

Accelerator Physics Particle Acceleration

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Lecture 6

Transfer Matrix

- For solenoid of length L , transfer matrix is

$$M_{sol} = M_{end}^{-1} M_{from\ rot} M_{dec} M_{to\ rot} M_{end}$$

- Decoupled matrix in rotating coordinate system (Eq. 17.34)

$$M_{dec} = \begin{pmatrix} \cos(\omega_L L / v_z) & (v_z / \omega_L) \sin(\omega_L L / v_z) & 0 & 0 \\ -(\omega_L / v_z) \sin(\omega_L L / v_z) & \cos(\omega_L L / v_z) & 0 & 0 \\ 0 & 0 & \cos(\omega_L L / v_z) & (v_z / \omega_L) \sin(\omega_L L / v_z) \\ 0 & 0 & -(\omega_L / v_z) \sin(\omega_L L / v_z) & \cos(\omega_L L / v_z) \end{pmatrix}$$

- Matrix from Rotating Coordinates (Eq. 17.36, corrected)

$$M_{from\ rot} = \begin{pmatrix} \cos(\omega_L L / v_z) & 0 & \sin(\omega_L L / v_z) & 0 \\ -(\omega_L / v_z) \sin(\omega_L L / v_z) & \cos(\omega_L L / v_z) & (\omega_L / v_z) \cos(\omega_L L / v_z) & \sin(\omega_L L / v_z) \\ -\sin(\omega_L L / v_z) & 0 & \cos(\omega_L L / v_z) & 0 \\ -(\omega_L / v_z) \cos(\omega_L L / v_z) & -\sin(\omega_L L / v_z) & -(\omega_L / v_z) \sin(\omega_L L / v_z) & \cos(\omega_L L / v_z) \end{pmatrix}$$

To/from rotating coordinates

$$v(z) = x(z) \cos(\omega_L z / v_z) - y(z) \sin(\omega_L z / v_z)$$

$$w(z) = x(z) \sin(\omega_L z / v_z) + y(z) \cos(\omega_L z / v_z)$$

$$\frac{dv}{dz} = \left(\frac{dx}{dz} - \frac{\omega_L}{v_z} y \right) \cos(\omega_L z / v_z) - \left(\frac{dy}{dz} + \frac{\omega_L}{v_z} x \right) \sin(\omega_L z / v_z)$$

$$\frac{dw}{dz} = \left(\frac{dx}{dz} - \frac{\omega_L}{v_z} y \right) \sin(\omega_L z / v_z) + \left(\frac{dy}{dz} + \frac{\omega_L}{v_z} x \right) \cos(\omega_L z / v_z)$$

$$\begin{pmatrix} v(z) \\ v'(z) \\ w(z) \\ w'(z) \end{pmatrix} = M_{to\ rot} \begin{pmatrix} x(z) \\ x'(z) \\ y(z) \\ y'(z) \end{pmatrix} = \begin{pmatrix} \cos(\omega_L z / v_z) & 0 & -\sin(\omega_L z / v_z) & 0 \\ -(\omega_L / v_z) \sin(\omega_L z / v_z) & \cos(\omega_L z / v_z) & -(\omega_L / v_z) \cos(\omega_L z / v_z) & -\sin(\omega_L z / v_z) \\ \sin(\omega_L z / v_z) & 0 & \cos(\omega_L z / v_z) & 0 \\ (\omega_L / v_z) \cos(\omega_L z / v_z) & \sin(\omega_L z / v_z) & -(\omega_L / v_z) \sin(\omega_L z / v_z) & \cos(\omega_L z / v_z) \end{pmatrix} \begin{pmatrix} x(z) \\ x'(z) \\ y(z) \\ y'(z) \end{pmatrix}$$

$$\begin{pmatrix} x'(z) \\ y'(z) \end{pmatrix} = \begin{pmatrix} (\omega_L z / v_z) y \\ -(\omega_L z / v_z) x \end{pmatrix} + \begin{pmatrix} \cos(\omega_L z / v_z) & \sin(\omega_L z / v_z) \\ -\sin(\omega_L z / v_z) & \cos(\omega_L z / v_z) \end{pmatrix} \begin{pmatrix} v'(z) \\ w'(z) \end{pmatrix}$$

$$\begin{pmatrix} x(z) \\ x'(z) \\ y(z) \\ y'(z) \end{pmatrix} = M_{from\ rot} \begin{pmatrix} v(z) \\ v'(z) \\ w(z) \\ w'(z) \end{pmatrix} = \begin{pmatrix} \cos(\omega_L z / v_z) & 0 & \sin(\omega_L z / v_z) & 0 \\ -(\omega_L / v_z) \sin(\omega_L z / v_z) & \cos(\omega_L z / v_z) & (\omega_L / v_z) \cos(\omega_L z / v_z) & \sin(\omega_L z / v_z) \\ -\sin(\omega_L z / v_z) & 0 & \cos(\omega_L z / v_z) & 0 \\ -(\omega_L / v_z) \cos(\omega_L z / v_z) & -\sin(\omega_L z / v_z) & -(\omega_L / v_z) \sin(\omega_L z / v_z) & \cos(\omega_L z / v_z) \end{pmatrix} \begin{pmatrix} v(z) \\ v'(z) \\ w(z) \\ w'(z) \end{pmatrix}$$

- Fringe effect by conservation canonical momentum

$$M_{end} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & v_z / \omega_L & 0 \\ 0 & 0 & 1 & 0 \\ -v_z / \omega_L & 0 & 0 & 1 \end{pmatrix} \quad M_{end}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -v_z / \omega_L & 0 \\ 0 & 0 & 1 & 0 \\ v_z / \omega_L & 0 & 0 & 1 \end{pmatrix}$$

- Match to Boundary Conditions at $z = 0$

$$M_{to\ rot}(z=0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\omega_L / v_z & 0 \\ 0 & 0 & 1 & 0 \\ \omega_L / v_z & 0 & 0 & 1 \end{pmatrix} = M_{end}^{-1}$$

Total Solenoid Transfer



$$M_{sol} = \begin{pmatrix} \cos^2 \Phi & (1/S)\sin 2\Phi & (1/2)\sin 2\Phi & (2/S)\sin^2 \Phi \\ -(S/4)\sin 2\Phi & \cos^2 \Phi & -(S/2)\sin^2 \Phi & (1/2)\sin 2\Phi \\ -(1/2)\sin 2\Phi & -(2/S)\sin^2 \Phi & \cos^2 \Phi & (1/S)\sin 2\Phi \\ (S/2)\sin^2 \Phi & -(1/2)\sin 2\Phi & -(S/4)\sin 2\Phi & \cos^2 \Phi \end{pmatrix}$$

Wiedemann 17.39

$$\Phi = \omega_L L / v_z$$

$$S = 2\omega_L / v_z$$

$$M_{sol} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/f_{sol} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/f_{sol} & 1 \end{pmatrix}$$

Thin Lens

Easy Calculation that Works

- Wiedemann points out the following simple calculation is OK

$$M_{sol} = \begin{pmatrix} \cos \Phi & 0 & \sin \Phi & 0 \\ 0 & \cos \Phi & 0 & \sin \Phi \\ -\sin \Phi & 0 & \cos \Phi & 0 \\ 0 & -\sin \Phi & 0 & \cos \Phi \end{pmatrix} \begin{pmatrix} \cos \Phi & 2 \sin \Phi / S & 0 & 0 \\ -\frac{S}{2} \sin \Phi & \cos \Phi & 0 & 0 \\ 0 & 0 & \cos \Phi & 2 \sin \Phi / S \\ 0 & 0 & -\frac{S}{2} \sin \Phi & \cos \Phi \end{pmatrix}$$

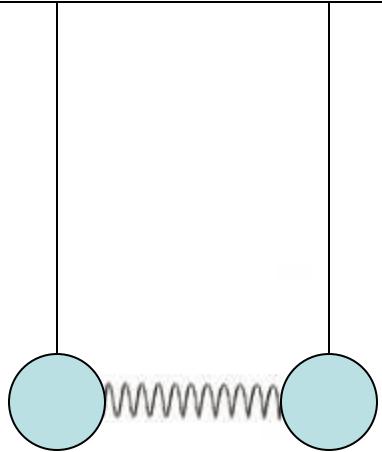
- Works because

$$M_{to\ rot} M_{end} = I$$

$$M_{end}^{-1} M_{from\ rot} = \begin{pmatrix} \cos \Phi & 0 & -\sin \Phi & 0 \\ 0 & \cos \Phi & 0 & -\sin \Phi \\ -\sin \Phi & 0 & \cos \Phi & 0 \\ 0 & -\sin \Phi & 0 & \cos \Phi \end{pmatrix}$$

- Explanation hard to follow

Coupled Pendula



$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega_g^2 x^2}{2} + \frac{m\omega_g^2 y^2}{2} + \frac{m\omega_s^2(x-y)^2}{2}$$

Equations of motion

$$\frac{d^2}{dt^2} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \omega_g^2 + \omega_s^2 & -\omega_s^2 \\ -\omega_s^2 & \omega_g^2 + \omega_s^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{i\omega t}$$

$$\text{Det} \begin{pmatrix} \omega_g^2 + \omega_s^2 - \omega^2 & -\omega_s^2 \\ -\omega_s^2 & \omega_g^2 + \omega_s^2 - \omega^2 \end{pmatrix} = 0$$

Solutions



Eigenvalues

$$(\omega^2 - \omega_g^2 - \omega_s^2)^2 - \omega_s^2 = 0$$

$$\omega^2 = \omega_g^2 + \omega_s^2 \pm \omega_s^2$$

Eigenvectors

$$\omega_- = \omega_g$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{symmetric mode}$$

$$\omega_+ = \sqrt{\omega_g^2 + 2\omega_s^2}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{antisymmetric mode}$$

Time Dependence

For an oscillation starting in x -direction

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{x_0}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_g t) + \frac{x_0}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\left(\sqrt{\omega_g^2 + 2\omega_s^2} t\right)$$

$$\omega_g = \frac{\omega_g + \sqrt{\omega_g^2 + 2\omega_s^2}}{2} + \frac{\omega_g - \sqrt{\omega_g^2 + 2\omega_s^2}}{2}$$

$$\sqrt{\omega_g^2 + 2\omega_s^2} = \frac{\omega_g + \sqrt{\omega_g^2 + 2\omega_s^2}}{2} - \frac{\omega_g - \sqrt{\omega_g^2 + 2\omega_s^2}}{2}$$

$$\therefore x(t) = x_0 \cos\left[\frac{\omega_g + \sqrt{\omega_g^2 + 2\omega_s^2}}{2} t\right] \cos\left[\frac{\omega_g - \sqrt{\omega_g^2 + 2\omega_s^2}}{2} t\right]$$

$$\therefore y(t) = -x_0 \sin\left[\frac{\omega_g + \sqrt{\omega_g^2 + 2\omega_s^2}}{2} t\right] \sin\left[\frac{\omega_g - \sqrt{\omega_g^2 + 2\omega_s^2}}{2} t\right]$$

Qualitatively

- Oscillation energy migrates $x \rightarrow y \rightarrow x$
- Period for a complete cycle is

$$\frac{4\pi}{\omega_g - \sqrt{\omega_g^2 + 2\omega_s^2}}$$

Becomes longer the weaker the coupling (\rightarrow compensation)

- If un-coupled oscillation periods different

$$\omega^2 = \frac{\omega_1^2 + \omega_2^2 + 2\omega_s^2 \pm \sqrt{(\omega_1^2 - \omega_2^2)^2 - 4\omega_s^4}}{2}$$

Eigenvectors no longer pure symmetric and antisymmetric
but “migration” is fairly generic behavior

Energy and Mode Invariants

- In normal mode coordinates

$$q_+ \frac{1}{\sqrt{2}}(x - y) \quad p_+ = \frac{1}{\sqrt{2}}(\dot{x} - \dot{y}) \quad q_- \frac{1}{\sqrt{2}}(x + y) \quad p_- = \frac{1}{\sqrt{2}}(\dot{x} + \dot{y})$$

- (uncoupled) Hamiltonian is

$$H = \frac{p_+^2}{2m} + \frac{m\omega_+^2 q_+^2}{2} + \frac{p_-^2}{2m} + \frac{m\omega_-^2 q_-^2}{2}$$

and is a constant of the motion.

- So are the individual mode excitation amplitudes

$$\epsilon_+ = \frac{2}{\omega_+} \left(\frac{p_+^2}{2m} + \frac{m\omega_+^2 q_+^2}{2} \right) \quad \epsilon_- = \frac{2}{\omega_-} \left(\frac{p_-^2}{2m} + \frac{m\omega_-^2 q_-^2}{2} \right)$$

Solutions Again

- In normal mode coordinates

$$q_+ = \sqrt{\varepsilon_+ \beta_+} \cos(\omega_+ t + \delta_+) \quad q_- = \sqrt{\varepsilon_- \beta_-} \cos(\omega_- t + \delta_-)$$

- betas and deltas from

$$\beta_{\pm} = \frac{1}{m\omega_{\pm}} \quad \delta_{\pm} = \tan^{-1} \frac{p_{\pm 0}}{m\omega_{\pm} q_0}$$

- back in coupled variables excitation looks like

$$x_1(z) = \sqrt{\varepsilon_+ \beta_+} \cos(\omega_+ t) \quad y_1(z) = \sqrt{\varepsilon_+ \beta_+} \cos(\omega_+ t + \pi)$$

$$x_2(z) = \sqrt{\varepsilon_+ \beta_+} \sin(\omega_+ t) \quad y_2(z) = \sqrt{\varepsilon_+ \beta_+} \sin(\omega_+ t + \pi)$$

$$x_3(z) = \sqrt{\varepsilon_- \beta_-} \cos(\omega_- t) \quad y_3(z) = \sqrt{\varepsilon_- \beta_-} \cos(\omega_- t)$$

$$x_4(z) = \sqrt{\varepsilon_- \beta_-} \sin(\omega_- t) \quad y_4(z) = \sqrt{\varepsilon_- \beta_-} \cos(\omega_- t)$$

Wiedemann Discussion



- 1-D know two pseudoharmonic trajectories

$$u_1 = \sqrt{\beta(z)} \cos \psi(z)$$

$$u_2 = \sqrt{\beta(z)} \sin \psi(z)$$

- z -derivative

$$u'_1 = \frac{1}{\sqrt{\beta(z)}} (-\alpha \cos \psi(z) - \sin \psi(z))$$

$$u'_2 = \frac{1}{\sqrt{\beta(z)}} (-\alpha \sin \psi(z) + \cos \psi(z))$$

- Express ellipse functions in terms of solutions

$$u_1^2 + u_2^2 = \beta(z)$$

$$u_1 u'_1 + u_2 u'_2 = -\alpha$$

$$u_1'^2 + u_2'^2 = \frac{\alpha^2 + 1}{\beta} = \gamma$$

- Change in phase (phase advance) is

$$\cos(\Delta\psi) = \cos(\psi - \psi_0) =$$

$$\frac{u_1 u_{10} + u_2 u_{20}}{\sqrt{u_1^2 + u_2^2} \sqrt{u_{10}^2 + u_{20}^2}}$$

Transfer Matrix

- General solution

$$\begin{pmatrix} x(z) \\ x'(z) \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ u'_1 & u'_2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

