's theorem, and is the starting point for instability analysis as the "unperturbed problem".

If $\psi = f\left(A, B, C, \cdots\right)$, where $A, B, C, \cdots$ are constants of the motion

$$\frac{d\vec{x}}{dt}\frac{\partial \psi}{\partial \vec{x}}+\frac{d\vec{p}}{dt}\frac{\partial \psi}{\partial \vec{p}}=\frac{\partial f}{\partial A}\left(\frac{d\vec{x}}{dt}\frac{\partial A}{\partial \vec{x}}+\frac{d\vec{p}}{dt}\frac{\partial A}{\partial \vec{p}}\right)+\frac{\partial f}{\partial B}\left(\frac{d\vec{x}}{dt}\frac{\partial B}{\partial \vec{x}}+\frac{d\vec{p}}{dt}\frac{\partial B}{\partial \vec{p}}\right)$$

$$+\frac{\partial f}{\partial C}\left(\frac{d\vec{x}}{dt}\frac{\partial C}{\partial \vec{x}}+\frac{d\vec{p}}{dt}\frac{\partial C}{\partial \vec{p}}\right)+\cdots=\frac{\partial f}{\partial A}\frac{dA}{dt}+\frac{\partial f}{\partial B}\frac{dB}{dt}+\frac{\partial f}{\partial C}\frac{dC}{dt}+\cdots=0$$

Examples

1-D Harmonic oscillator Hamiltonian. Bi-Maxwellian distribution is a stationary distribution

$$\psi = \frac{1}{2\pi}\exp\left(-H/kT\right) = \frac{m\omega}{2\pi kT}\exp\left(-p_x^2/2mkT\right)\exp\left(-mx^2\omega^2/2kT\right),$$

As is any other function of the Hamiltonian. Contours of constant ψ line up with contours of constant H

2 D transverse Gaussians, including focusing structure in ring

$$\psi\left(s;x,x';y,y'\right) \propto \exp\left(-\left(\gamma_x\left(s\right)x^2 + 2\alpha_x\left(s\right)xx' + \beta_x\left(s\right)x'^2\right)/\varepsilon_x\right)$$
$$\times\exp\left(-\left(\gamma_y\left(s\right)y^2 + 2\alpha_y\left(s\right)yy' + \beta_y\left(s\right)y'^2\right)/\varepsilon_y\right)$$

Contours of constant ψ line up with contours of constant Courant-Snyder invariant. Stationary as particles move on ellipses!

Solution by Characteristics

More subtle: a solution to the full Vlasov equation may be obtained from the distribution function at some the initial condition, provided the particle orbits may be found unambiguously from the initial conditions throughout phase space. Example: 1-D harmonic oscillator Hamiltonian.

$$\begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} = \begin{pmatrix} \cos\omega(t-t_0) & \sin\omega(t-t_0)/\omega \\ -\omega\sin\omega(t-t_0) & \cos\omega(t-t_0) \end{pmatrix} \begin{pmatrix} x(t_0) \\ x'(t_0) \end{pmatrix} \rightarrow \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} \cos\omega(t-t_0) & -\sin\omega(t-t_0)/\omega \\ \omega\sin\omega(t-t_0) & \cos\omega(t-t_0) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

$$\psi(x,x';t=t_0) = f_0(x,x')$$

Let $\psi(x,x';t) = f_0\left(\cos\omega(t-t_0)x - \sin\omega(t-t_0)x'/\omega, \omega\sin\omega(t-t_0)x + \cos\omega(t-t_0)x'\right)$

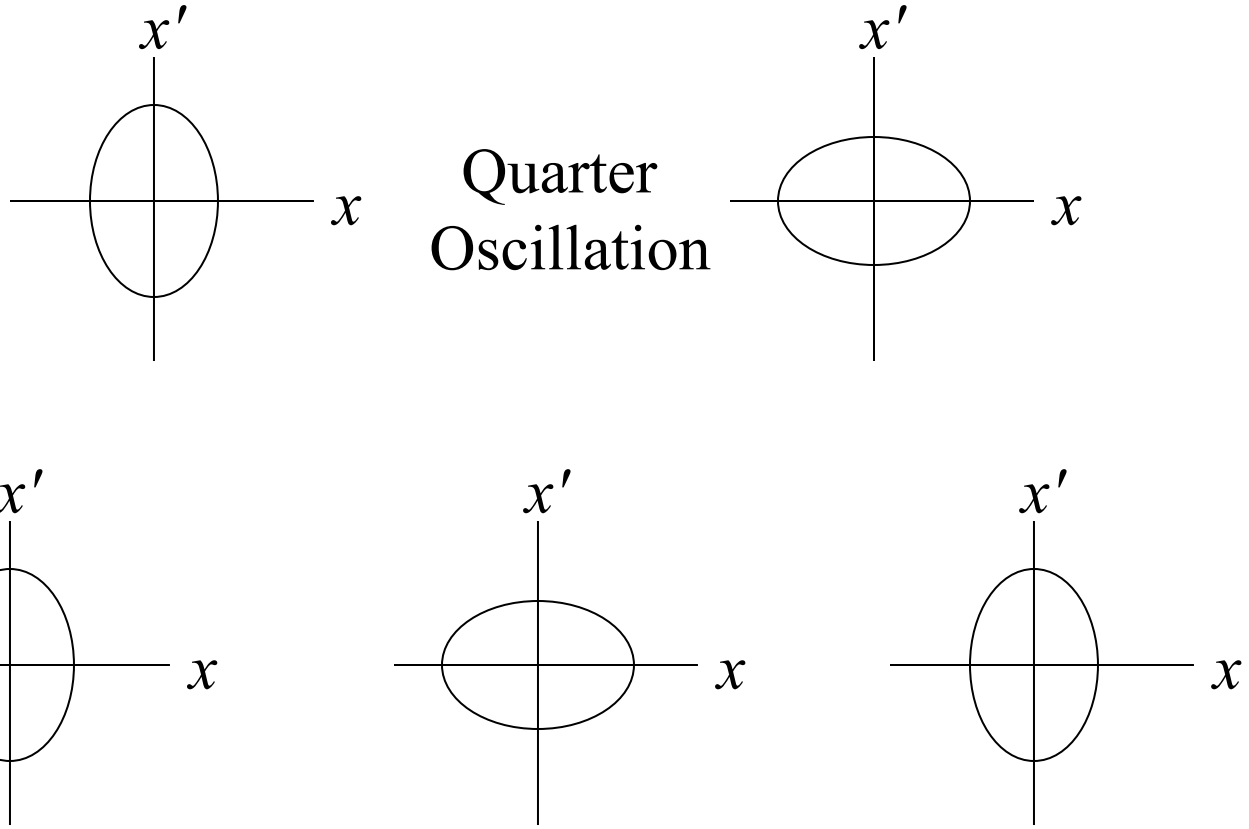
$$\frac{\partial\psi}{\partial t} = \frac{\partial f_0}{\partial x}\frac{dx(t;x,x')}{dt} + \frac{\partial f_0}{\partial x'}\frac{dx'(t;x,x')}{dt}$$

