Accelerator Physics
Particle Acceleration

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Lecture 4

Clarifications from Last Time

- On Crest,

$$V_c = \frac{2\sqrt{\beta}}{\sqrt{1+\beta}}\sqrt{P_g R_L} - R_L I_0 = \frac{2\sqrt{\beta}}{\sqrt{1+\beta}}\sqrt{P_g R_L}\left(1 - \frac{R_L I_0}{\sqrt{P_g R_L}}\frac{\sqrt{1+\beta}}{2\sqrt{\beta}}\right) \rightarrow K = \frac{\sqrt{R_a} I_0}{2\sqrt{P_g}}$$

- Off Crest with Detuning

$$P_g = \frac{Z_0 \left|I^+\right|^2}{2} = \frac{Z_0 k^2 \left|i_g\right|^2}{8}$$

$$i_g = \frac{2V_c}{R_L}\left(1 - i\tan\Psi\right) + 2I_0\left(\cos\psi_b + i\sin\psi_b\right)$$

$$P_g = \frac{Z_0 k^2 \left|V_c\right|^2}{2R_L^2}\left[\left(1 + \frac{I_0 R_L}{V_c}\cos\psi_b\right)^2 + \left(\tan\Psi - \frac{I_0 R_L}{V_c}\sin\psi_b\right)^2\right]$$

Off Crest/Off Resonance Power

- Typically electron storage rings operate off crest in order to ensure stability against phase oscillations.
- As a consequence, the rf cavities must be detuned off resonance in order to minimize the reflected power and the required generator power.
- Longitudinal gymnastics may also impose off crest operation in recirculating linacs.
- We write the beam current and the cavity voltage as

$$\tilde{I}_b = 2I_0 e^{i\psi_b}$$

$$\tilde{V}_c = V_c e^{i\psi_c} \quad \text{and set } \psi_c = 0$$

- The generator power can then be expressed as:

$$P_g = \frac{V_c^2}{R_L}\frac{(1+\beta)}{4\beta}\left\{\left[1+\frac{I_0 R_L}{V_c}\cos\psi_b\right]^2+\left[\tan\Psi-\frac{I_0 R_L}{V_c}\sin\psi_b\right]^2\right\}$$

Wiedemann 16.31

Optimal Detuning and Coupling

- Condition for optimum tuning with beam:

$$\tan\Psi = \frac{I_0 R_L}{V_c}\sin\psi_b$$

- Condition for optimum coupling with beam:

$$\beta_{\text{opt}} = 1 + \frac{I_0 R_a}{V_c}\cos\psi_b$$

- Minimum generator power:

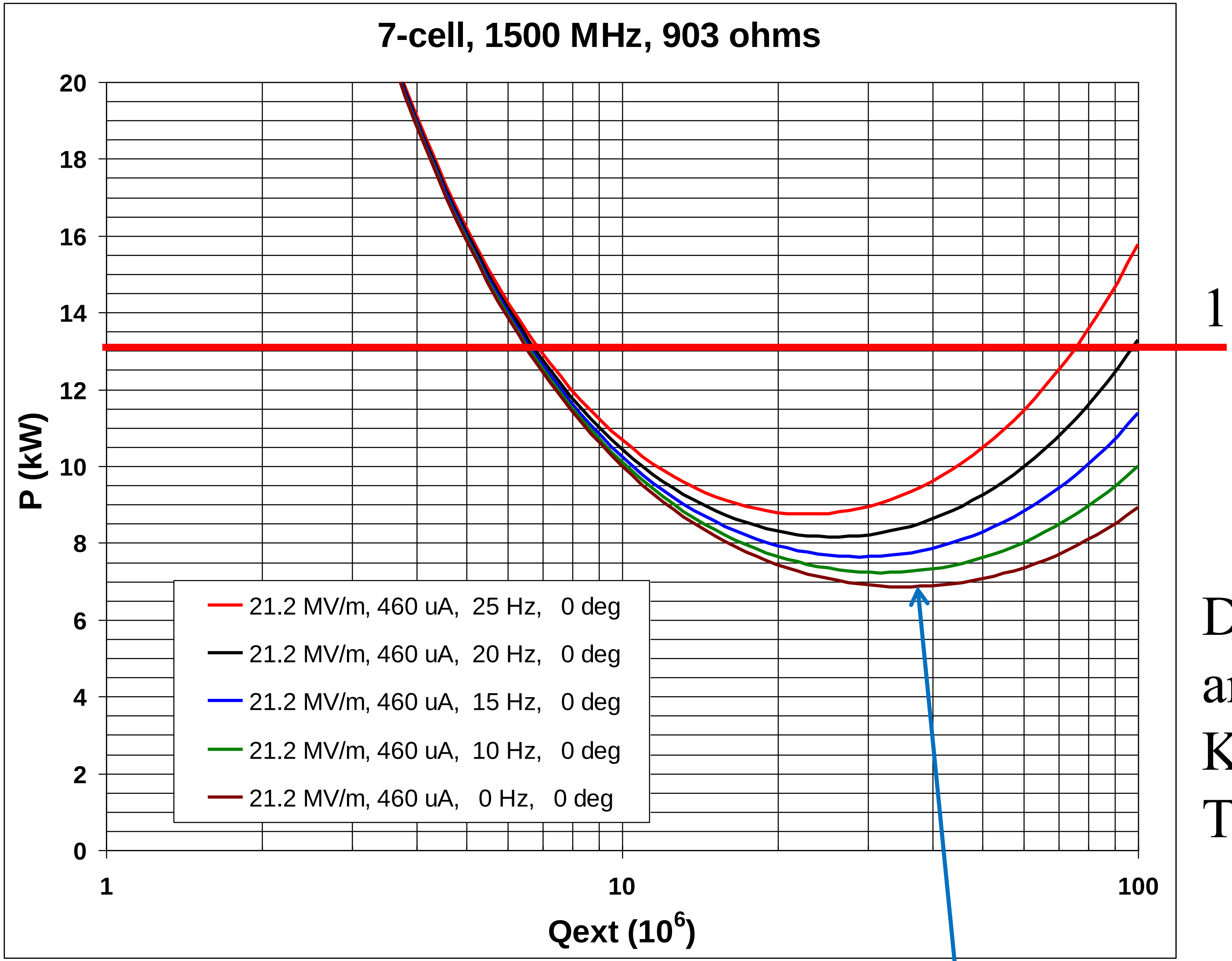
$$P_{g,\min} = \frac{V_c^2\beta_{\text{opt}}}{R_a}$$