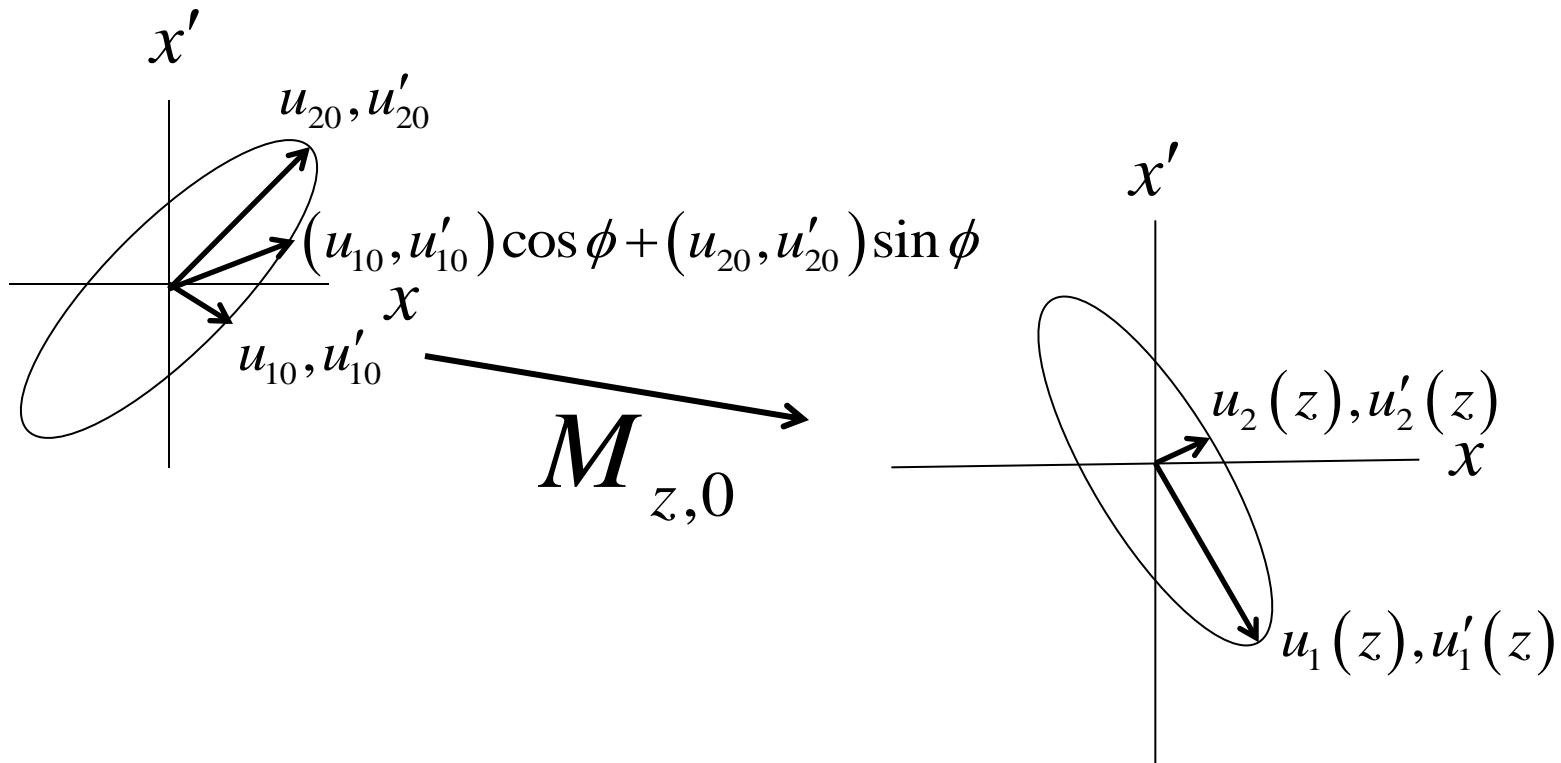
- Initial conditions

$$\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = \begin{pmatrix} u_{10} & u_{20} \\ u'_{10} & u'_{20} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{u_{10}u'_{20} - u_{20}u'_{10}} \begin{pmatrix} u'_{20} & -u_{20} \\ -u'_{10} & u_{10} \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- Transfer matrix

$$M = \begin{pmatrix} u_1 & u_2 \\ u'_1 & u'_2 \end{pmatrix} \begin{pmatrix} u'_{20} & -u_{20} \\ -u'_{10} & u_{10} \end{pmatrix} = \begin{pmatrix} u_1u'_{20} - u_2u'_{10} & u_2u_{10} - u_1u_{20} \\ u'_1u'_{20} - u'_2u'_{10} & u'_2u_{10} - u'_1u_{20} \end{pmatrix}$$

Ellipse Transformation



In 2×2 D



- As in toy problem, invariant is hyperellipsoid in 4 D.
Parameterize individual state points as

$$\vec{v}(z) = \sqrt{\epsilon_I} [v_1(z)\cos\vartheta_I - v_2(z)\sin\vartheta_I] \cos\chi$$
$$+ \sqrt{\epsilon_{II}} [v_3(z)\cos\vartheta_{II} - v_4(z)\sin\vartheta_{II}] \sin\chi$$

- Compared to uncoupled case have twice the number of matrix elements. Need twice the number of solutions.
Wiedemann chooses

$$x_1(z) = \sqrt{\beta_{x_I}(z)} \cos\phi_{x_I}(z) \quad y_1(z) = \sqrt{\beta_{y_I}(z)} \cos\phi_{y_I}(z)$$
$$x_2(z) = \sqrt{\beta_{x_I}(z)} \sin\phi_{x_I}(z) \quad y_2(z) = \sqrt{\beta_{y_I}(z)} \sin\phi_{y_I}(z)$$
$$x_3(z) = \sqrt{\beta_{x_{II}}(z)} \cos\phi_{x_{II}}(z) \quad y_3(z) = \sqrt{\beta_{y_{II}}(z)} \cos\phi_{y_{II}}(z)$$
$$x_4(z) = \sqrt{\beta_{x_{II}}(z)} \sin\phi_{x_{II}}(z) \quad y_4(z) = \sqrt{\beta_{y_{II}}(z)} \sin\phi_{y_{II}}(z)$$

As Before



$$\beta_{x_I} = x_1^2 + x_2^2$$

$$\beta_{y_I} = y_1^2 + y_2^2$$

$$\beta_{x_{II}} = x_3^2 + x_4^2$$

$$\beta_{y_{II}} = y_3^2 + y_4^2$$

$$\cos \phi_{x_I} = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$$

$$\cos \phi_{y_I} = \frac{y_1}{\sqrt{y_1^2 + y_2^2}}$$

$$\cos \phi_{x_{II}} = \frac{x_3}{\sqrt{x_3^2 + x_4^2}}$$

$$\cos \phi_{y_I} = \frac{y_3}{\sqrt{y_3^2 + y_4^2}}$$

⋮

Convert Fundamental Solutions

- Convert usual phase space variables

$$x_1(z) = \sqrt{\beta_{x_I}} \cos \phi_{x_I} \quad x'_1(z) = \sqrt{\gamma_{x_I}} \cos \psi_{x_I}$$

$$\rightarrow \gamma_{x_I} = \frac{\beta_{x_I}^2 \phi'^2 + \alpha_{x_I}^2}{\beta_{x_I}} \quad \psi_{x_I} = \phi_{x_I} - \tan^{-1} \frac{\beta_{x_I} \phi'_{x_I}}{\alpha_{x_I}}$$

- Points of hyper-ellipsoid

$$v(z) = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \sqrt{\epsilon_I} \begin{pmatrix} \sqrt{\beta_{x_I}(z)} \cos(\phi_{x_I} + \vartheta_I) \\ \sqrt{\gamma_{x_I}(z)} \cos(\psi_{x_I} + \vartheta_I) \\ \sqrt{\beta_{y_I}(z)} \cos(\phi_{y_I} + \vartheta_I) \\ \sqrt{\gamma_{y_I}(z)} \cos(\psi_{y_I} + \vartheta_I) \end{pmatrix} \cos \chi + \sqrt{\epsilon_{II}} \begin{pmatrix} \sqrt{\beta_{x_{II}}(z)} \cos(\phi_{x_{II}} + \vartheta_{II}) \\ \sqrt{\gamma_{x_{II}}(z)} \cos(\psi_{x_{II}} + \vartheta_{II}) \\ \sqrt{\beta_{y_{II}}(z)} \cos(\phi_{y_{II}} + \vartheta_{II}) \\ \sqrt{\gamma_{y_{II}}(z)} \cos(\psi_{y_{II}} + \vartheta_{II}) \end{pmatrix} \sin \chi$$

Beam size

- Set all derivatives zero

$$\phi_{x,y_{I,II}}(z) = -\vartheta_{I,II}$$

$$\frac{d(x,y)}{d\chi} = 0 \rightarrow \sin^2 \chi = \frac{\varepsilon_{II} \beta_{(x,y)}}{E_{(x,y)}} \quad E_{(x,y)} = \sqrt{\varepsilon_I \beta_{(x,y)_I} + \varepsilon_{II} \beta_{(x,y)_{II}}}$$

$$\psi_{x,y_{I,II}}(z) = -\vartheta_{I,II}$$

$$\frac{d(x',y')}{d\chi} = 0 \rightarrow \sin^2 \chi = \frac{\varepsilon_{II} \beta_{(x,y)}}{A_{(x,y)}} \quad A_{(x,y)} = \sqrt{\varepsilon_I \gamma_{(x,y)_I} + \varepsilon_{II} \gamma_{(x,y)_{II}}}$$

- Projected Ellipse

$$E'_{(x,y)} = -\frac{\varepsilon_I \alpha_{(x,y)_I} + \varepsilon_{II} \alpha_{(x,y)_{II}}}{\sqrt{\varepsilon_I \beta_{(x,y)_I} + \varepsilon_{II} \beta_{(x,y)_{II}}}}$$

$$A_{(x,y)}^2 (x,y)^2 - 2E'_{(x,y)} E_{(x,y)} (x,y)(x',y') + E_{(x,y)}^2 (x',y')^2 = \varepsilon_{(x,y)}^2$$

$$\varepsilon_{(x,y)}^2 = E_{(x,y)} \sqrt{A_{(x,y)}^2 - E'^2_{(x,y)}}$$

Beam Tilt



- Project x - y plane

$$E_{(x,y)} = \sqrt{\varepsilon_I \beta_{(x,y)_I} + \varepsilon_{II} \beta_{(x,y)_{II}}}$$
$$E_{xy} = \frac{\varepsilon_I \sqrt{\beta_{x_I} \beta_{y_I}} \cos \Delta\phi_I + \varepsilon_{II} \sqrt{\beta_{x_{II}} \beta_{y_{II}}} \cos \Delta\phi_{II}}{E_x}$$
$$\Delta\phi_{I,II} = \phi_{x_{I,II}} - \phi_{x_{I,II}}$$

$$\tan 2\psi = \frac{2E_x E_y}{E_x^2 - E_y^2} = 2 \frac{\varepsilon_I \sqrt{\beta_{x_I} \beta_{y_I}} \cos \Delta\phi_I + \varepsilon_{II} \sqrt{\beta_{x_{II}} \beta_{y_{II}}} \cos \Delta\phi_{II}}{\varepsilon_{x_I} \Delta\beta_I + \varepsilon_{x_{II}} \Delta\beta_{II}}$$

Beam sizes

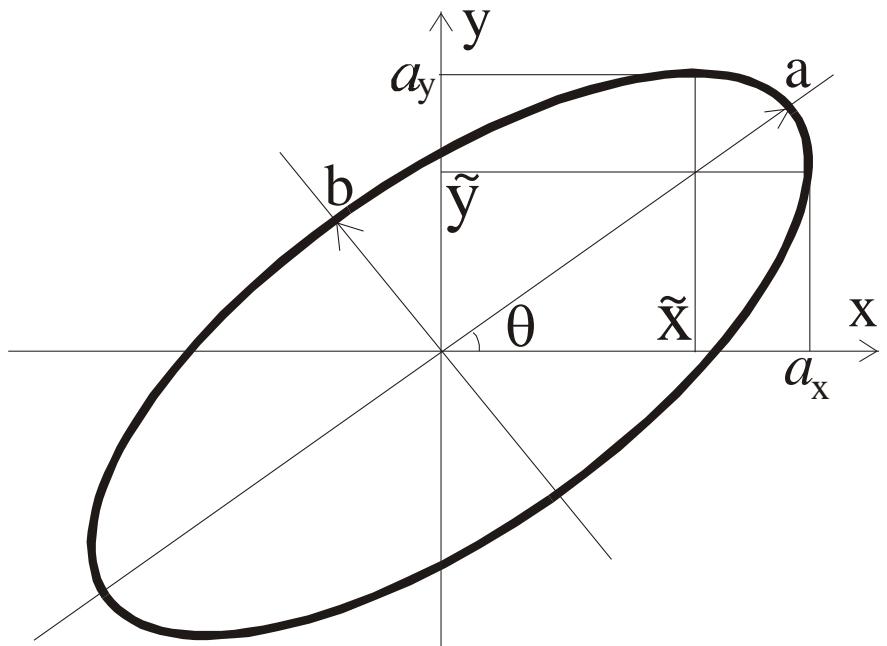


$$a_x = \sqrt{\varepsilon_1 \beta_{1x} + \varepsilon_2 \beta_{2x}}$$

$$a_y = \sqrt{\varepsilon_1 \beta_{1y} + \varepsilon_2 \beta_{2y}}$$

❖ Ellipse equation

$$\frac{x^2}{a_x^2} - \frac{2\tilde{\alpha}xy}{a_x a_y} + \frac{y^2}{a_y^2} = 1 - \tilde{\alpha}^2$$



◆ Ellipse rotation parameter

$$\tilde{\alpha} \equiv \frac{\langle xy \rangle}{\sqrt{\langle x^2 \rangle \langle y^2 \rangle}} = \frac{\tilde{y}}{a_y} = \frac{\tilde{x}}{a_x} = \frac{\sqrt{\beta_{1x} \beta_{1y}} \varepsilon_1 \cos \nu_1 + \sqrt{\beta_{2x} \beta_{2y}} \varepsilon_2 \cos \nu_2}{\sqrt{\varepsilon_1 \beta_{1x} + \varepsilon_2 \beta_{2x}} \sqrt{\varepsilon_1 \beta_{1y} + \varepsilon_2 \beta_{2y}}}$$

Decoupling



- Counter Wound Solenoids
- Compensating Solenoids or Skew Quads
 - Jefferson Lab
 - MEIC Application
 - Rings
- Flat Beams