$$= \frac{\partial f}{\partial x}\left[-\omega\sin\omega(t-t_0)x - \cos\omega(t-t_0)x'\right] + \frac{\partial f}{\partial x'}\left[\omega^2\cos\omega(t-t_0)x - \omega\sin\omega(t-t_0)x'\right]$$

$$\frac{dx}{dt}\frac{\partial\psi}{\partial x} = x'\left[\frac{\partial f}{\partial x}\cos\omega(t-t_0) + \frac{\partial f}{\partial x'}\omega\sin\omega(t-t_0)\right]$$

$$\frac{dx'}{dt}\frac{\partial\psi}{\partial x'} = -\omega^2 x\left[-\frac{\partial f}{\partial x}\sin\omega(t-t_0)/\omega + \frac{\partial f}{\partial x'}\cos\omega(t-t_0)\right] \therefore \frac{d\psi}{dt} = 0$$

Breathing Mode



The particle envelope "breaths" at twice the revolution frequency!

Sacherer Theory

Assume beam is acted on by a linear focusing force plus additional linear or non-linear forces

$$x'' + k_x^2 x - F_x = 0$$

$$y'' + k_y^2 x - F_y = 0$$

For space charge example we'll see

$$F_{x(y)} = \frac{qE_{x(y)}\left(1-\beta^2\right)}{\gamma mc^2\beta^2} = \frac{qE_{x(y)}}{\gamma^3 mc^2\beta^2}$$

Now

$$\left\langle xx''\right\rangle + k_x^2\left\langle x^2\right\rangle - \left\langle F_x x\right\rangle = 0$$

$$\left\langle yy''\right\rangle + k_y^2\left\langle y^2\right\rangle - \left\langle F_y y\right\rangle = 0$$

Assume distributions zero-centered and let

$$\tilde{x}^2 = \left\langle x^2\right\rangle \quad \tilde{x}'^2 = \left\langle x'^2\right\rangle \qquad \tilde{y}^2 = \left\langle y^2\right\rangle \quad \tilde{y}'^2 = \left\langle y'^2\right\rangle$$

$$\left\langle x^2 \right\rangle' = 2\left\langle xx' \right\rangle = \tilde{x}^{2\prime} = 2\tilde{x}\tilde{x}'$$

$$\left\langle x^2 \right\rangle'' = \tilde{x}^{2\prime\prime} = \left(2\tilde{x}\tilde{x}'\right)' = 2\left(\tilde{x}\tilde{x}'' + \tilde{x}'^2\right)$$

Also

$$\left\langle xx' \right\rangle' = \left\langle x'^2 \right\rangle + \left\langle xx'' \right\rangle = \left\langle x'^2 \right\rangle - k_x^2 \left\langle x^2 \right\rangle + \left\langle F_x x \right\rangle$$

$$\frac{1}{2}\left\langle x^2 \right\rangle'' = \left\langle xx' \right\rangle' = \tilde{x}\tilde{x}'' + \tilde{x}'^2 = \left\langle x'^2 \right\rangle - k_x^2 \left\langle x^2 \right\rangle + \left\langle F_x x \right\rangle$$

$$\tilde{x}' = \left\langle xx' \right\rangle / \tilde{x} \rightarrow \tilde{x}\tilde{x}'' + \frac{\left\langle xx' \right\rangle^2 - \left\langle x'^2 \right\rangle \left\langle x^2 \right\rangle}{\tilde{x}^2} + k_x^2 \tilde{x}^2 - \left\langle F_x x \right\rangle = 0$$

$$\tilde{x}'' + \frac{\left\langle xx' \right\rangle^2 - \left\langle x'^2 \right\rangle \left\langle x^2 \right\rangle}{\tilde{x}^3} + k_x^2 \tilde{x} - \frac{\left\langle F_x x \right\rangle}{\tilde{x}} = 0$$