Wiedemann
16.36

C75 Power Estimates

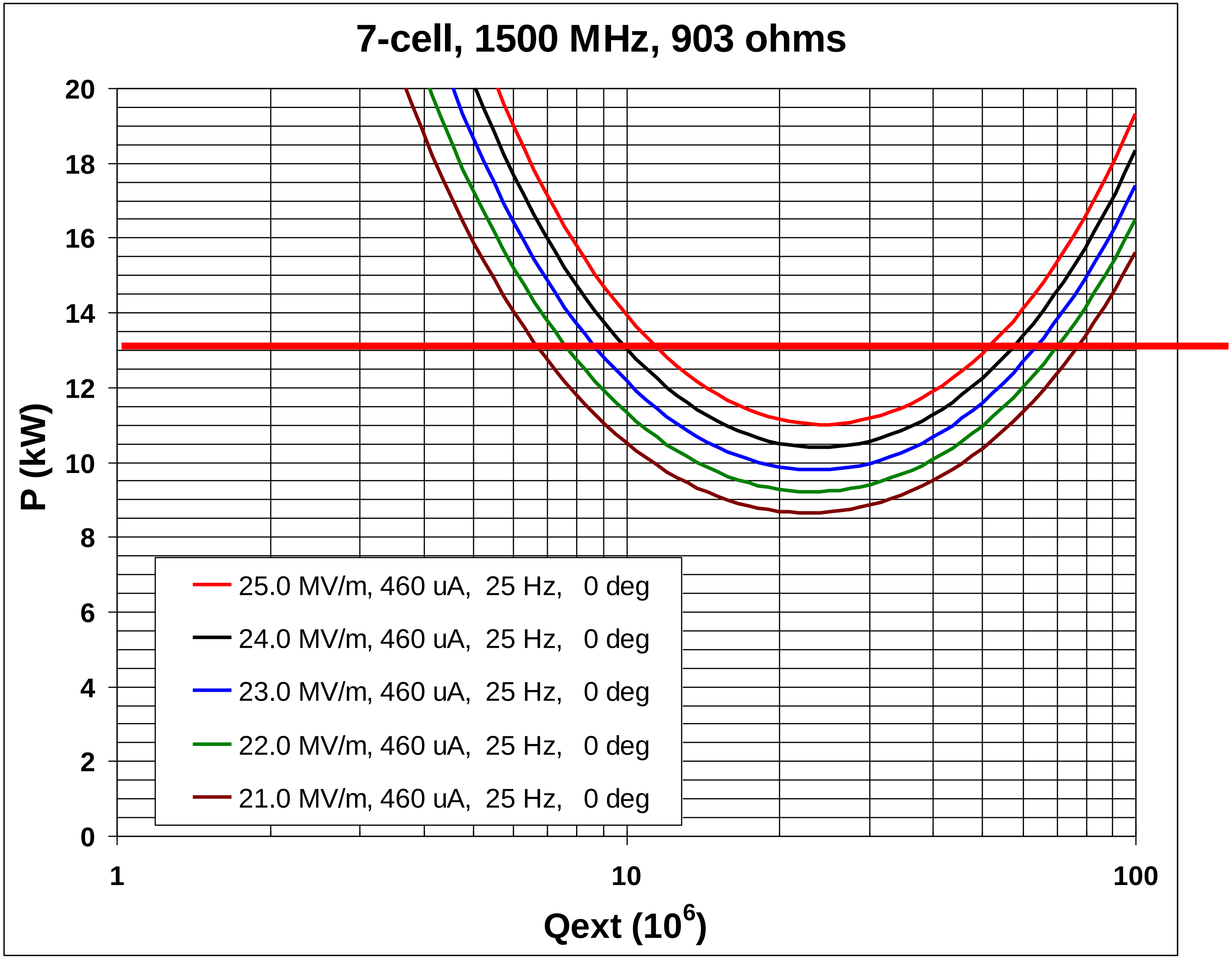
G. A. Krafft

12 GeV Project Specs



Delayen and Krafft TN-07-29

460 muA*15 MV=6.8 kW

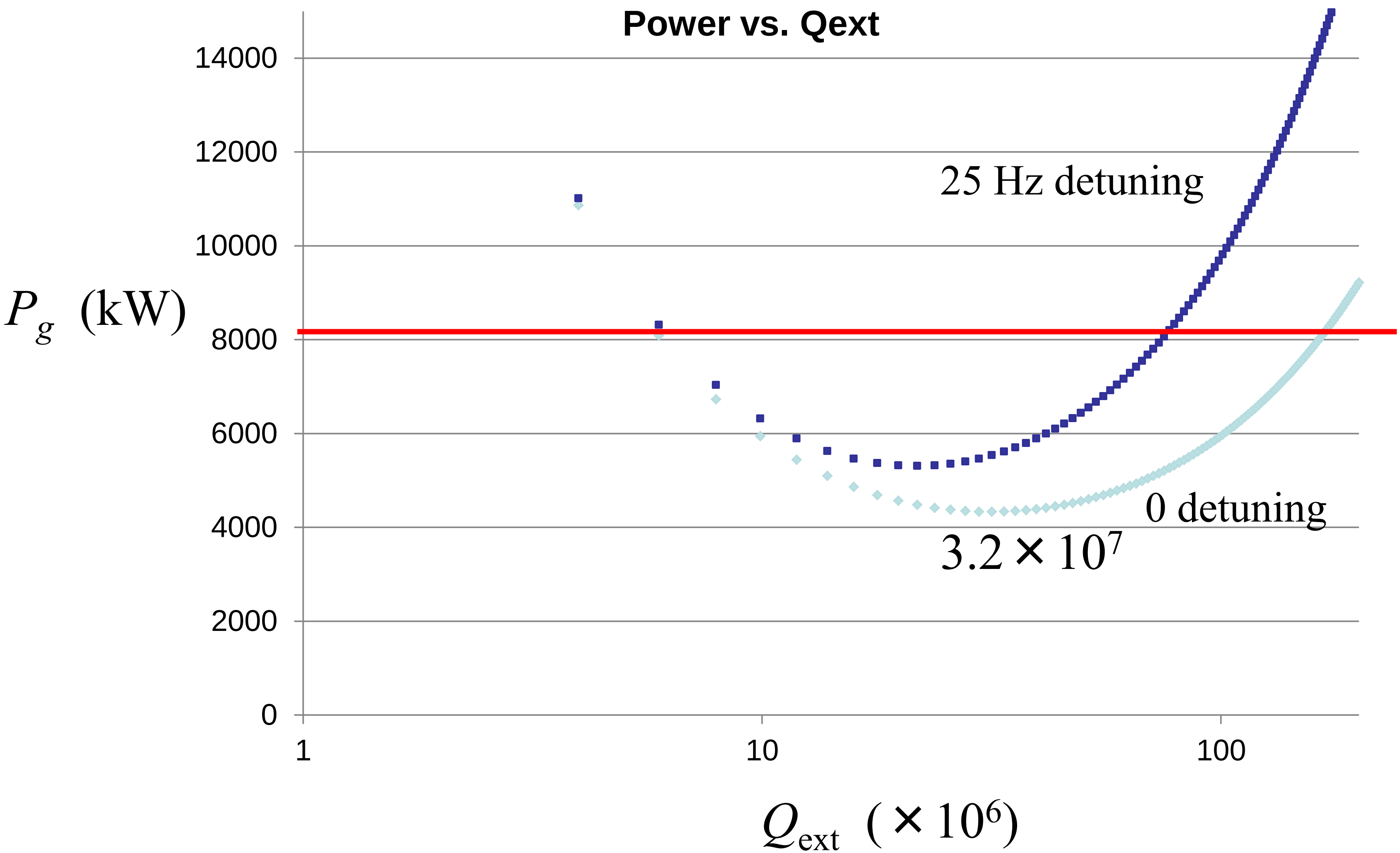

7-cell, 1500 MHz, 903 ohms
P (kW)
Qext (10^6)
0
2
4
6
8
10
12
14
16
18
20
1
10
100
25.0 MV/m, 460 uA, 25 Hz, 0 deg
24.0 MV/m, 460 uA, 25 Hz, 0 deg
23.0 MV/m, 460 uA, 25 Hz, 0 deg
22.0 MV/m, 460 uA, 25 Hz, 0 deg
21.0 MV/m, 460 uA, 25 Hz, 0 deg

Assumptions

- Low Loss R/Q = 903*5/7 = 645 Ω
- Max Current to be accelerated 460 μA
- Compute 0 and 25 Hz detuning power curves
- 75 MV/cryomodule (18.75 MV/m)
- Therefore matched power is 4.3 kW

(Scale increase 7.4 kW tube spec)

- Q_{ext} adjustable to 3.18×10^7(if not need more RF power!)