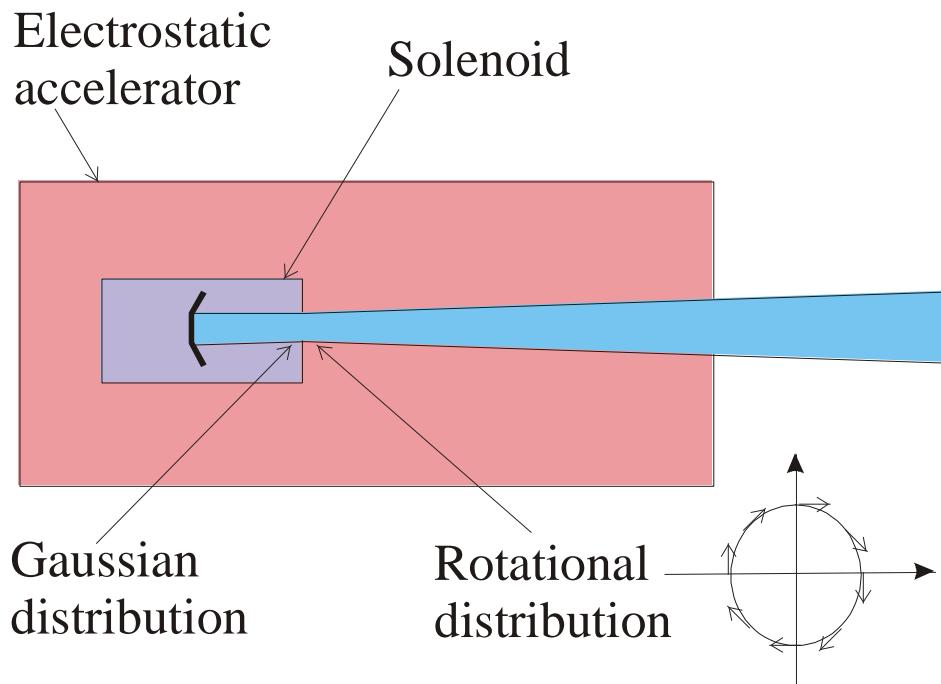
Counter-Wound Solenoids

$$\begin{aligned}
 & \left(\begin{array}{cccc}
 \cos^2 \Phi & (1/S) \sin 2\Phi & -(1/2) \sin 2\Phi & -(2/S) \sin^2 \Phi \\
 -(S/4) \sin 2\Phi & \cos^2 \Phi & (S/2) \sin^2 \Phi & -(1/2) \sin 2\Phi \\
 (1/2) \sin 2\Phi & (2/S) \sin^2 \Phi & \cos^2 \Phi & (1/S) \sin 2\Phi \\
 -(S/2) \sin^2 \Phi & (1/2) \sin 2\Phi & -(S/4) \sin 2\Phi & \cos^2 \Phi
 \end{array} \right) \\
 \times & \left(\begin{array}{cccc}
 \cos^2 \Phi & (1/S) \sin 2\Phi & (1/2) \sin 2\Phi & (2/S) \sin^2 \Phi \\
 -(S/4) \sin 2\Phi & \cos^2 \Phi & -(S/2) \sin^2 \Phi & (1/2) \sin 2\Phi \\
 -(1/2) \sin 2\Phi & -(2/S) \sin^2 \Phi & \cos^2 \Phi & (1/S) \sin 2\Phi \\
 (S/2) \sin^2 \Phi & -(1/2) \sin 2\Phi & -(S/4) \sin 2\Phi & \cos^2 \Phi
 \end{array} \right) \\
 = & \left(\begin{array}{cccc}
 \cos 2\Phi & (2/S) \sin 2\Phi & 0 & 0 \\
 -(S/2) \sin 2\Phi & \cos 2\Phi & 0 & 0 \\
 0 & 0 & \cos 2\Phi & (2/S) \sin 2\Phi \\
 0 & 0 & -(S/2) \sin 2\Phi & \cos 2\Phi
 \end{array} \right)
 \end{aligned}$$

Axisymmetric Rotational Distribution



❖ Fermilab electron cooling



The electron beam distribution is axially symmetric, and uncoupled at the cathode:

$$\mathbf{E}_B = \frac{1}{\varepsilon_T} \begin{bmatrix} \gamma_0 & \alpha_0 & 0 & 0 \\ \alpha_0 & \beta_0 & 0 & 0 \\ 0 & 0 & \gamma_0 & \alpha_0 \\ 0 & 0 & \alpha_0 & \beta_0 \end{bmatrix}$$

where $\varepsilon_T = r_c \sqrt{mkT_c} / P_0$ is the thermal emittance of the beam

Axisymmetric Rotational Distribution



- ◆ At the exit of the solenoid the electron beam distribution is still axially symmetric

$$\boldsymbol{\Xi}_{in} = \boldsymbol{\Phi}^T \boldsymbol{\Xi}_B \boldsymbol{\Phi} = \frac{1}{\varepsilon_T} \begin{bmatrix} \gamma_0 + \Phi^2 \beta_0 & \alpha_0 & 0 & -\Phi \beta_0 \\ \alpha_0 & \beta_0 & \Phi \beta_0 & 0 \\ 0 & \Phi \beta_0 & \gamma_0 + \Phi^2 \beta_0 & \alpha_0 \\ -\Phi \beta_0 & 0 & \alpha_0 & \beta_0 \end{bmatrix}$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \Phi & 0 \\ 0 & 0 & 1 & 0 \\ -\Phi & 0 & 0 & 1 \end{bmatrix}$$

- ♠ $\Phi = eB / 2P_0c$ is the rotational focusing strength of the solenoid
- ♠ B is the solenoid magnetic field.

Axisymmetric Rotational Distribution



- ◆ The eigen-vectors of the rotational distribution:

$$\hat{\mathbf{v}}_1 = \begin{bmatrix} \sqrt{\beta} \\ -\frac{i+2\alpha}{2\sqrt{\beta}} \\ i\sqrt{\beta} \\ -i\frac{i+2\alpha}{2\sqrt{\beta}} \end{bmatrix}, \quad \hat{\mathbf{v}}_2 = \begin{bmatrix} i\sqrt{\beta} \\ -i\frac{i+2\alpha}{2\sqrt{\beta}} \\ \sqrt{\beta} \\ -\frac{i+2\alpha}{2\sqrt{\beta}} \end{bmatrix}$$

♠ It corresponds to $u = 1/2$, $\nu_1 = \nu_2 = \pi/2$

- ◆ Then, the matrix $\hat{\mathbf{V}}$ is

$$\hat{\mathbf{V}} = \begin{bmatrix} \sqrt{\beta} & 0 & 0 & -\sqrt{\beta} \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{2\sqrt{\beta}} & \frac{1}{2\sqrt{\beta}} & \frac{\alpha}{\sqrt{\beta}} \\ 0 & -\sqrt{\beta} & \sqrt{\beta} & 0 \\ \frac{1}{2\sqrt{\beta}} & \frac{\alpha}{\sqrt{\beta}} & -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{2\sqrt{\beta}} \end{bmatrix}$$

Axisymmetric Rotational Distribution



- ◆ Comparing left and right hand sides of the equation

$$\hat{\Xi}_{in} = \mathbf{U}\hat{\mathbf{V}} \begin{bmatrix} 1/\varepsilon_1 & 0 & 0 & 0 \\ 0 & 1/\varepsilon_1 & 0 & 0 \\ 0 & 0 & 1/\varepsilon_2 & 0 \\ 0 & 0 & 0 & 1/\varepsilon_2 \end{bmatrix} \hat{\mathbf{V}}^T \mathbf{U}^T$$

♠ One obtains

$$\beta = \frac{\beta_0}{2\sqrt{1+\Phi^2\beta_0^2}} ,$$

$$\alpha = \frac{\alpha_0}{2\sqrt{1+\Phi^2\beta_0^2}} ,$$

$$\varepsilon_1 = \frac{\varepsilon_T}{\sqrt{1+\Phi^2\beta_0^2} - \Phi\beta_0} \xrightarrow{\Phi\beta_0 \gg 1} 2\Phi\beta_0\varepsilon_T ,$$

$$\varepsilon_2 = \frac{\varepsilon_T}{\sqrt{1+\Phi^2\beta_0^2} + \Phi\beta_0} \xrightarrow{\Phi\beta_0 \gg 1} \frac{\varepsilon_T}{2\Phi\beta_0} .$$

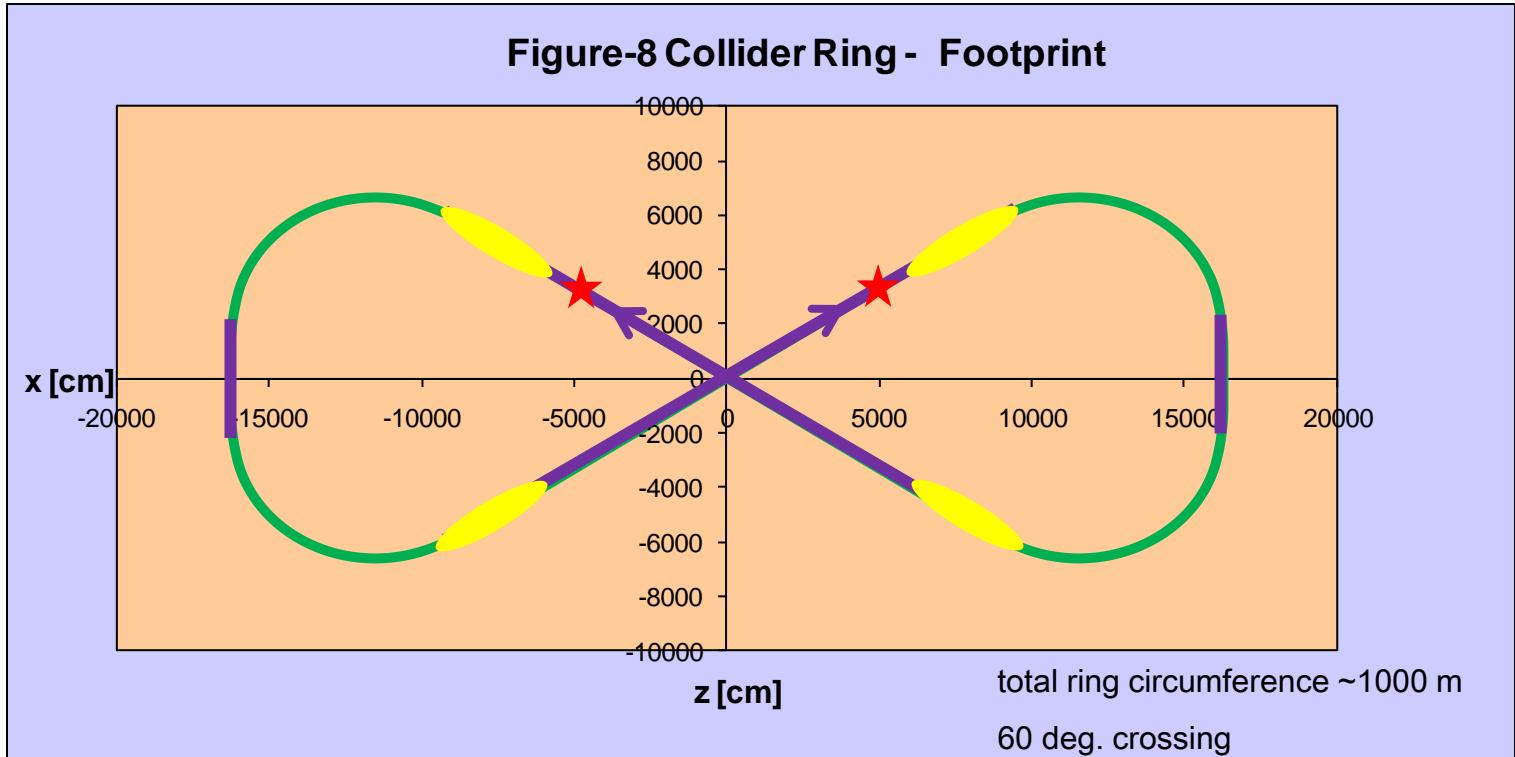
- 4D-emmitance conservation:

$$\varepsilon_1\varepsilon_2 = \varepsilon_T^2$$

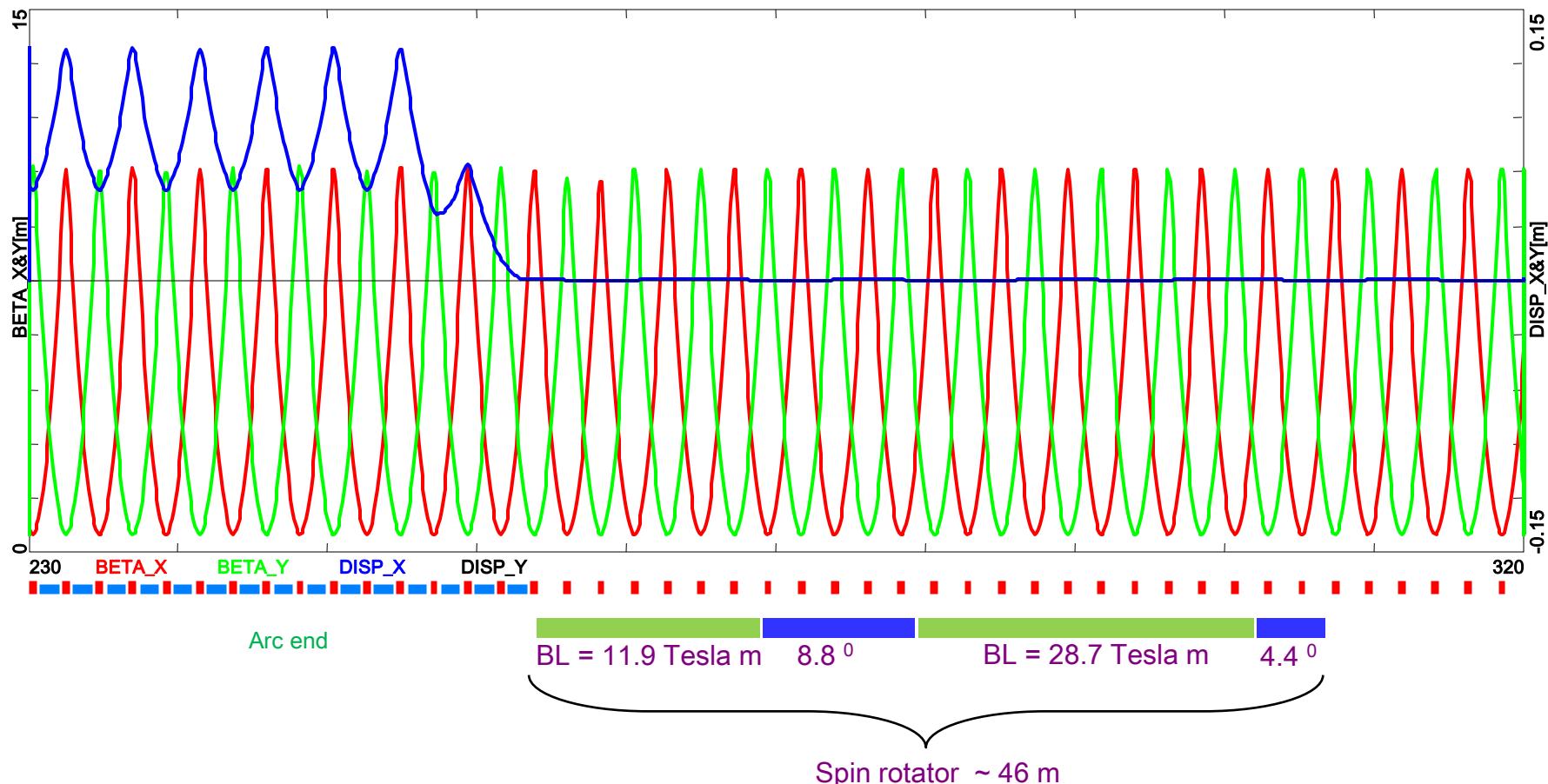
- Rotational emittance estimate

$$\varepsilon_{rot} = r\theta = r(r\Phi) = r^2\Phi = (\varepsilon_T\beta_0)\Phi$$