$$\tilde{x}'' - \frac{\varepsilon_{rms}^2}{\tilde{x}^3} + k_x^2 \tilde{x} - \frac{\left\langle F_x x \right\rangle}{\tilde{x}} = 0 \qquad \text{"Envelope" equation}$$

rms Emittance Conserved

$$\left(\left\langle x^2\right\rangle\left\langle x'^2\right\rangle-\left\langle xx'\right\rangle^2\right)'$$

$$=\left\langle x^2\right\rangle'\left\langle x'^2\right\rangle+\left\langle x^2\right\rangle\left\langle x'^2\right\rangle'-2\left\langle xx'\right\rangle\left\langle xx'\right\rangle'$$

$$=2\left\langle xx'\right\rangle\left\langle x'^2\right\rangle+2\left\langle x^2\right\rangle\left\langle x'x''\right\rangle-2\left\langle xx'\right\rangle\left(\left\langle x'^2\right\rangle-k_x^2\left\langle x^2\right\rangle+\left\langle F_x x\right\rangle\right)$$

$$=2\left\langle x^2\right\rangle\left(-k_x^2\left\langle x'x\right\rangle+\left\langle F_x x'\right\rangle\right)+2\left\langle xx'\right\rangle\left(k_x^2\left\langle x^2\right\rangle-\left\langle F_x x\right\rangle\right)$$

$$=2\left\langle x^2\right\rangle\left\langle F_x x'\right\rangle-2\left\langle xx'\right\rangle\left\langle F_x x\right\rangle$$

For linear forces derivative vanishes and *rms* emittance conserved. Emittance growth implies *non-linear forces*.

Space Charge and Collective Effects

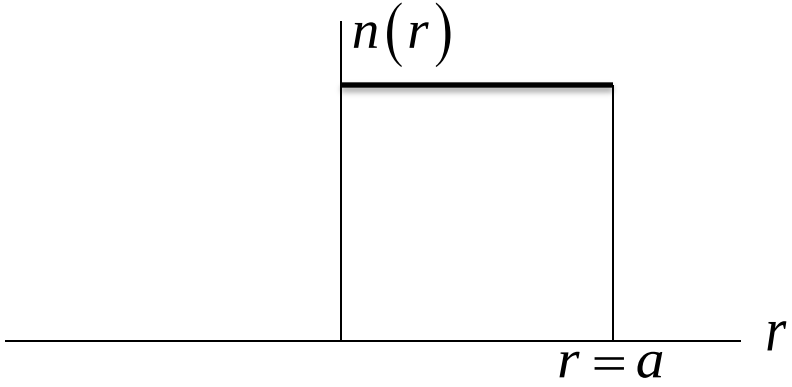
- Collective Effects
  - Brillouin Flow
  - Self-consistent Field
  - KV Equation
  - Bennet Pinch
  - Landau Damping

Simple Problem

- How to account for interactions between particles
- Approach 1: Coulomb sums
  - Use Coulomb's Law to calculate the interaction between each particle in beam
  - Unfavorable N^2 in calculation but perhaps most realistic
  - more and more realistic as computers get better
- Approach 2: Calculate EM field using ME
  - Need procedure to define charge and current densities
  - Track particles in resulting field

Uniform Beam Example

- Assume beam density is uniform and axi-symmetric going into magnetic field



Electric Field

$$\frac{1}{r}\frac{\partial}{\partial r} rE_r = \frac{qn}{\varepsilon_0} \rightarrow E_r = \frac{qn}{2\varepsilon_0} r$$

Self-Magnetic Field by Ampere's Law

$$2\pi r B_\theta = \mu_0 qn\beta c \pi r^2 \rightarrow B_\theta = \mu_0 \frac{qn\beta c}{2} r$$

Brillouin Flow

Total Collective Force on beam particle

$$\vec{F} = q\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right) \doteq \left(\frac{q^2 n}{2\varepsilon_0}\left(1-\beta^2\right)\right) r$$

effective (de)focussing strength

$k = \dfrac{\omega_p^2}{2\beta^2 c^2 \gamma^3}$ where the non-relativistic "plasma frequency" is $\omega_p^2 = \dfrac{q^2 n}{\varepsilon_0 m}$

By previous work with solenoids in the rotating frame, can have equilibrium (force balance) when

$\dfrac{\omega_p^2}{2\gamma} = \Omega_c^2$ non-relativistic plasma and cyclotron frequencies $\omega_L = \dfrac{\Omega_c}{2\gamma}$

This state is known as Brillouin Flow and neglects beam temperature (fluid flow)

Comments

- Some authors, Reiser in particular, define a relativistic plasma frequency

$$\omega_p^2 = \frac{q^2 n}{\varepsilon_0 \gamma^3 m}$$

- Lawson's book has a nice discussion about why it is impossible to establish a relativistic Brillouin flow in a device where beam is extracted from a single cathode at an equipotential surface. In this case one needs to have either sheering of the rotation or non-uniform density in the self-consistent solution.