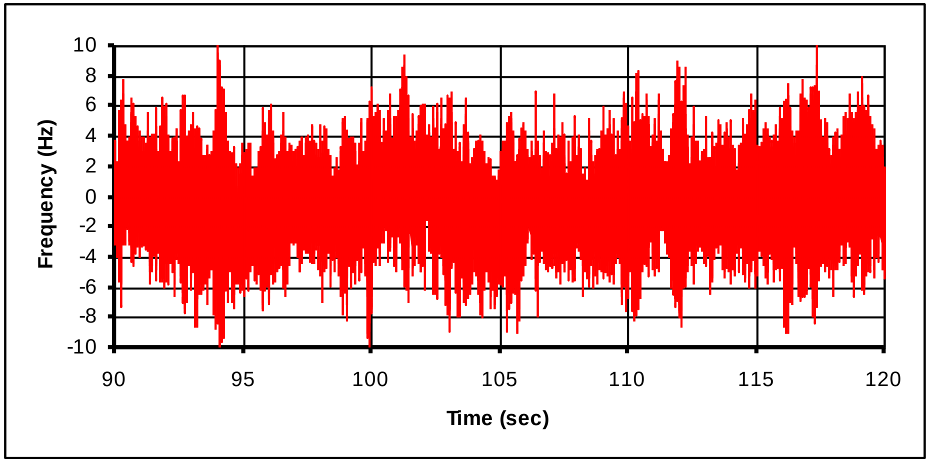

Power vs. Qext
14000
12000
10000
8000
6000
4000
2000
0
P_g (kW)
25 Hz detuning
0 detuning
3.2×10^7
1
10
100
Q_{ext} $(\times 10^6)$

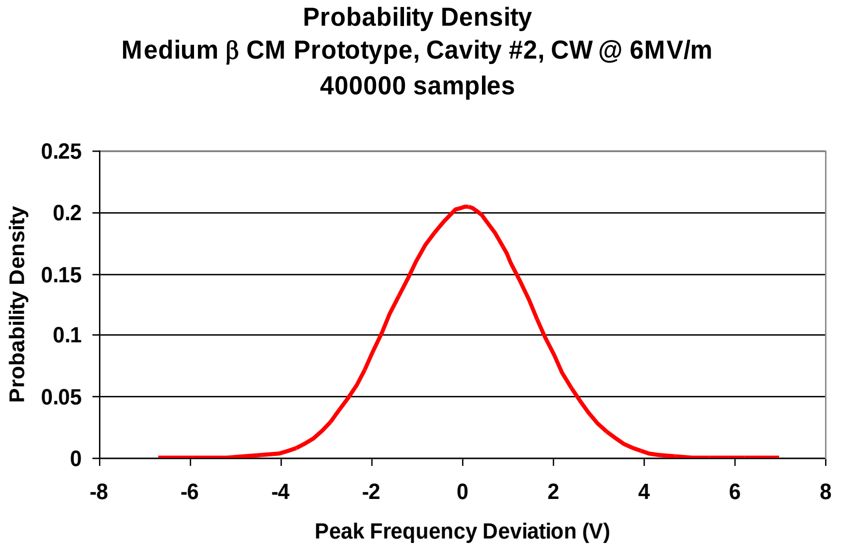
RF Cavity with Beam and Microphonics

The detuning is now: $\tan\Psi = -2Q_L \dfrac{\delta f_0 \pm \delta f_m}{f_0}$ $\qquad \tan\psi_0 = -2Q_L \dfrac{\delta f_0}{f_0}$

where δf_0 is the static detuning (controllable)

and δf_m is the random dynamic detuning (uncontrollable)





Q_{ext} Optimization with Microphonics

$$P_g = \frac{V_c^2}{R_L}\frac{(1+\beta)}{4\beta}\left\{\left[1+\frac{I_{tot}R_L}{V_c}\cos\psi_{tot}\right]^2+\left[\tan\Psi-\frac{I_{tot}R_L}{V_c}\sin\psi_{tot}\right]^2\right\}$$

$$\tan\Psi = -2Q_L\frac{\delta f}{f_0}$$

where δf is the total amount of cavity detuning in Hz, including static detuning and microphonics.

- Optimizing the generator power with respect to coupling gives:

$$\beta_{opt} = \sqrt{(b+1)^2+\left[2Q_0\frac{\delta f}{f_0}+b\tan\psi_{tot}\right]^2}$$

$$\text{where} \quad b \equiv \frac{I_{tot}R_a}{V_c}\cos\psi_{tot}$$

where I_{tot} is the magnitude of the resultant beam current vector in the cavity and ψ_{tot} is the phase of the resultant beam vector with respect to the cavity voltage.