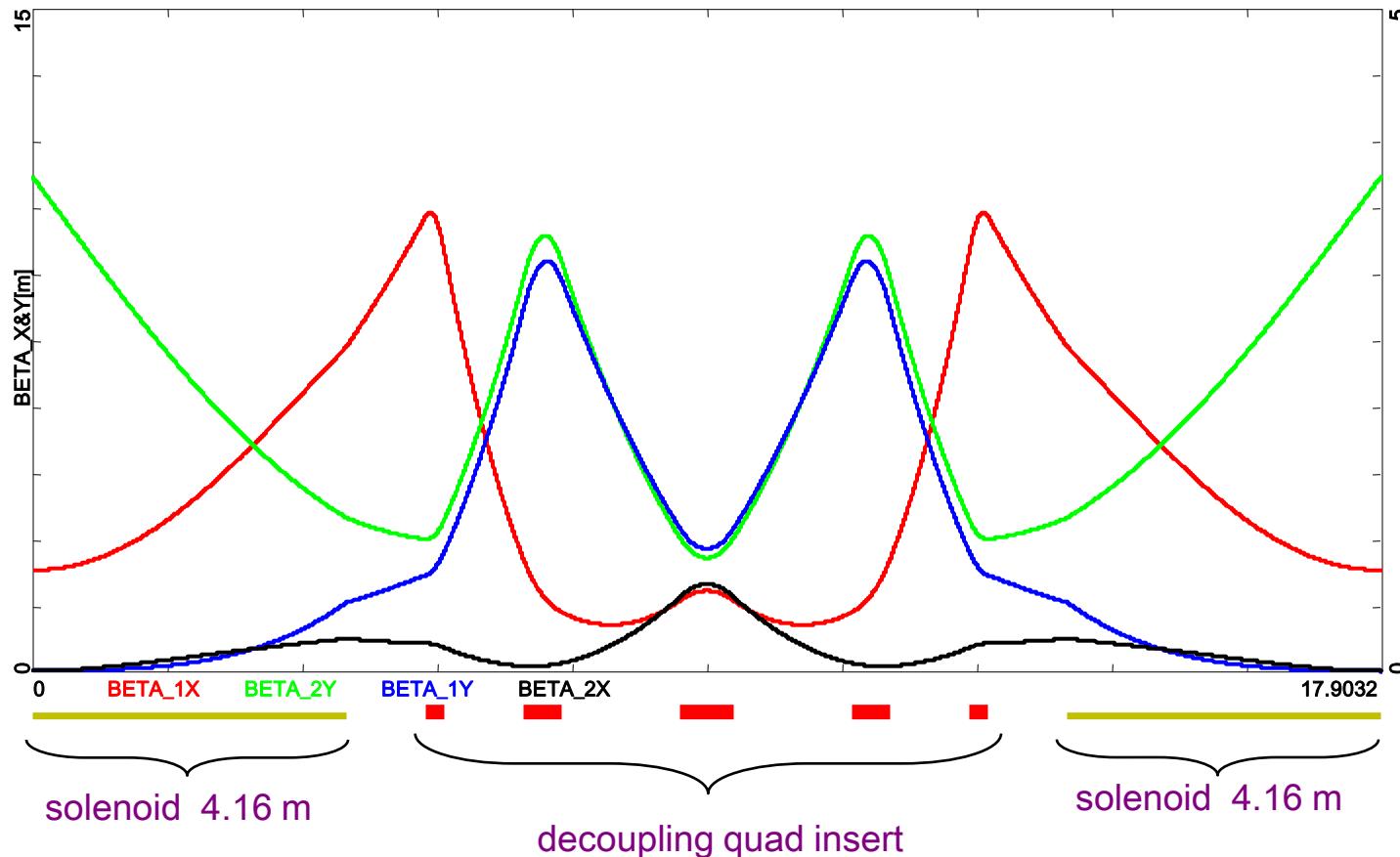
Spin Rotators for Figure-8 Collider Ring



Spin Rotator – Ingredients...



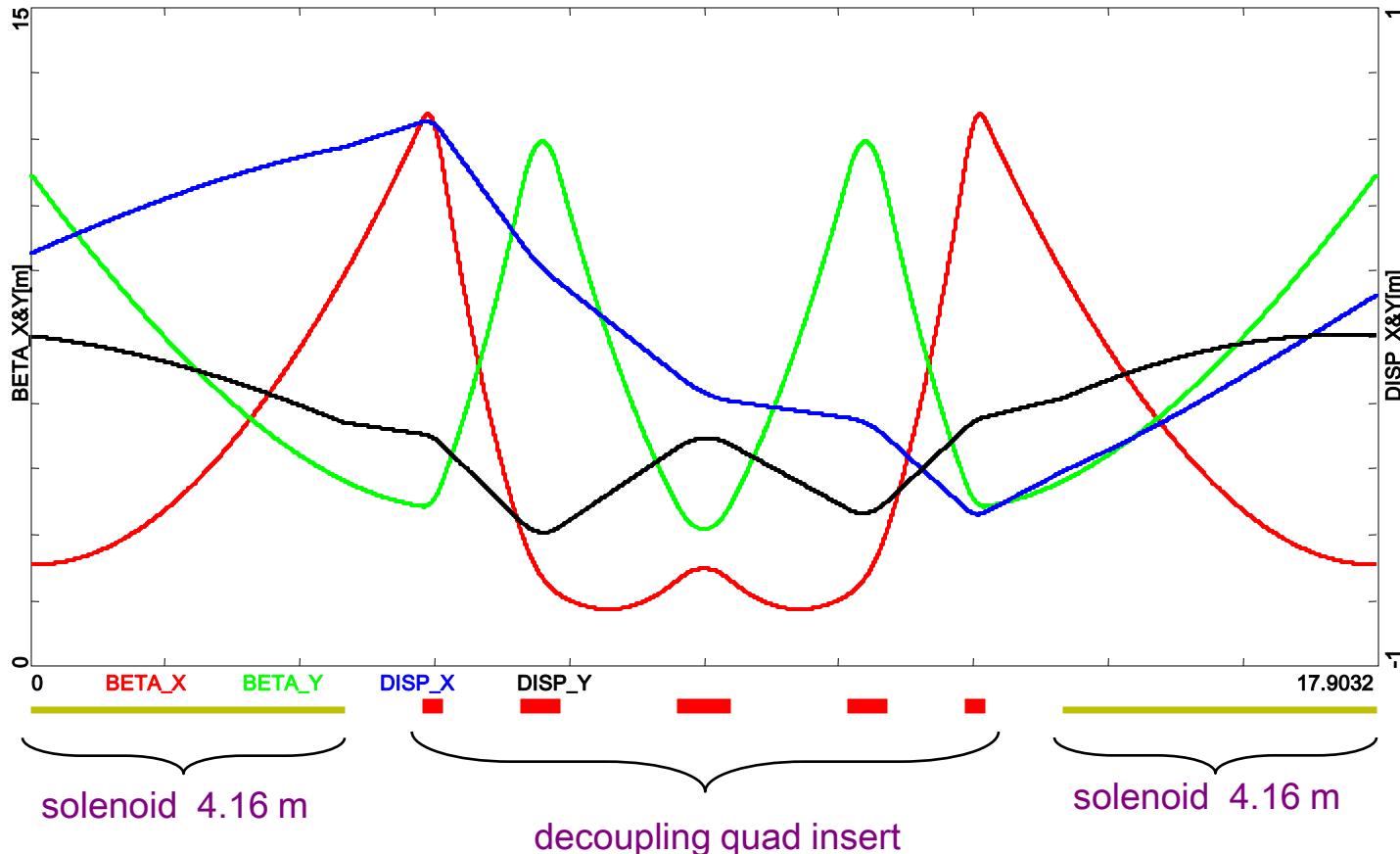
Locally Decoupled Solenoid Pair



$$M = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}$$

Hisham Sayed, PhD Thesis
ODU, 2011

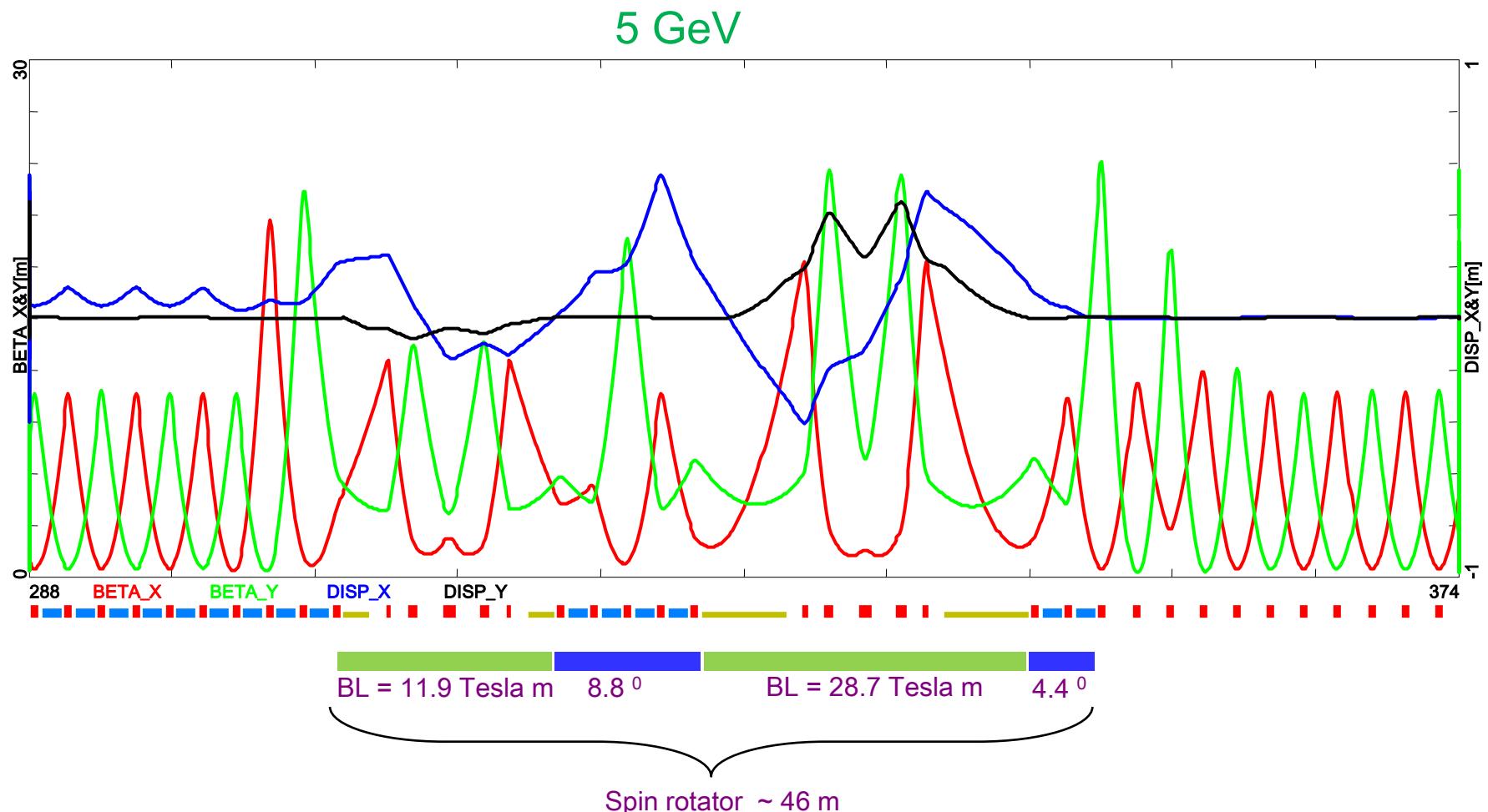
Locally Decoupled Solenoid Pair



$$M = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}$$

Hisham Sayed, PhD Thesis
ODU, 2011

Universal Spin Rotator - Optics





Ionization Cooling in an Axially Symmetric Channel

- ❖ A single-particle phase-space trajectory along the beam orbit can be expressed as:

$$\hat{\mathbf{x}}(s) = \operatorname{Re} \left(\sqrt{\epsilon_1} \hat{\mathbf{v}}_1(s) e^{-i(\psi_1 + \mu_1(s))} + \sqrt{\epsilon_2} \hat{\mathbf{v}}_2(s) e^{-i(\psi_2 + \mu_2(s))} \right) ,$$

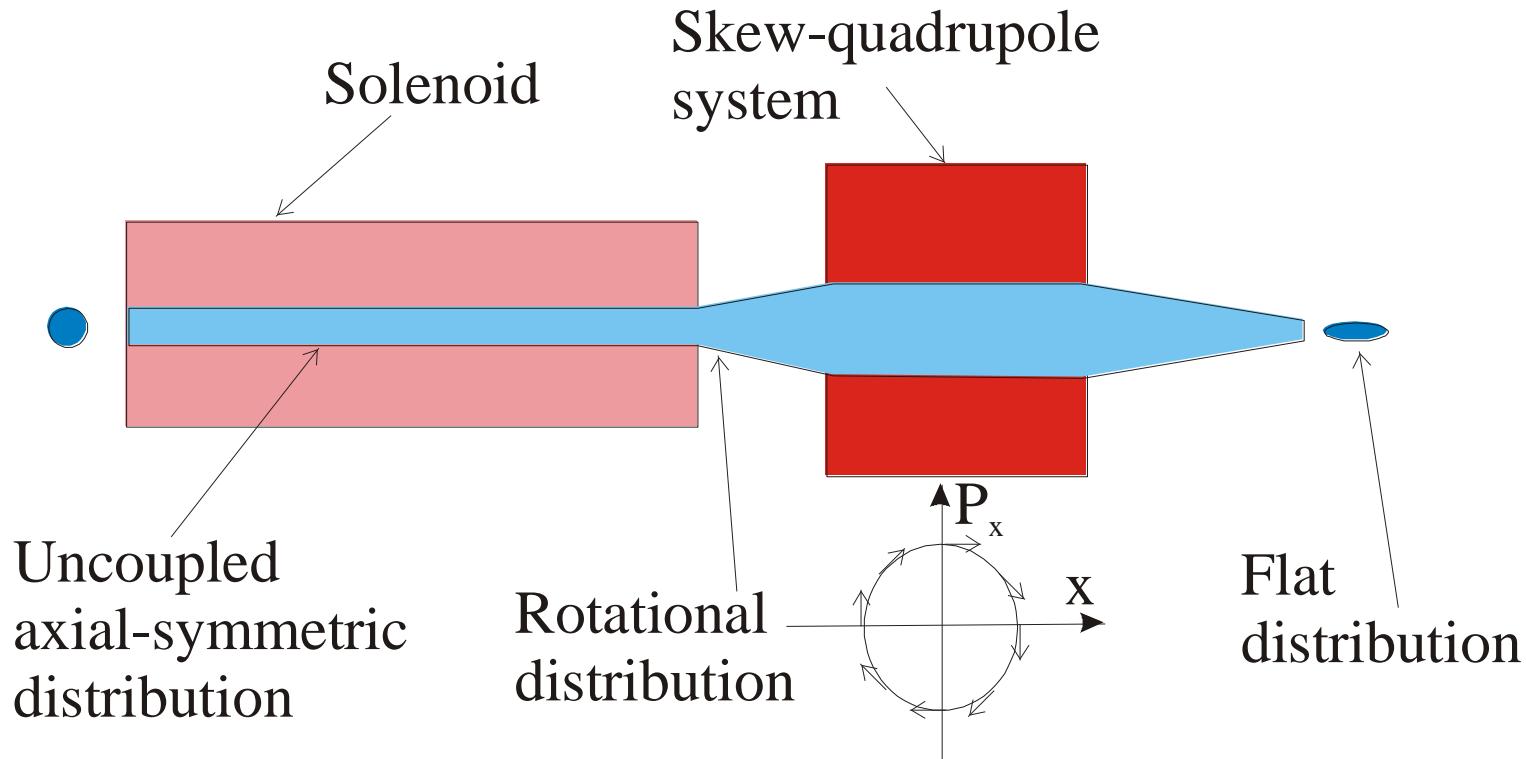
- ◆ One can rewrite the above equations in the following compact form

$$\hat{\mathbf{x}}(s) = \hat{\mathbf{V}}(s) \mathbf{a}(s)$$

where

$$\hat{\mathbf{V}}(s) = \begin{bmatrix} \hat{\mathbf{v}}_1'(s), -\hat{\mathbf{v}}_1''(s), \hat{\mathbf{v}}_2'(s), -\hat{\mathbf{v}}_2''(s) \end{bmatrix} \quad \mathbf{a}(s) = \begin{bmatrix} \sqrt{\epsilon_1} \cos(\psi_1 + \mu_1(s)) \\ \sqrt{\epsilon_1} \sin(\psi_1 + \mu_1(s)) \\ \sqrt{\epsilon_2} \cos(\psi_2 + \mu_2(s)) \\ \sqrt{\epsilon_2} \sin(\psi_2 + \mu_2(s)) \end{bmatrix}$$