Vlasov-Poisson System

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \dot{q}\frac{\partial \psi}{\partial x} + \dot{p}\frac{\partial \psi}{\partial p} = 0$$

$$\dot{q} = p / m$$

$$\dot{p}_i = q\left(E_i + \left(\vec{v} \times \vec{B}\right)_i\right)$$

- Self-consistent Field

$$\vec{\nabla} \cdot \vec{E} = -\nabla^2 \phi = \frac{en}{\varepsilon_0}$$

$$n = \int \psi d^3 \dot{\vec{q}}$$

K-V Distribution

- Single value for the transverse Hamiltonian

$$\frac{1}{\varepsilon_x}\left(\frac{x^2+\left(\alpha_x x+\beta_x x'\right)^2}{\beta_x}\right)+\frac{1}{\varepsilon_y}\left(\frac{y^2+\left(\alpha_y y+\beta_y y'\right)^2}{\beta_y}\right)=C$$

$$\psi\left(x,x',y,y'\right)\propto\delta\left(C-1\right)$$

$$\rho\left(z\right)=\frac{I}{\pi\beta cXY}$$

$$K=\frac{I}{I_0}\frac{2}{\beta^3\gamma^3}\qquad I_0=\frac{4\pi\varepsilon_0 mc^3}{q}$$

$$E_x=\frac{I}{\pi\varepsilon_0\beta c}\frac{x}{Y\left(X+Y\right)}$$

K-V Envelope Equation

$$E_x = \frac{I}{\pi\varepsilon_0\beta c}\frac{x}{X\left(X+Y\right)}$$

$$E_y = \frac{I}{\pi\varepsilon_0\beta c}\frac{y}{Y\left(X+Y\right)}$$

$$x'' + k_y x - \frac{2K}{X\left(X+Y\right)}x = 0$$

$$y'' + k_y x - \frac{2K}{Y\left(X+Y\right)}x = 0$$

Envelope Equation

$$X'' + k_x X - \frac{2K}{\left(X+Y\right)} - \frac{\varepsilon_x^2}{X^3} = 0$$

$$Y'' + k_y Y - \frac{2K}{\left(X+Y\right)} - \frac{\varepsilon_y^2}{Y^3} = 0$$

Waterbag Distribution

- Lemons and Thode were first to point out SC field is solved as Bessel Functions for a certain equation of state. Later, others, including my advisor and I showed the equation of state was exact for the waterbag distribution.

$$H_T = \frac{p_z^2}{2m}\left(x'^2 + y'^2\right) + \frac{m\omega_0^2\left(x^2 + y^2\right)}{2} + e\phi_{SC}$$

$$\psi = A\Theta\left(H_0 - H_T\right)$$

$$n(r) = \iint \psi\, dx'dy' = n_0\left(1 - \frac{m\omega_0^2\left(x^2 + y^2\right)}{2H_0} - \frac{\phi_{SC}}{\phi_0}\right) \qquad \phi_0 = H_0 / e$$

$$\sigma_v^2 = \frac{\iint \psi\, p_z^2\left(x'^2 + y'^2\right) dx'dy'}{m^2 \iint \psi\, dx'dy'} = \frac{H_0}{m}\left(1 - \frac{m\omega_0^2\left(x^2 + y^2\right)}{2H_0} - \frac{\phi_{SC}}{\phi_0}\right)$$

Self-consistent potential solves

$$\nabla^2\phi_{SC} + \frac{\phi_{SC}}{\lambda_D^2} = \frac{en_0}{\varepsilon_0}\left[\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0} - 1\right]$$

$$\lambda_D = \frac{\sigma_v}{\omega_p} = \sqrt{\frac{\varepsilon_0 m}{e^2 n_0}\frac{H_0}{m}} = \sqrt{\frac{\varepsilon_0 H_0}{e^2 n_0}}$$ Debye Length

Analytic solutions in terms of Modified Bessel Functions

$$e\phi(r) = \frac{m\omega_0^2\left(x^2+y^2\right)}{2} + A\left(I_0\left(r/\lambda_D\right) - 1\right) + BK_0\left(r/\lambda_D\right)$$

$B = 0$ by boundary condition

A chosen so that solution without I_0 solution to inhomogeneous eqn.

Equation for Beam Radius

Now

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right]\frac{r^2}{2}=2$$

$$\therefore A=m\lambda_D^2\left(2\omega_0^2-\omega_p^2\right)$$

At $r=r_b$ the density vanishes

$$H_0=m\lambda_D^2\left(2\omega_0^2-\omega_p^2\right)\left(1-I_0\left(r_b/\lambda_D\right)\right)$$

$$1+\frac{\omega_p^2}{2\omega_0^2-\omega_p^2}=I_0\left(r_b/\lambda_D\right)$$

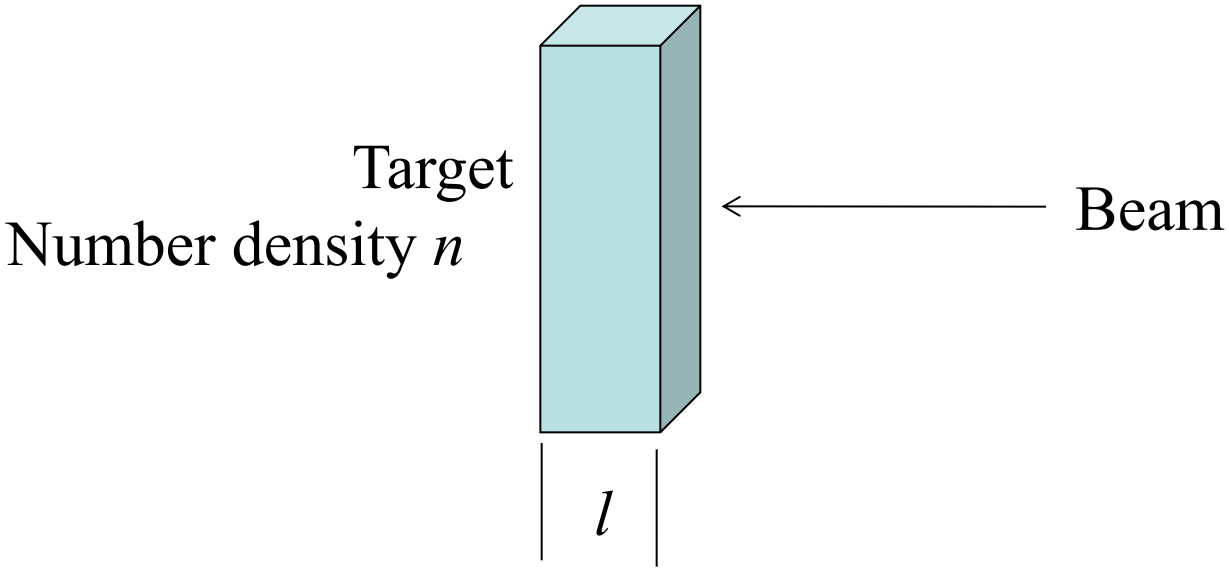
$$n_b\left(r\right)=\hat{n}_b\frac{I_0\left(r_b/\lambda_D\right)-I_0\left(r/\lambda_D\right)}{I_0\left(r_b/\lambda_D\right)-1}$$