Correct Static Detuning

- To minimize generator power with respect to tuning:

$$\delta f_0 = -\frac{f_0}{2Q_0} b \tan \psi_{tot}$$

$$P_g = \frac{V_c^2}{R_a} \frac{1}{4\beta} \left\{ (1+b+\beta)^2 + \left[2Q_0 \frac{\delta f_m}{f_0} \right]^2 \right\}$$

- The resulting power is

$$P_g = \frac{V_c^2}{R_a} \frac{1}{4\beta_{opt}} \left\{ (1+b)^2 + 2(1+b)\beta_{opt} + \beta_{opt}^2 + \left[2Q_0 \frac{\delta f_m}{f_0} \right]^2 \right\}$$

$$= \frac{V_c^2}{2R_a} \left\{ (1+b) + \beta_{opt} \right\}$$

Optimal Q_{ext} and Power

- Condition for optimum coupling:

$$\beta_{opt} = \sqrt{(b+1)^2 + \left(2Q_0 \frac{\delta f_m}{f_0}\right)^2}$$

and

$$P_g^{opt} = \frac{V_c^2}{2R_a}\left[b+1+\sqrt{(b+1)^2 + \left(2Q_0 \frac{\delta f_m}{f_0}\right)^2}\right]$$

- In the absence of beam (b=0):

$$\beta_{opt} = \sqrt{1 + \left(2Q_0 \frac{\delta f_m}{f_0}\right)^2}$$

and

$$P_g^{opt} = \frac{V_c^2}{2R_a}\left[1+\sqrt{1 + \left(2Q_0 \frac{\delta f_m}{f_0}\right)^2}\right]$$

Problem for the Reader

- Assuming no microphonics, plot β_{opt} and P^g_{opt} as function of b (beam loading), b=-5 to 5, and explain the results.
- How do the results change if microphonics is present?

Example

- ERL Injector and Linac:

  δf_m=25 Hz, Q_0=1x10^{10} , f_0=1300 MHz, I_0=100 mA, V_c=20 MV/m, L=1.04 m, R_a/Q_0=1036 ohms per cavity

- ERL linac: Resultant beam current, I_{tot} = 0 mA (energy recovery)

  and β_{opt}=385 $\Rightarrow$ Q_L=2.6x10^7 $\Rightarrow$ P_g = 4 kW per cavity.