Vertex-to-plane Transformer Insert



Vertex-to-plane Transformer Insert



- ◆ Eigen-vectors of the decoupled motion in the coordinate system rotated by 45^0

$$\begin{bmatrix} \sqrt{\beta_1} \\ -\frac{i+\alpha_1}{\sqrt{\beta_1}} \\ \sqrt{\beta_1} \\ 0 \\ 0 \end{bmatrix} \xrightarrow[45 \text{ deg. rotation}]{1/\sqrt{2}} \begin{bmatrix} \sqrt{\beta_1} \\ -\frac{i+\alpha_1}{\sqrt{\beta_1}} \\ \sqrt{\beta_1} \\ \sqrt{\beta_1} \\ -\frac{i+\alpha_1}{\sqrt{\beta_1}} \end{bmatrix} \xrightarrow[\substack{\beta_1=2\beta \\ \alpha_1=2\alpha}]{\quad} \begin{bmatrix} \sqrt{\beta} \\ -\frac{i+2\alpha}{2\sqrt{\beta}} \\ \sqrt{\beta} \\ \sqrt{\beta} \\ -\frac{i+2\alpha}{2\sqrt{\beta}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_2 \\ i\mathbf{F}_2 \end{bmatrix}$$

♠ Rotational eigen-vectors

$$\begin{bmatrix} \mathbf{F}_2 \\ i\mathbf{F}_2 \end{bmatrix} \quad \begin{bmatrix} i\mathbf{F}_2 \\ \mathbf{F}_2 \end{bmatrix}$$

Vertex-to-plane Transformer Insert

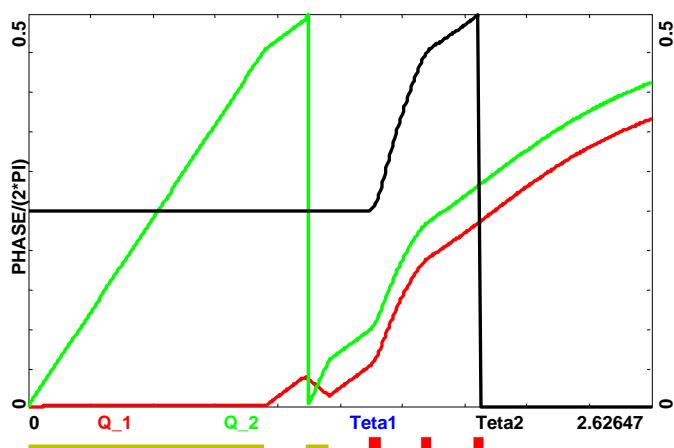
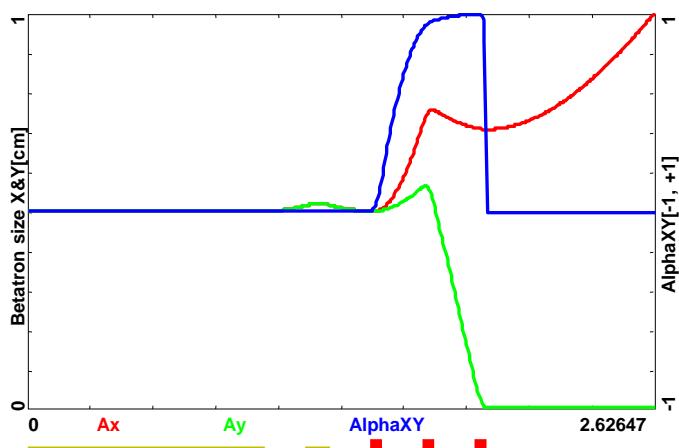
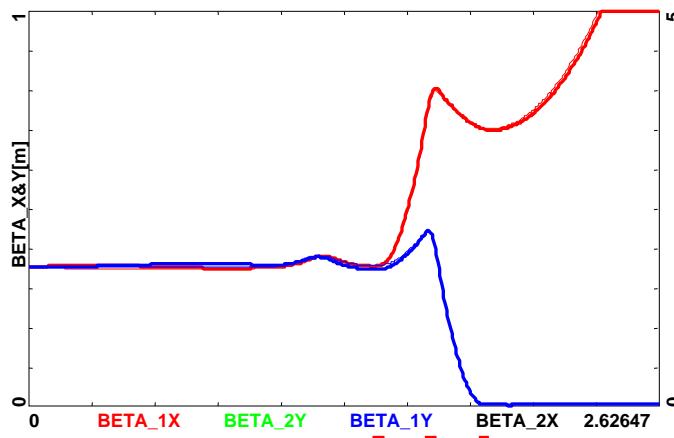
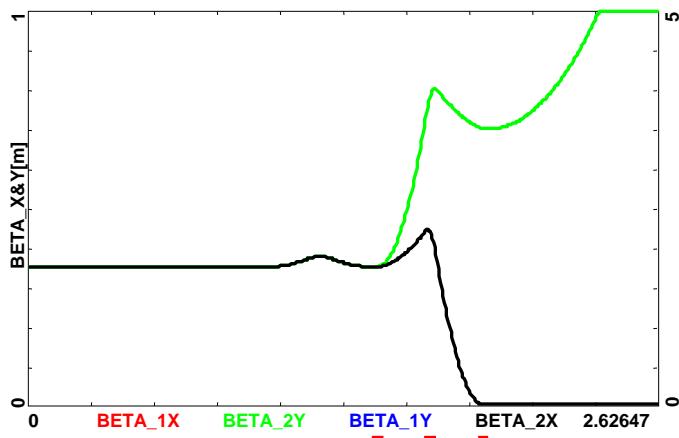


- ◆ Focusing system with 45° difference between the horizontal and vertical betatron phase advances will transform the initial vertex distribution into the flat one
- ◆ The resulting 2D emittances are as follows

$$\varepsilon_1 = \frac{\varepsilon_T}{\sqrt{1 + \Phi^2 \beta_0^2} - \Phi \beta_0} \quad , \quad \varepsilon_2 = \frac{\varepsilon_T}{\sqrt{1 + \Phi^2 \beta_0^2} + \Phi \beta_0} \quad .$$

- ◆ Lattice implementation – Twiss functions, beam sizes etc.

Vertex-to-plane Transformer Insert



$$E_{\text{kin}} = 10 \text{ MeV},$$

$$T_c = 0.2 \text{ eV},$$

$$R_c = 0.5 \text{ cm},$$

$$B_{\text{sol}} = 1 \text{ kG},$$

$$\Rightarrow \varepsilon_1 = 7.14 \cdot 10^{-3} \text{ cm},$$

$$\varepsilon_2 = 3.24 \cdot 10^{-8} \text{ cm}$$



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Northern Illinois Center for Accelerator
and Detector Development



Generation and Dynamics of Magnetized Beams for High-Energy Electron Cooling*

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**International Workshop on Accelerator Science & Technologies
for future electron-ion colliders (EIC'14), Jefferson Lab, March 17-21, 2014**

*sponsored by the DOE awards DE-FG02-08ER41532 to Northern Illinois University and
DE-AC02-07CH11359 to the Fermi Research Alliance LLC.

Outline

- Introduction
- Features and parameterization of magnetized beams
- Formation of magnetized bunches:
 - methods and limitations,
 - experiments in rf gun.
- Transport and Manipulation:
 - transverse matching,
 - longitudinal manipulations,
 - decoupling into flat beams.
- Outlook

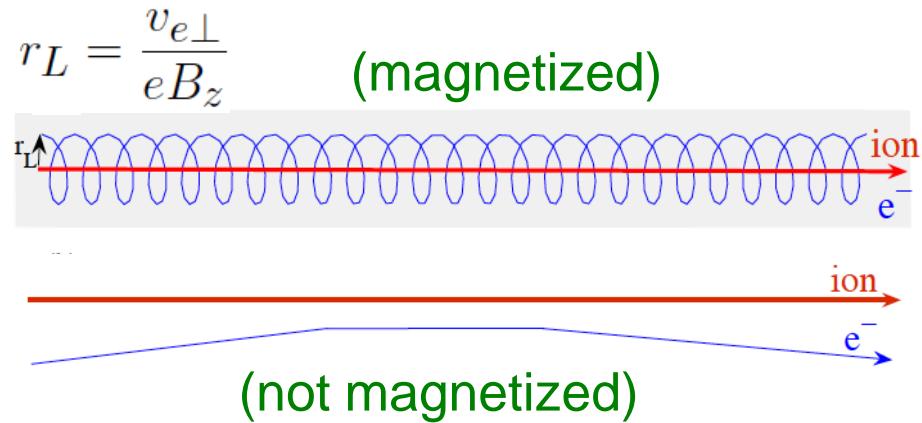
Required Electron-Beam Parameters

- Cooling interaction time

$$\tau \approx \rho / v_{e\perp} \quad (\text{not magnetized})$$

$$\tau \approx \frac{\rho}{v - v_{e\parallel}} \quad (\text{magnetized})$$

- magnetized cooling less dependent on e- beam transverse emittance (to what extent?)

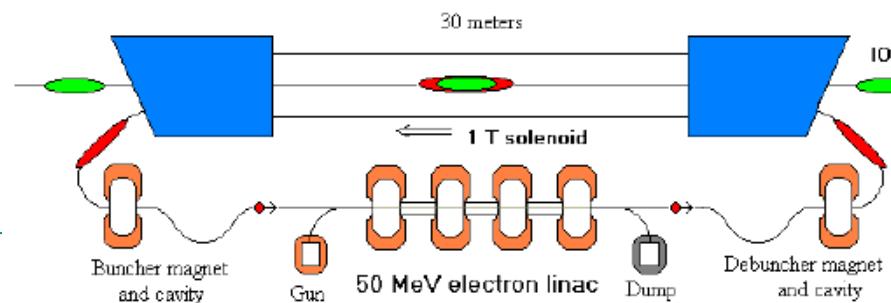
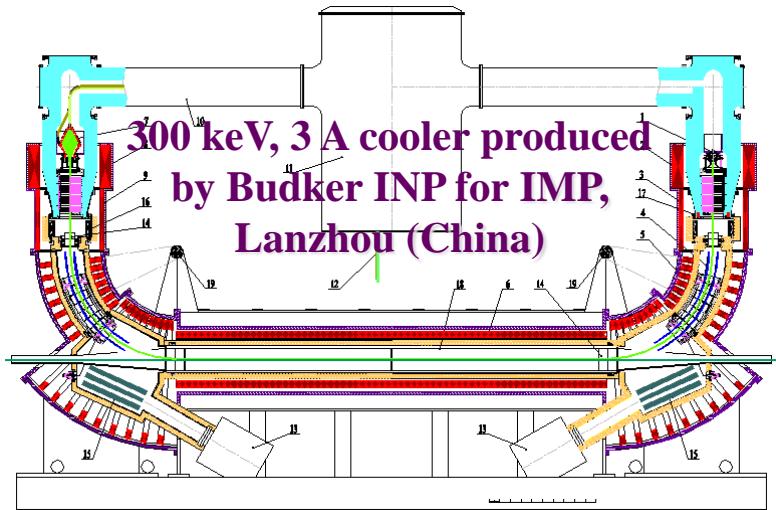


- electron-cooling accelerator provides beam eventually matched to cooling-solenoid section

Cooler configurations

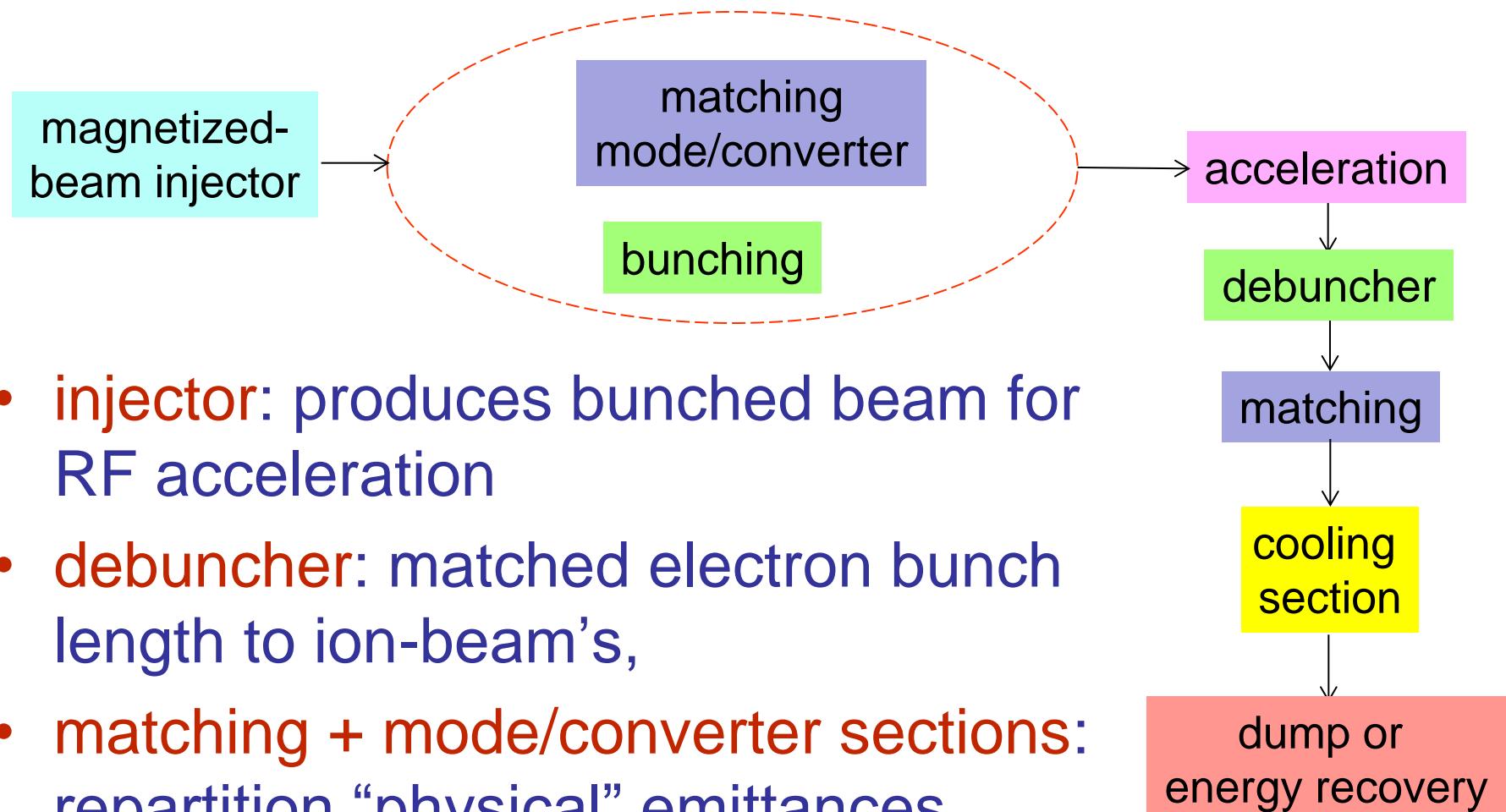
e.g. see S. Nagaitsev, et al., PRL96, 044801 (2006)

- low-energy coolers:
 - lattice (bends) embedded in magnetic fields,
 - based on DC electron sources,
 - no further acceleration or bunching, needed.
- high-energy coolers:
 - medium energies required (50-100 MeV),
 - acceleration in SCRF linac → bunching
 - lumped solenoidal fields → matching