Luminosity and Beam-Beam Effect

- Luminosity Defined
- Beam-Beam Tune Shift
- Luminosity Tune-shift Relationship (Krafft-Ziemann Thm.)
- Beam-Beam Effect

Events per Beam Crossing

- In a nuclear physics experiment with a beam crossing through a thin fixed target

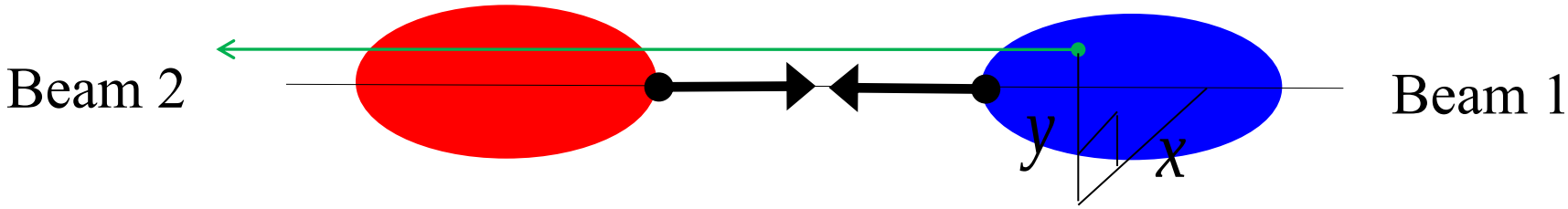


- Probability of single event, per beam particle passage is

$$P = n\sigma l$$

- σ is the "cross section" for the process (area units)

Collision Geometry



- Probability an event is generated by a single particle of Beam 1 crossing Beam 2 bunch with Gaussian density*

$$P = \sigma \frac{N_2 \exp\left(-x^2/2\sigma_{2x}^2\right)\exp\left(-y^2/2\sigma_{2y}^2\right)}{(2\pi)^{3/2}\sigma_{2x}\sigma_{2y}\sigma_{2z}} \int_{-\infty}^{\infty} \exp\left(-z^2/2\sigma_{2z}^2\right)dz$$

$$= \frac{N_2 \exp\left(-x^2/2\sigma_{2x}^2\right)\exp\left(-y^2/2\sigma_{2y}^2\right)}{2\pi\sigma_{2x}\sigma_{2y}}\sigma$$

* This expression still correct when relativity done properly

Collider Luminosity

- **Probability an event is generated by a Beam 1 bunch with Gaussian density crossing a Beam 2 bunch with Gaussian density**

$$P = \frac{N_1 N_2}{2\pi\sqrt{\sigma_{1x}^2 + \sigma_{2x}^2}\sqrt{\sigma_{1y}^2 + \sigma_{2y}^2}}\sigma$$

- **Event rate with equal transverse beam sizes**

$$\frac{dN}{dt} = \frac{fN_1 N_2}{4\pi\sigma_x\sigma_y}\sigma = \mathcal{L}\sigma$$

- **Luminosity**

$$\mathcal{L} = \frac{fN_1 N_2}{4\pi\sigma_x\sigma_y} \sim 10^{33}\text{sec}^{-1}\text{cm}^{-2},$$

$$\text{for } f = 100 \text{ MHz}, N_1 = N_2 = 10^{10}, \sigma_x = \sigma_y = 10 \text{ microns}$$

Beam-Beam Tune Shift

- As we've seen previously, in a ring accelerator the number of transverse oscillations a particle makes in one circuit is called the "betatron tune" Q.
- Any deviation from the design values of the tune (in either the horizontal or vertical directions), is called a "tune shift". For long term stability of the beam in a ring accelerator, the tune must be highly controlled.

$$M_{tot} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} \cos\mu & \beta^* \sin\mu \\ -\sin\mu/\beta^* & \cos\mu \end{pmatrix}$$

$$= \begin{pmatrix} \cos\mu & \beta^* \sin\mu \\ -\cos\mu/f - \sin\mu/\beta^* & \cos\mu - (\beta^*/f)\sin\mu \end{pmatrix}$$

$$\cos(\mu + \Delta\mu) = \frac{\mathrm{Tr}(M_{tot})}{2} = \cos\mu - \frac{\beta^*}{2f}\sin\mu$$

$$\xi = \Delta Q = \frac{\Delta\mu}{2\pi} = \frac{\beta^*}{4\pi f} \qquad \beta^* << f$$