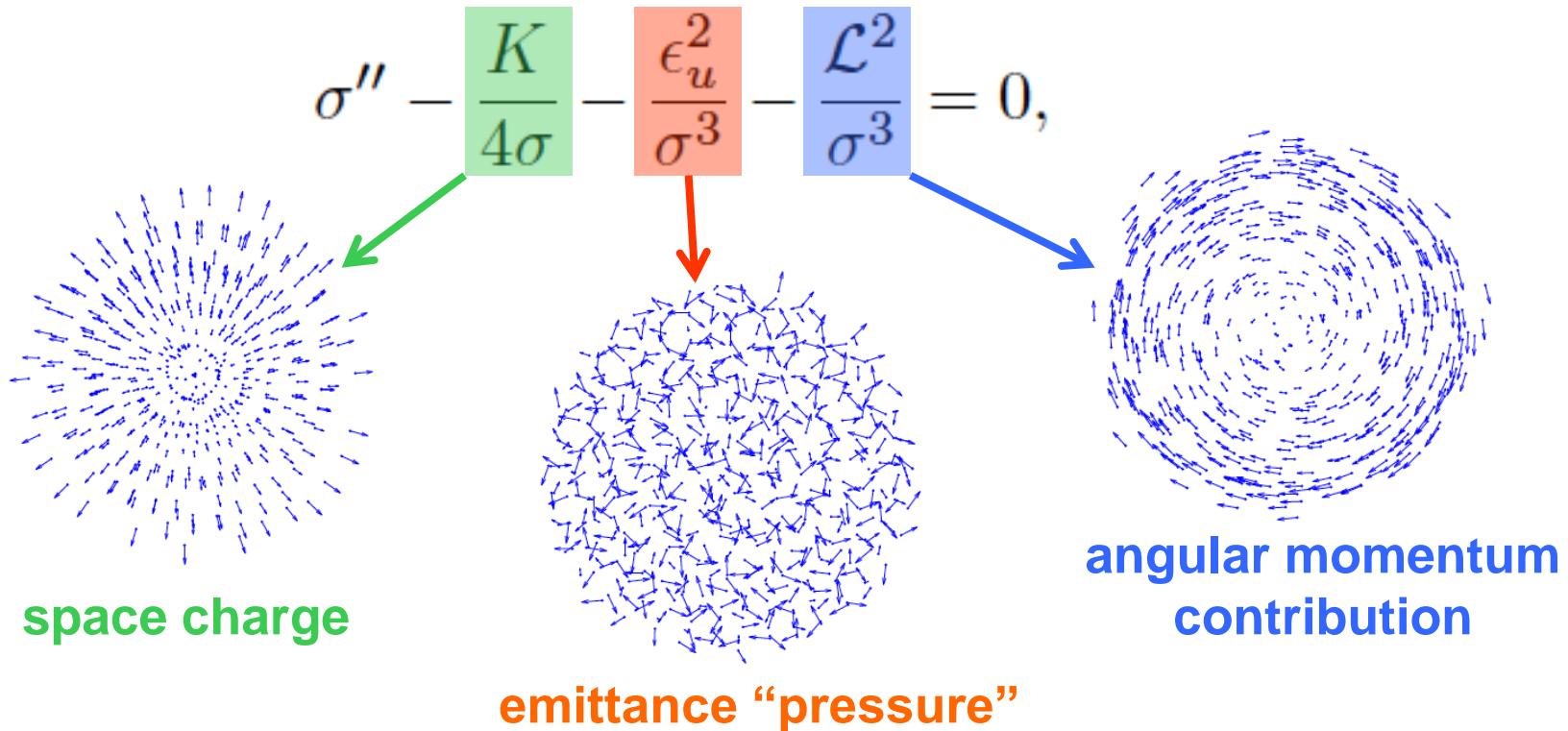
early concept for RHIC e-cooling

High-energy coolers



Beam dynamics regimes (round beams)

- Radial envelope (σ) equation in a drift (Lawson):



K : generalized permeance

ϵ_u : uncorrelated geometric emittance

\mathcal{L} : magnetization

Features & Parameterization

- possible parameterization of coupled motion between 2 degrees of freedom has been extensively discussed; see:
 - D.A. Edwards and L.C. Teng, IEEE Trans. Nucl. Sci. 20, 3, pp. 885-889 (1973).
 - I. Borchardt, E. Karantzoulis, H. Mais, G. Ripken, DESY 87-161 (1987).
 - V. Lebedev, S. A. Bogacz, ArXiV:1207.5526 (2007).
 - A. Burov, S. Nagaitsev, A. Shemyakin, Ya. Derbenev, PRSTAB 3, 094002 (2000).
 - A. Burov, S. Nagaitsev, Ya. Derbenev, PRE 66, 016503 (2002).
- Simpler description that provides the necessary insights..

A simple description of coupled motion

- Consider the 4x4 beam matrix

$$\Sigma \equiv \begin{bmatrix} \langle \mathbf{X}\tilde{\mathbf{X}} \rangle & \langle \mathbf{X}\tilde{\mathbf{Y}} \rangle \\ \langle \mathbf{Y}\tilde{\mathbf{X}} \rangle & \langle \mathbf{Y}\tilde{\mathbf{Y}} \rangle \end{bmatrix} \quad \text{where} \quad \begin{aligned} \tilde{\mathbf{X}} &\equiv (x, x') \\ \tilde{\mathbf{Y}} &\equiv (y, y') \end{aligned}$$

- Introduce the “correlation” matrix: $C \equiv \langle \mathbf{Y}\tilde{\mathbf{X}} \rangle \langle \mathbf{X}\tilde{\mathbf{X}} \rangle^{-1}$
- Beam matrix takes the form:

$$\Sigma = \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & C^{-1} \\ C & 0 \end{bmatrix} \right) \begin{bmatrix} \langle \mathbf{X}\tilde{\mathbf{X}} \rangle & 0 \\ 0 & \langle \mathbf{Y}\tilde{\mathbf{Y}} \rangle \end{bmatrix}$$

- The correlation subjects to $R = \begin{bmatrix} H & G \\ U & V \end{bmatrix}$ transforms as $C_0 \rightarrow C$

$$C = (U + VC_0)(H + GC_0)^{-1}$$

- C provides information on the coupling only.

Beam matrix for a round magnetized beam

- At a waist, the matrix of a magnetized (round) beam is

$$\Sigma_0 = \begin{bmatrix} \varepsilon T_0 & \mathcal{L}J \\ -\mathcal{L}J & \varepsilon T_0 \end{bmatrix}. \quad \text{where} \quad T_0 = \begin{bmatrix} \beta & -\alpha \\ -\alpha & \frac{1+\alpha^2}{\beta} \end{bmatrix}$$

and the magnetization is

$$\mathcal{L} = \langle xy' \rangle = -\langle x'y \rangle = \frac{L}{2p_z}$$

- The eigen-emittances of this beam matrix are:

$$\varepsilon_{\pm} = \varepsilon \pm \mathcal{L}. \quad \text{where} \quad \varepsilon^2 = \mathcal{L}^2 + \varepsilon_u^2 = |\Sigma|$$

- the eigen-emittances can be mapped into “physical” emittances using a skewed beamline

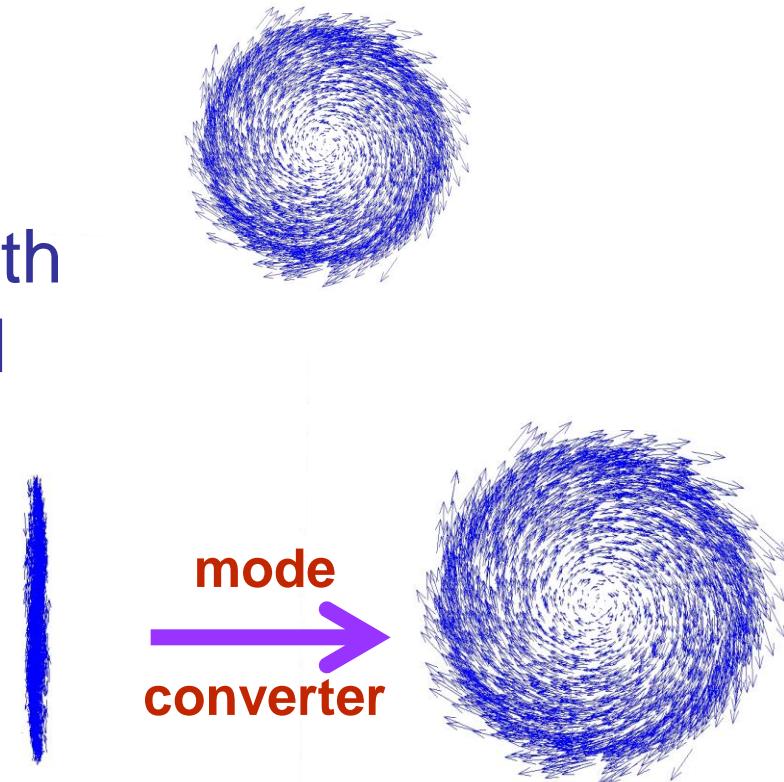
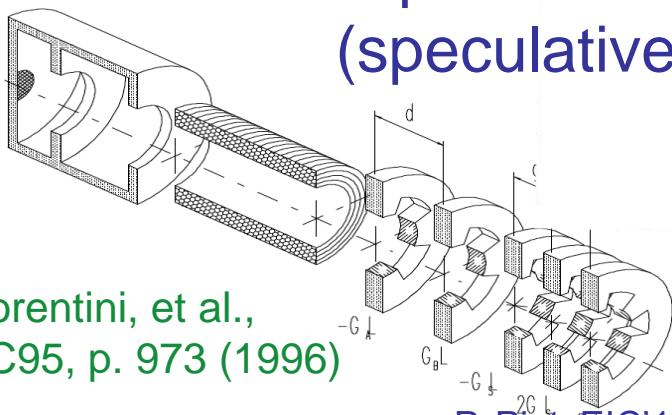
$$\begin{bmatrix} M_+ & M_- \\ M_- & M_+ \end{bmatrix}$$

decoupling
when

$$M_- + M_+ C_0 = 0.$$

Formation of magnetized bunches

- Cathode immersed in an axial **B** field
- Sheet beams at birth (with subsequent flat-to-round beam converter)
 - shaped cathode,
 - line-laser focus
 - Nonlinear optics (speculative)



Y. Derbenev, University of Michigan
report UM-HE-98-04 (1998)

G. Florentini, et al.,
Proc. PAC95, p. 973 (1996)

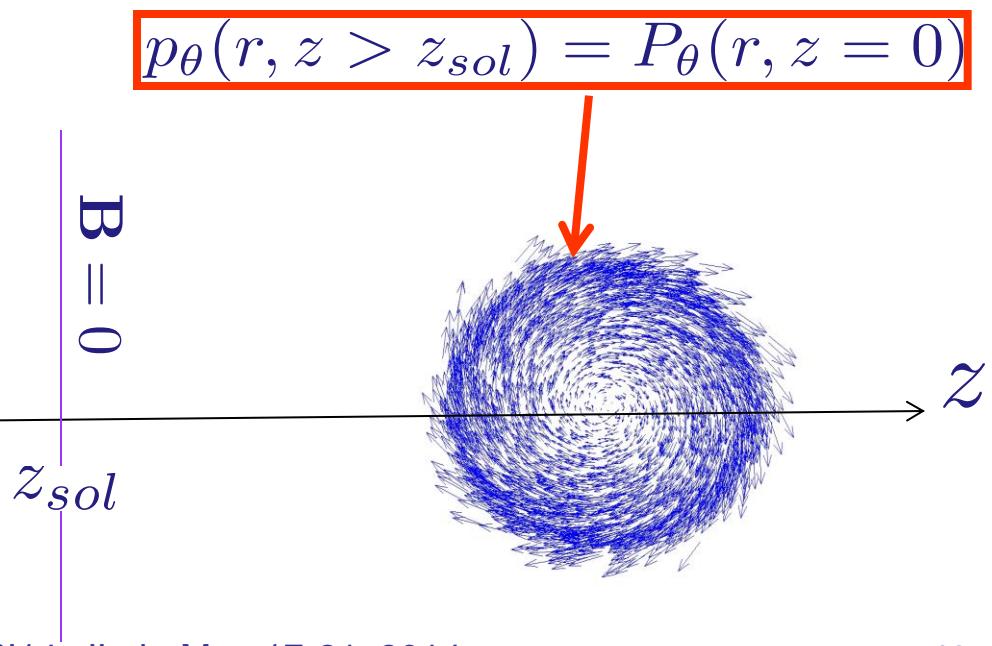
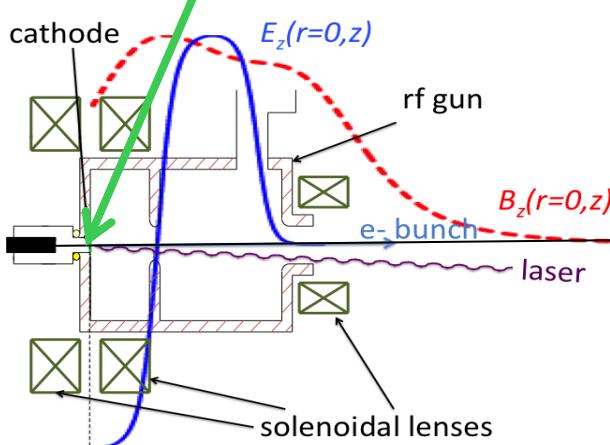
Cathode in a magnetic field

- electrons born in an axial B field $B_z \rightarrow$ CAM

$$L(r) = erA_\theta \simeq \frac{er^2}{2}B_{z,0} + \mathcal{O}(r^4)$$

- upon exit of solenoid field ($A_\theta = 0$): CAM becomes purely kinetic.

$$P_\theta(r, z = 0) = eA_\theta(r, z = 0)$$



Emittance vs magnetization

- “effective emittance” $\varepsilon^2 = \mathcal{L}^2 + \varepsilon_u^2$

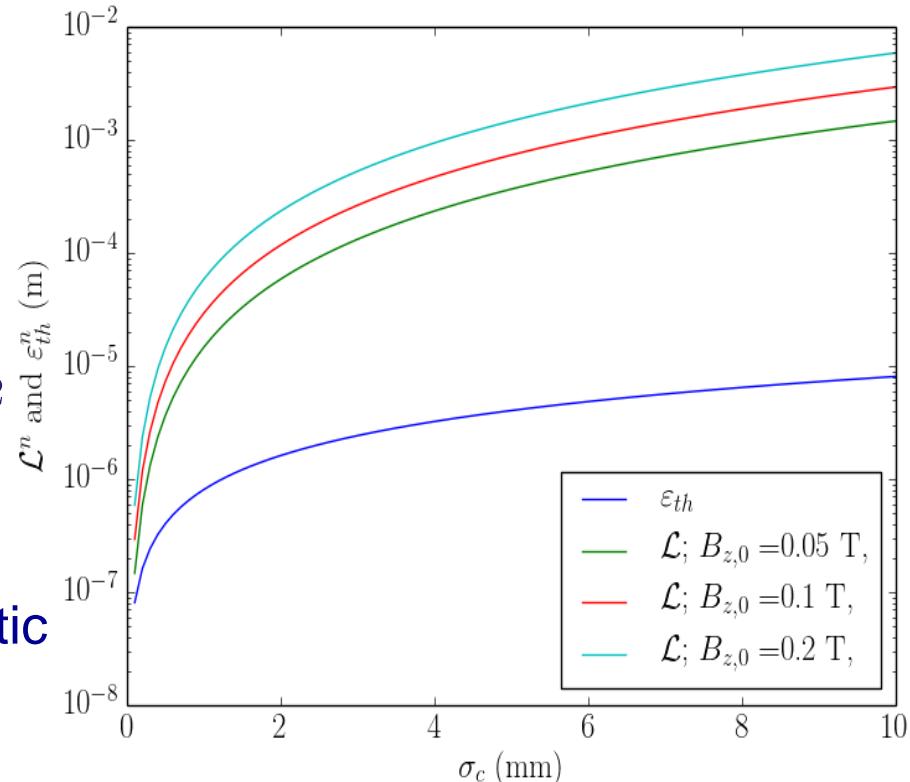
- magnetization

$$\mathcal{L} = \frac{eB_0}{2mc} \sigma_c^2$$

- The emittance has a lower-bound value :

$$\varepsilon_u^n \equiv \beta\gamma\varepsilon_u \geq \varepsilon_{th} = \sigma_c \left(\frac{2\delta E}{3mc^2} \right)^{1/2}$$

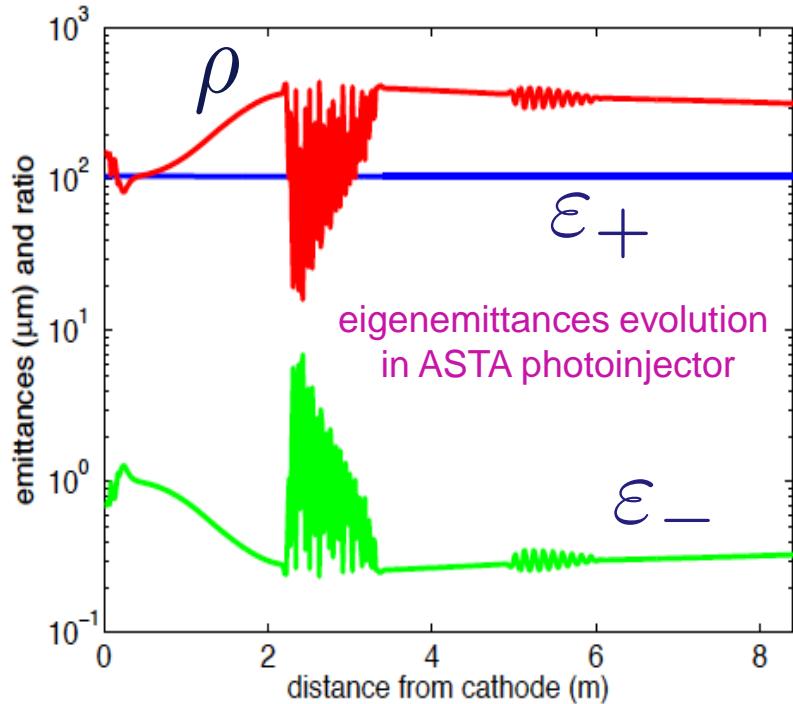
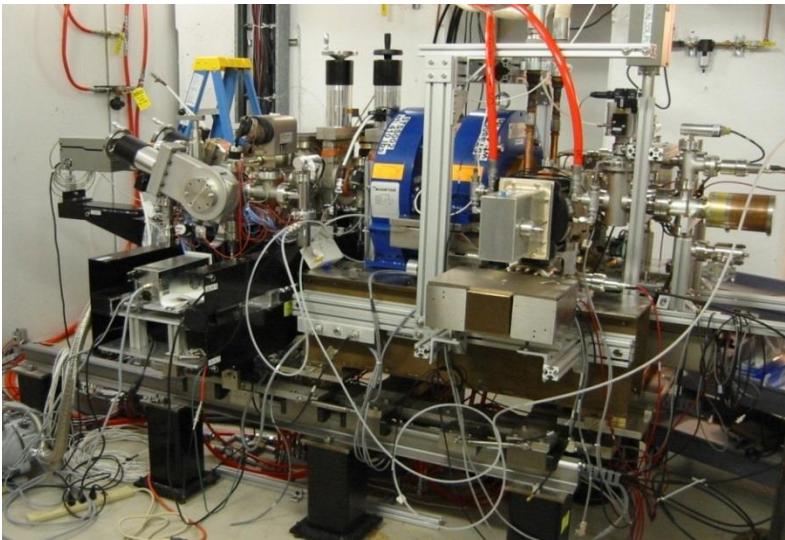
where δE is the excess in kinetic energy during emission



- Practically, ε_u includes other contributions.

Example of 3.2-nC magnetized bunch

- high-charge bunch subject to emittance degradation
- proper optimization (emittance compensation)
→ 4-D emittance comparable to round beams.

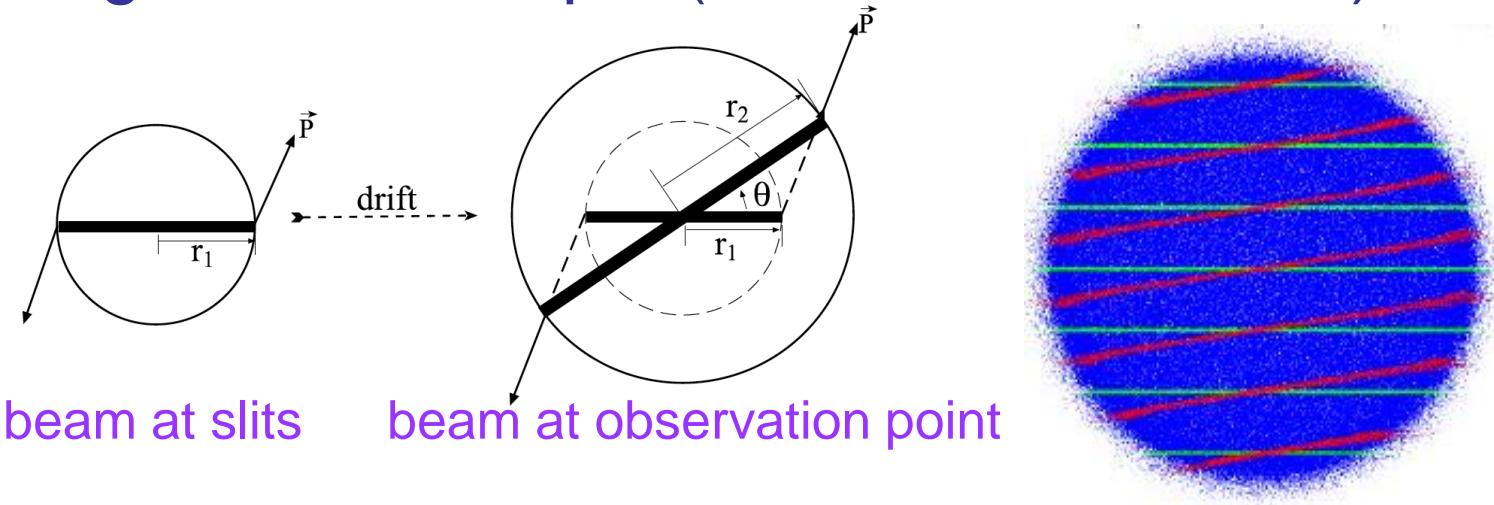


parameter	flat-beam configuration	round-beam configuration	units
Q	3.2	3.2	nC
E	47.18	48.77	MeV
ϵ_x	105.04	5.43	μm
ϵ_y	0.31	5.44	μm
ϵ_{4D}	5.53	5.44	μm
ρ	$\simeq 334$	$\simeq 1$	-

P. Piot et al. IPAC13; C-X. Wang, FEL06, 721 (2006)
X. Chang, I. Ben-Zvi, J. Kewisch AAC04 (2004)

Measuring (kinetic) angular momentum

- Kinetic angular momentum can be measured using a slit technique (similar to emittance)



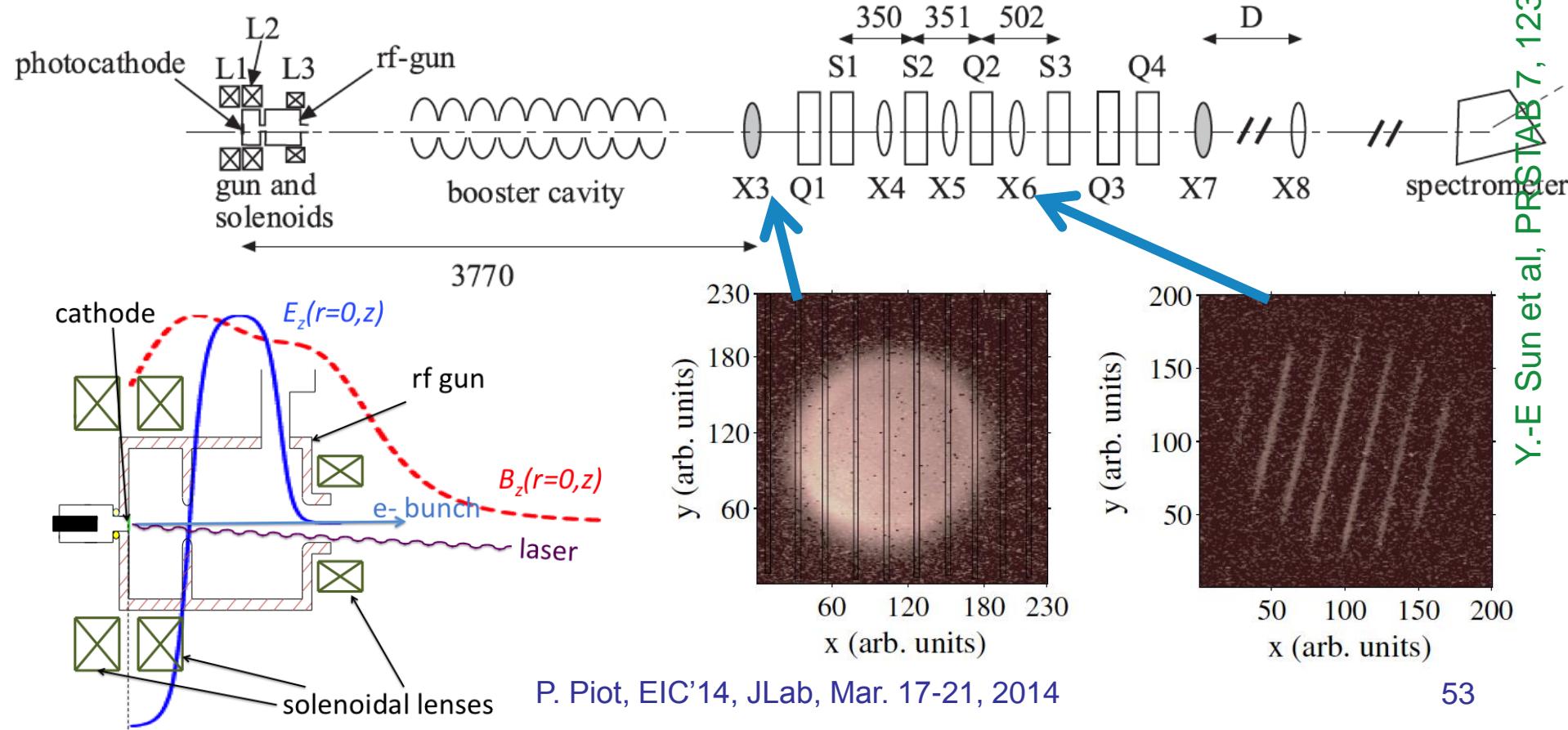
- The beam's average angular momentum is given by

$$\langle L \rangle = 2P_z \frac{\sigma_1 \sigma_2 \sin \theta}{D}$$

$\sigma_{1,2}$: rms beam size at slit (1) and observation screen (2),
 P_z : axial momentum
 D : drift length between locations (1) and (2).

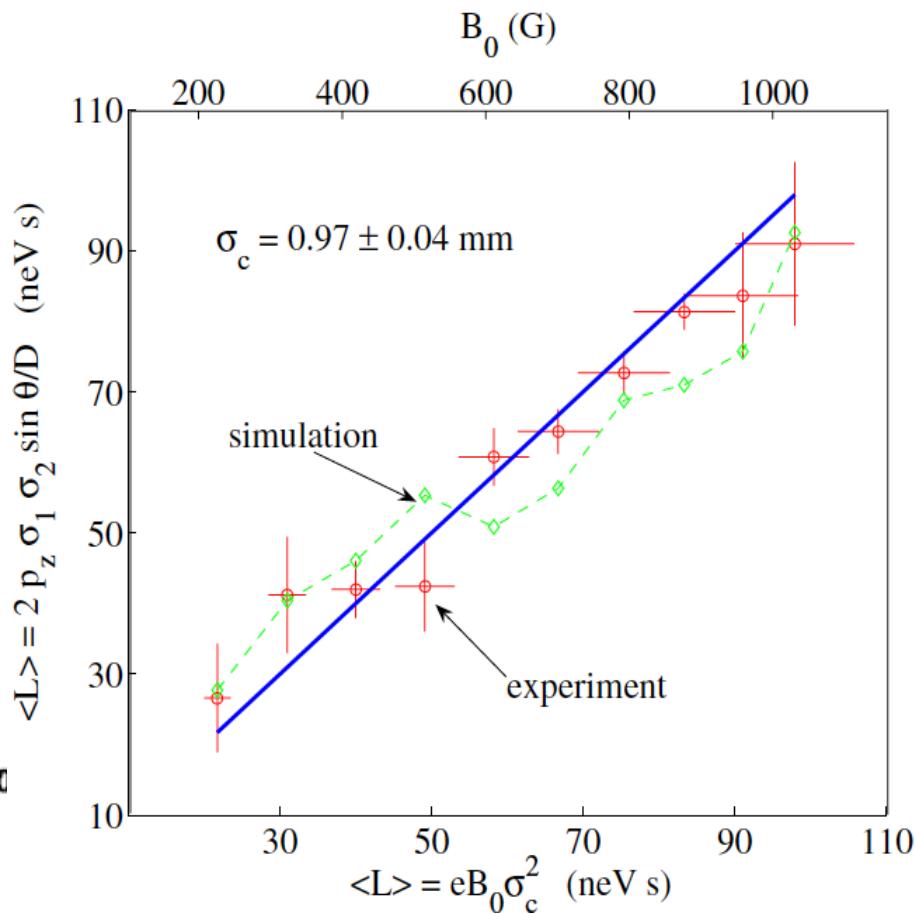
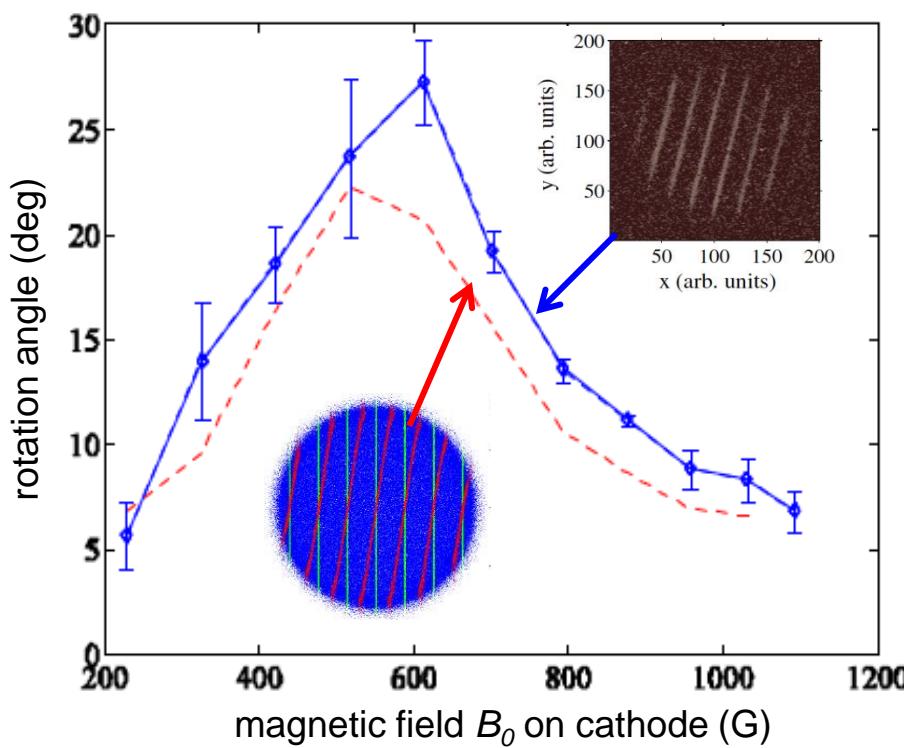
Experimental generation in a photoinjector