Bessetti-Erskine Solution

- 2-D potential of Bi-Gaussian transverse distribution

$$\rho(x,y) = \frac{Q'}{2\pi\sigma_x\sigma_y}\exp\left(-\frac{x^2}{2\sigma_x^2}\right)\exp\left(-\frac{y^2}{2\sigma_y^2}\right)$$

- Potential Theory gives solution to Poisson Equation

$$\nabla^2\phi = \frac{\rho(x,y)}{\varepsilon_0}$$

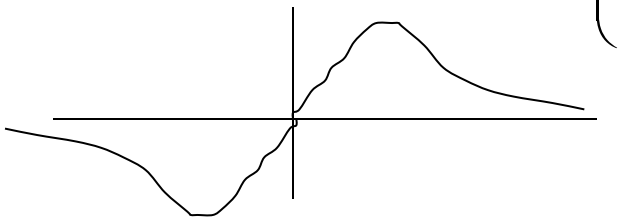
$$\phi(x,y) = \frac{Q'}{4\pi\varepsilon_0}\int_0^\infty \frac{\exp\left(-\frac{x^2}{2\sigma_x^2+q}\right)\exp\left(-\frac{y^2}{2\sigma_y^2+q}\right)}{\sqrt{2\sigma_x^2+q}\sqrt{2\sigma_y^2+q}}dq$$

- Bassetti and Erskine manipulate this to

$$E_x = \frac{Q'}{2\varepsilon_0\sqrt{2\pi\left(\sigma_x^2-\sigma_y^2\right)}}\,\mathrm{Im}\left[w\left(\frac{x+iy}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right)-\exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{x^2}{2\sigma_y^2}\right)w\left(\frac{x\left(\frac{\sigma_y}{\sigma_x}\right)+iy\left(\frac{\sigma_x}{\sigma_y}\right)}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right)\right]$$

$$E_y = \frac{Q'}{2\varepsilon_0\sqrt{2\pi\left(\sigma_x^2-\sigma_y^2\right)}}\,\mathrm{Re}\left[w\left(\frac{x+iy}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right)-\exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{x^2}{2\sigma_y^2}\right)w\left(\frac{x\left(\frac{\sigma_y}{\sigma_x}\right)+iy\left(\frac{\sigma_x}{\sigma_y}\right)}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right)\right]$$

$w(z)$ Complex error function



- We need 2-D linear field for small displacements

$$E_x(x,0) = -\frac{\partial\phi}{\partial x} \doteq \frac{Q'x}{2\pi\varepsilon_0}\int_0^\infty \frac{1}{\left(\sqrt{2\sigma_x^2+q}\right)^{3/2}\sqrt{2\sigma_y^2+q}}dq$$

- Can do the integral analytically

$$\int_0^\infty \frac{1}{\left(\sqrt{2\sigma_x^2+q}\right)^3\sqrt{2\sigma_y^2+q}}dq = \int_{\sigma_x^2+\sigma_y^2}^\infty \frac{\sigma_y^2-\sigma_x^2+q'}{\left(\sqrt{\sigma_x^2-\sigma_y^2+q'}\right)^3\left(\sqrt{\sigma_y^2-\sigma_x^2+q'}\right)^3}dq'$$

$$= \int_{\sigma_x^2+\sigma_y^2}^\infty \frac{\sigma_y^2-\sigma_x^2+q'}{\left(q'^2-\left(\sigma_y^2-\sigma_x^2\right)^2\right)^{3/2}}dq' = \left[-\frac{\left(\sigma_y^2-\sigma_x^2\right)q'}{\left(\sigma_y^2-\sigma_x^2\right)^2\left(q'^2-\left(\sigma_y^2-\sigma_x^2\right)^2\right)^{1/2}} - \frac{1}{\left(q'^2-\left(\sigma_y^2-\sigma_x^2\right)^2\right)^{1/2}}\right]_{\sigma_x^2+\sigma_y^2}^\infty$$

$$= -\frac{1}{\sigma_y^2-\sigma_x^2} + \frac{\sigma_y^2+\sigma_x^2}{\sigma_y^2-\sigma_x^2}\frac{1}{2\sigma_x\sigma_y} + \frac{1}{2\sigma_x\sigma_y} = \frac{-2\sigma_x\sigma_y+\sigma_y^2+\sigma_x^2+\sigma_y^2-\sigma_x^2}{\left(\sigma_y^2-\sigma_x^2\right)2\sigma_x\sigma_y} = \frac{1}{\left(\sigma_x+\sigma_y\right)\sigma_x}$$

- Similarly for the *y*-direction

$$E_y(0,y) = -\frac{\partial\phi}{\partial y} \doteq \frac{Q'y}{2\pi\varepsilon_0}\frac{1}{\sigma_y\left(\sigma_x+\sigma_y\right)}$$

Linear Beam-Beam Kick

- Linear kick received after interaction with bunch

$$\Delta\left(\gamma_1\beta_{1x}mc\right)=q_1\int_{-\infty}^{\infty}\left(\vec{E}_{2x}+\left(\vec{\mathrm{v}}\times\vec{B}\right)_{2x}\right)\left(\vec{x}_1\left(t\right),t\right)dt$$

by relativity, for oppositely moving beams

$$\Delta\gamma\beta_{1x}mc=q_1\left(1+\beta_{1z}\beta_{2z}\right)\int_{-\infty}^{\infty}\left(\vec{E}_{2x}\right)\left(\vec{x}_1\left(t\right),t\right)dt$$

Following linear Bassetti-Erskine model

$$E_{2x}\left(x,0,z,t\right)=\frac{q_2x}{2\pi\varepsilon_0}\frac{1}{\sigma_x\left(\sigma_x+\sigma_y\right)}\frac{1}{\sqrt{2\pi}\sigma_2}\exp\left(-\frac{\left(z-\beta_{2z}ct\right)^2}{2\sigma_z^2}\right)$$

q_1 moves with $\vec{x}\left(t\right)=\left(x,0,-\beta_{1z}ct\right)$

Linear Beam-Beam Tune Shift

$$\therefore \Delta\gamma\beta_{1x}mc = q_1 \frac{\left(1+\beta_{1z}\beta_{2z}\right)}{\beta_{1z}+\beta_{2z}} \frac{q_2 x}{2\pi\varepsilon_0 c} \frac{1}{\sigma_x\left(\sigma_x+\sigma_y\right)}$$

$$1/f = \frac{2N_2}{\gamma_1} \frac{\left(1+\beta_{1z}\beta_{2z}\right)}{\beta_{1z}+\beta_{2z}} \frac{r_1}{\sigma_x\left(\sigma_x+\sigma_y\right)} \qquad r_1 = \frac{e^2}{4\pi\varepsilon_0 m_1 c^2}$$

$$1/f \doteq \frac{2N_2 r_1}{\gamma_1\sigma_x\left(\sigma_x+\sigma_y\right)}$$

Both beams relativistic

From linear Bassetti-Erskine model, and replacing the beam size

$$\xi_x^1 = \frac{N_2 r_1}{2\pi\gamma_1} \frac{1}{\varepsilon_x^1\left(1+\sigma_y/\sigma_x\right)} \qquad \xi_y^1 = \frac{N_2 r_1}{2\pi\gamma_1} \frac{1}{\varepsilon_y^i\left(1+\sigma_y/\sigma_x\right)\left(\sigma_x/\sigma_y\right)}$$

Argument entirely symmetric wrt choice of bunch 1 and 2

$$\xi_x^i = \frac{N_{\bar{i}} r_i}{2\pi\gamma_i} \frac{1}{\varepsilon_x^i\left(1+\sigma_y/\sigma_x\right)} \qquad \xi_y^i = \frac{N_{\bar{i}} r_i}{2\pi\gamma_i} \frac{1}{\varepsilon_y^i\left(1+\sigma_y/\sigma_x\right)\left(\sigma_x/\sigma_y\right)}$$

Luminosity Beam-Beam tune-shift relationship

- Express Luminosity in terms of the (larger!) vertical tune shift (*i* either 1 or 2)

$$\mathcal{L} = \frac{f N_i \xi_y^i \gamma_i}{2 r_i \beta_{iy}^*}\left(1+\sigma_y/\sigma_x\right) = \frac{I_i}{e}\frac{\xi_y^i \gamma_i}{2 r_i \beta_{iy}^*}\left(1+\sigma_y/\sigma_x\right)$$

- Necessary, but not sufficient, for self-consistent design
- Expressed in this way, and given a known limit to the beam-beam tune shift, the only variables to manipulate to increase luminosity are the stored current, the aspect ratio, and the β^* (beta function value at the interaction point)
- Applies to ERL-ring colliders, stored beam (ions) only

Luminosity-Deflection Theorem

- Luminosity-tune shift formula is linearized version of a much more general formula discovered by Krafft and generalized by V. Ziemann.

- Relates easy calculation (luminosity) to a hard calculation (beam-beam force), and contains all the standard results in beam-beam interaction theory.

- Based on the fact that the relativistic beam-beam force is almost entirely transverse, i. e., 2-D electrostatics applies.

2-D Electrostatics Theorem

$$\vec{E}(\vec{x}) = \frac{2Q'}{4\pi\varepsilon_0}\frac{\vec{x}-\vec{x}'}{\left|\vec{x}-\vec{x}'\right|^2}$$

$$\vec{F}'_{21} = -\vec{F}'_{12} = \frac{1}{2\pi\varepsilon_0}\iint \rho_2(\vec{x}_2)\frac{\vec{x}_2-\vec{x}_1}{\left|\vec{x}_2-\vec{x}_1\right|}\rho_1(\vec{x}_1)\,d^2\vec{x}_1 d^2\vec{x}_2 \quad \text{1 on 2}$$

$$n_1(\vec{x}_1) = \rho_1(\vec{x}_1)/Q_1' \quad n_2(\vec{x}_2) = \rho_1\left(\vec{x}_2+\vec{b}\right)/Q_1' \text{ zero centerred}$$

$$Q_i' = \iint \rho_i(\vec{x})\,d^2\vec{x} \qquad \vec{b} = \iint \vec{x}\rho_2(\vec{x})\,d^2\vec{x}/Q_2'$$

$$\vec{F}'_{21} = -\vec{F}'_{12} = \frac{Q_1'Q_2'}{2\pi\varepsilon_0}\iint n_2(\vec{x}_2)\frac{\vec{x}_1+\vec{b}-\vec{x}_2}{\left|\vec{x}_1+\vec{b}-\vec{x}_2\right|^2}n_1(\vec{x}_1)\,d^2\vec{x}_1 d^2\vec{x}_2$$

$$\vec{\nabla}_{\vec{b}} \cdot \frac{\vec{x}_1 + \vec{b} - \vec{x}_2}{\left|\vec{x}_1 + \vec{b} - \vec{x}_2\right|^2} = 2\pi\delta\left(x_2 + b_x + x_1\right)\delta\left(y_2 + b_y + y_1\right)$$

$$\vec{\nabla}_{\vec{b}} \cdot \vec{F}'_{21} = \frac{1}{\varepsilon_0}\iint \rho_2\left(\vec{x} + \vec{b}\right)\rho_1\left(\vec{x}\right)d^2\vec{x}$$