- Fermilab A0 normal-conducting photoinjector (decommissioned),
- 15 MeV, charge up to 2 nC, ~3-10 ps bunch



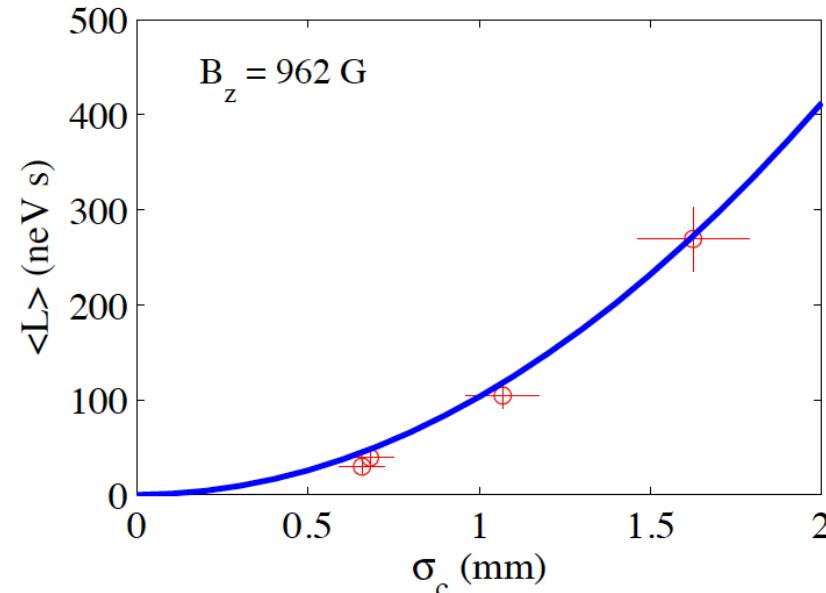
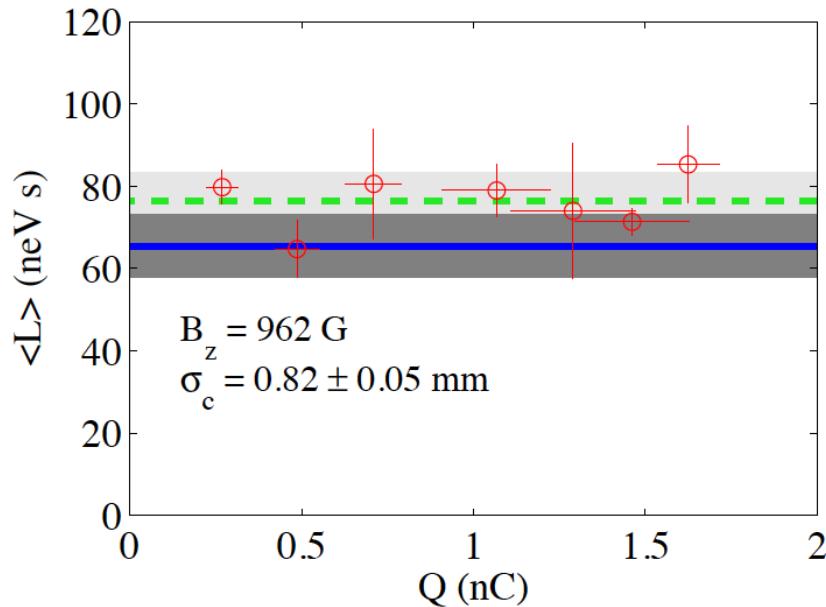
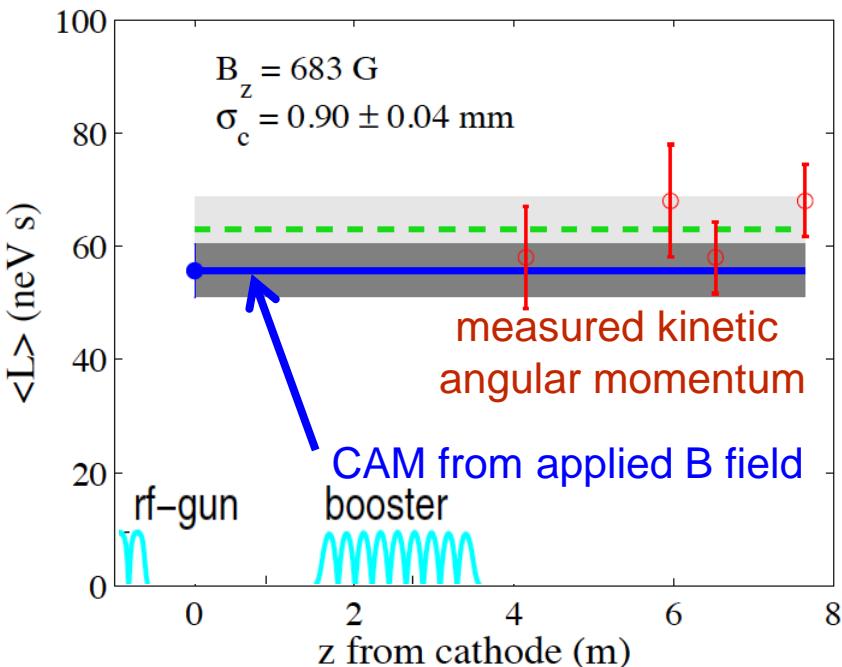
Experimental generation in a photoinjector

- linear scaling with B field on photocathode



Experimental generation in a photoinjector

- weak Q dependence,
- quadratic scaling with laser spot size σ_c on photocathode.



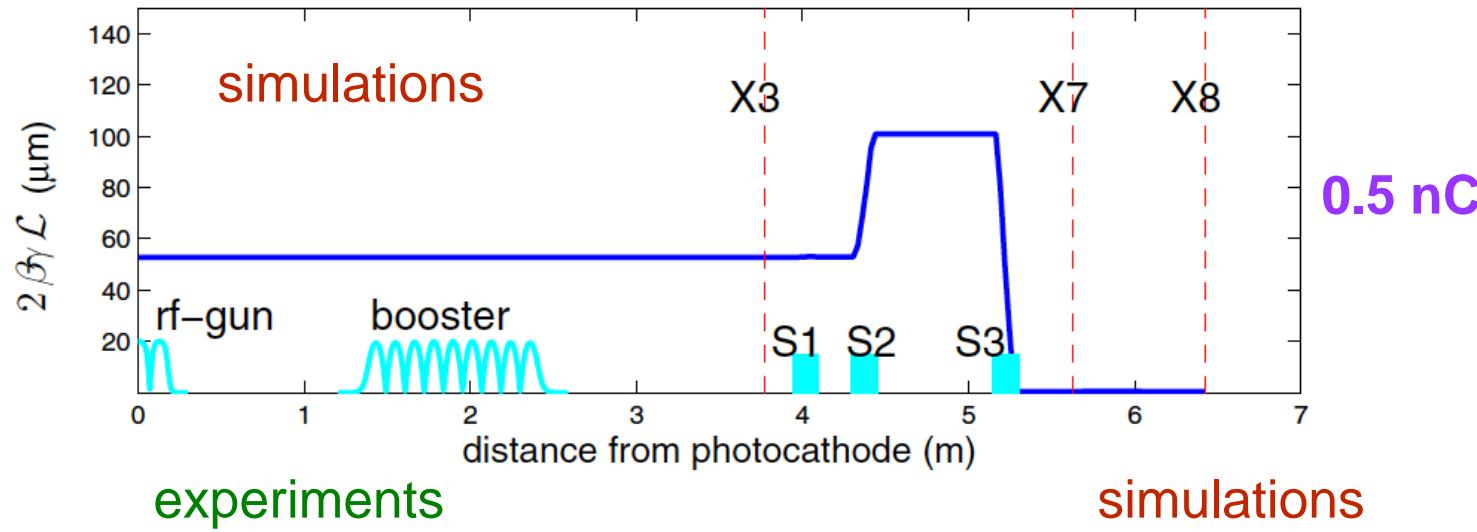
Decoupling into flat ($\varepsilon_x/\varepsilon_y \neq 1$) beam

- Transport of magnetized bunches while preserving \mathcal{L} is challenging,
- Use of round-to-flat beam transformer to convert into uncoupled (flat) beam
→ eigen-emittances maps into “physical” transverse emittances:

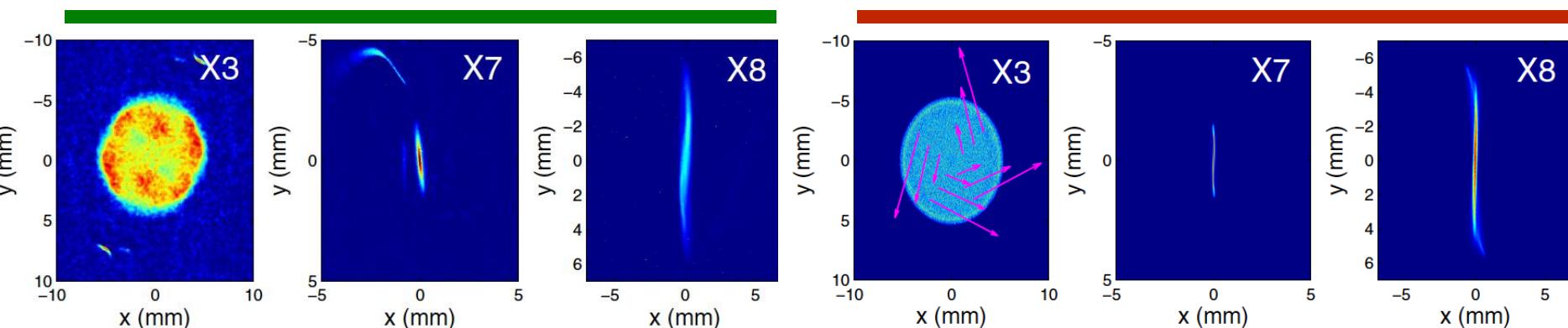
$$\varepsilon_n^\pm = \sqrt{(\varepsilon_n^u)^2 + (\beta\gamma\mathcal{L})^2}$$
$$\pm (\beta\gamma\mathcal{L})^{\beta\gamma\mathcal{L} \gg \varepsilon_n^u} \left\{ \begin{array}{l} \varepsilon_n^+ \approx 2\beta\gamma\mathcal{L}, \\ \varepsilon_n^- \approx \frac{(\varepsilon_n^u)^2}{2\beta\gamma\mathcal{L}}, \end{array} \right.$$

Decoupling into flat beam: experiments (1)

- Same experimental setup as used for generation of CAM-dominated beams



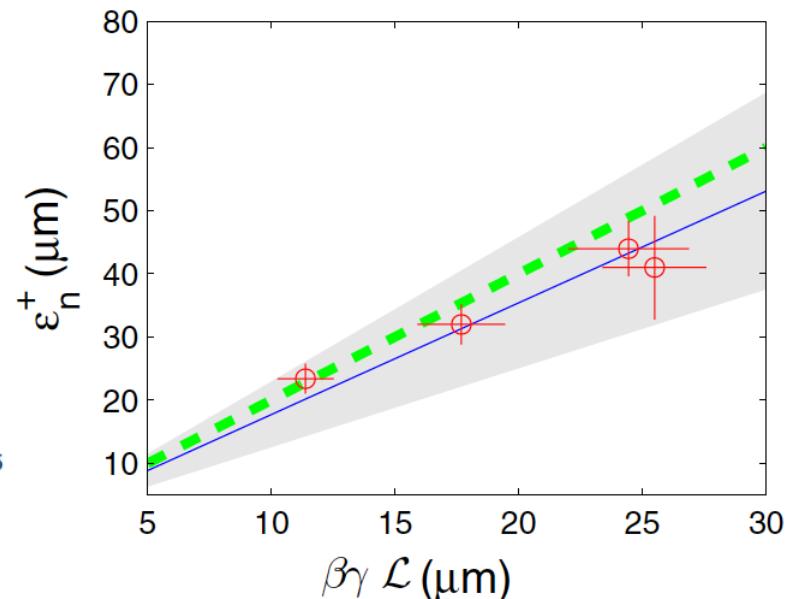
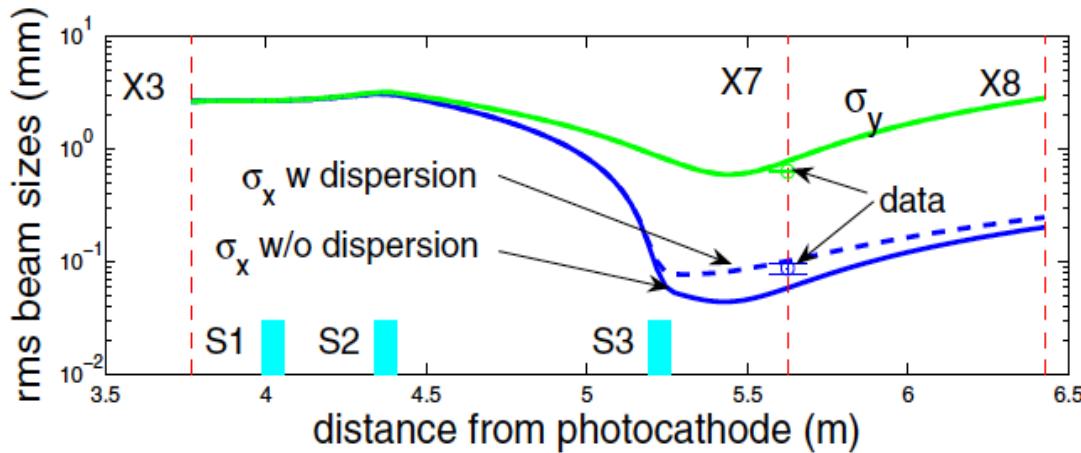
P. Piot, et al, PRSTAB 9
, 031001 (2006)



Decoupling into flat beam: experiments (2)

- normal emittances map into the flat-beam emittance
- large experimental uncertainties for smallest emittance meas.

Parameter	Experiment	Simulation	Unit
σ_x^{X7}	$0.088 \pm 0.01 (\pm 0.01)$	0.058	mm
σ_y^{X7}	$0.63 \pm 0.01 (\pm 0.01)$	0.77	mm
$\sigma_x^{X8,v}$	$0.12 \pm 0.01 (\pm 0.01)$	0.11	mm
$\sigma_y^{X8,h}$	$1.68 \pm 0.09 (\pm 0.01)$	1.50	mm
ε_n^x	$0.41 \pm 0.06 (\pm 0.02)$	0.27	μm
ε_n^y	$41.1 \pm 2.5 (\pm 0.54)$	53	μm
$\varepsilon_n^y / \varepsilon_n^x$	$100.2 \pm 20.2 (\pm 5.2)$	196	



Outlook + open questions

- magnetized beam from a SCRF gun:
 - flux concentrator around cathode?
 - flat beam at cathode
[J. Rosenzweig, PAC93 showed $(\varepsilon_+, \varepsilon_-) = (95, 4.5) \mu\text{m}$]
- needed ϵ_u and \mathcal{L} ? and limit on 4-D emittance?
- planned future experiment at ASTA