Generalizes $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ $\left(\text{take } \rho_2\left(\vec{x}\right) \propto \delta^2\left(\vec{x} + \vec{b}\right)\right)$

Transverse interaction in the beam-beam problem

$$\Delta p_1 = \frac{q_1 q_2}{2\pi\varepsilon_0 c}\frac{\vec{x}_1 - \vec{x}_2}{\left|\vec{x}_1 - \vec{x}_2\right|^2}$$

$$\vec{D}\left(\vec{b}\right) = \Delta\gamma_1\vec{\beta}_1 = -\Delta m_2\gamma_2\vec{\beta}_2 \,/\, m_1$$

$$= \frac{q_1 q_2}{m_1 c^2}\iint n_2\left(\vec{x}_2\right)\frac{\vec{x}_1 - \vec{x}_2 - \vec{b}}{\left|\vec{x}_1 - \vec{x}_2 - \vec{b}\right|^2} n_1\left(\vec{x}_1\right) d^2\vec{x}_1 d^2\vec{x}_2$$

$$\vec{\nabla}_{\vec{b}} \cdot \vec{D}\left(\vec{b}\right) = 4\pi N_2 r_e \iint n_2\left(\vec{x} - \vec{b}\right) n_1\left(\vec{x}\right) d^2\vec{x} \qquad r_e = \frac{e^2}{4\pi\varepsilon_0 m c^2}$$

$$L\left(\vec{b}\right) = N_1 N_2 \iint n_2\left(\vec{x} - \vec{b}\right) n_1\left(\vec{x}\right) d^2\vec{x}$$

$$L\left(\vec{b}\right) = \frac{N_1}{4\pi r_e}\vec{\nabla}_{\vec{b}} \cdot \vec{D}\left(\vec{b}\right)$$

$$L\left(\vec{b}\right) = -\frac{N_2}{4\pi r_e}\vec{\nabla}_{\vec{b}} \cdot \left(\Delta\gamma_2\vec{\beta}_2\right)$$

$$\begin{pmatrix} D_x \\ D_y \end{pmatrix} = \frac{\gamma_1}{2f} \begin{pmatrix} \sigma_y / \sigma_x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_x \\ b_y \end{pmatrix}$$

$$L = \frac{N_1 \gamma \xi}{2 r_e \beta^*} \left(1 + \sigma_y / \sigma_x\right) \qquad \text{as before}$$

Maximum when

$$\frac{\partial}{\partial b_x}\left[\frac{\partial D_x}{\partial b_x}\right] = 0, \qquad \frac{\partial}{\partial b_y}\left[\frac{\partial D}{\partial b_y}\right] = 0$$

Luminosity-Deflection Pairs

- Round Beam Fast Model

$$\vec{D}\left(\vec{b}\right)=\frac{2N_2 r_e \vec{b}}{\sigma^2+b^2} \qquad L\left(\vec{b}\right)=\frac{N_1 N_2 \sigma^2}{\pi\left(\sigma^2+b^2\right)^2}$$

- Gaussian Macroparticles

$$\vec{D}\left(\vec{b}\right)=\vec{D}_{Bassetti\_Erskine}\left(\vec{b};\sqrt{\sigma_{1x}^2+\sigma_{2x}^2};\sqrt{\sigma_{1y}^2+\sigma_{2y}^2}\right)$$

$$L\left(\vec{b}\right)=\frac{N_1 N_2}{2\pi\sqrt{\sigma_{1x}^2+\sigma_{2x}^2}\sqrt{\sigma_{1y}^2+\sigma_{2y}^2}}\exp\left(-\frac{b_x^2}{\sqrt{\sigma_{1x}^2+\sigma_{2x}^2}}\right)\exp\left(-\frac{b_y^2}{\sqrt{\sigma_{1y}^2+\sigma_{2y}^2}}\right)$$

- Smith-Laslett Model

$$\vec{D}\left(\vec{b}\right)=\frac{2N_2 r_e \vec{b}}{\hat{b}^2 AB}\left\{\frac{\left(4\hat{b}^2+2\hat{b}^4\right)}{\left(4\hat{b}^2+\hat{b}^4\right)}-\frac{4\hat{b}^2}{\left(4\hat{b}^2+\hat{b}^4\right)^{3/2}}\left\{\sinh^{-1}\left[\frac{\hat{b}^3}{2}+\frac{3\hat{b}}{2}\right]+\sinh^{-1}\left[\frac{\hat{b}}{2}\right]\right\}\right\}$$

$$L\left(\vec{b}\right)=\frac{N_1 N_2}{\pi AB}\left\{\frac{\left(2\hat{b}^2-4\right)\hat{b}^2}{\left(4\hat{b}^2+\hat{b}^4\right)^2}-\frac{4\hat{b}^2\left(1+\hat{b}^2\right)}{\left(4\hat{b}^2+\hat{b}^4\right)^{5/2}}\left\{\sinh^{-1}\left[\frac{\hat{b}^3}{2}+\frac{3\hat{b}}{2}\right]+\sinh^{-1}\left[\frac{\hat{b}}{2}\right]\right\}\right\}$$

$$\hat{b}^2=\left(\frac{b_x}{A}\right)^2+\left(\frac{b_y}{B}\right)^2$$