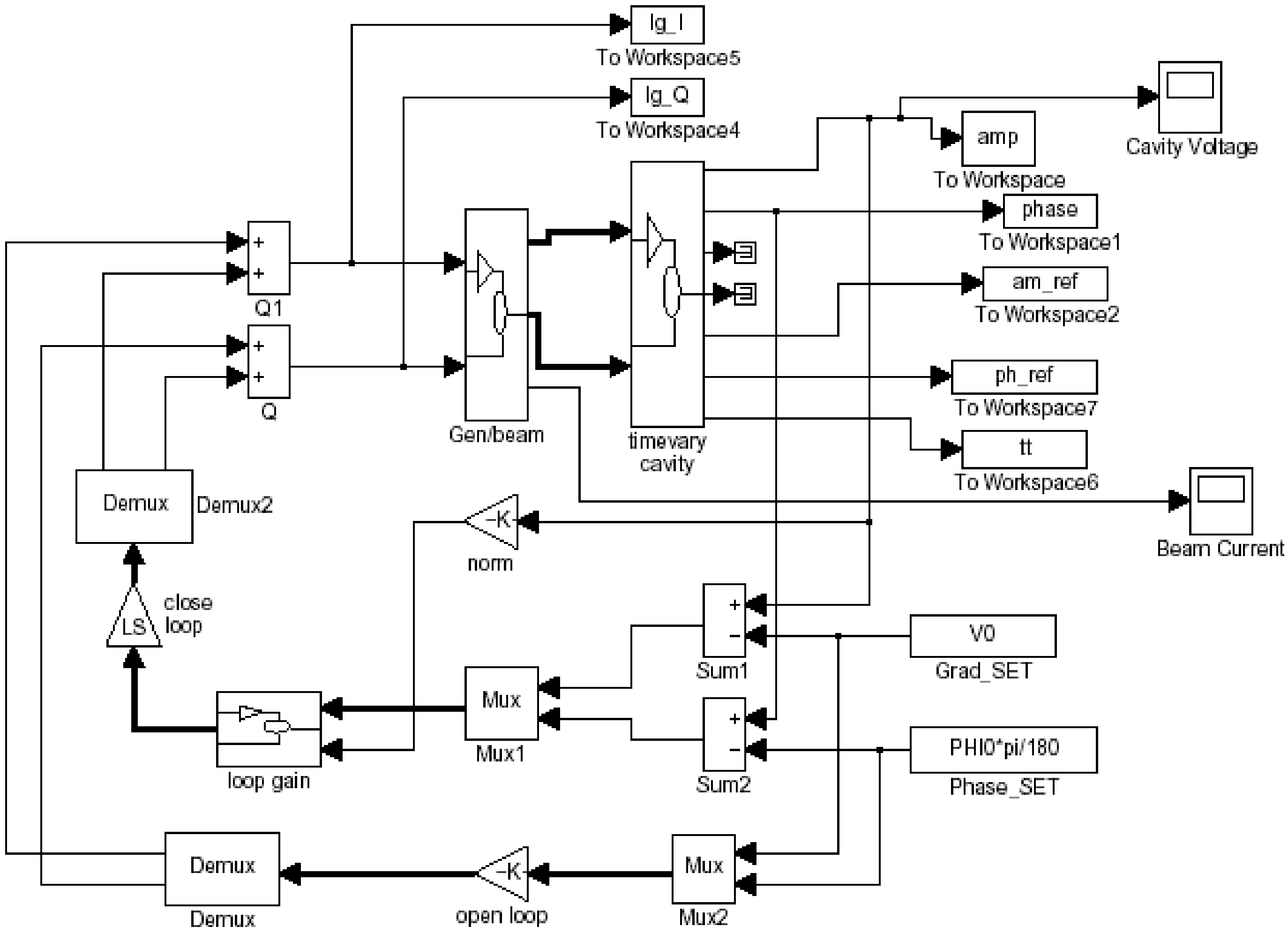
- ERL Injector: I_0=100 mA and β_{opt}= 5x10^4 ! $\Rightarrow$ Q_L= 2x10^5 $\Rightarrow$ P_g = 2.08 MW per cavity!

  Note: $I_0 V_a$ = 2.08 MW $\Rightarrow$ optimization is entirely dominated by beam loading.

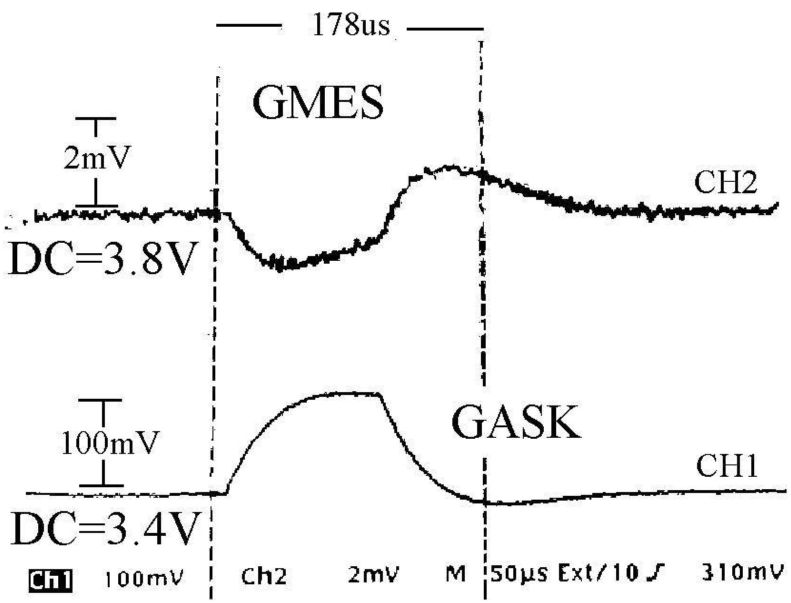
RF System Modeling

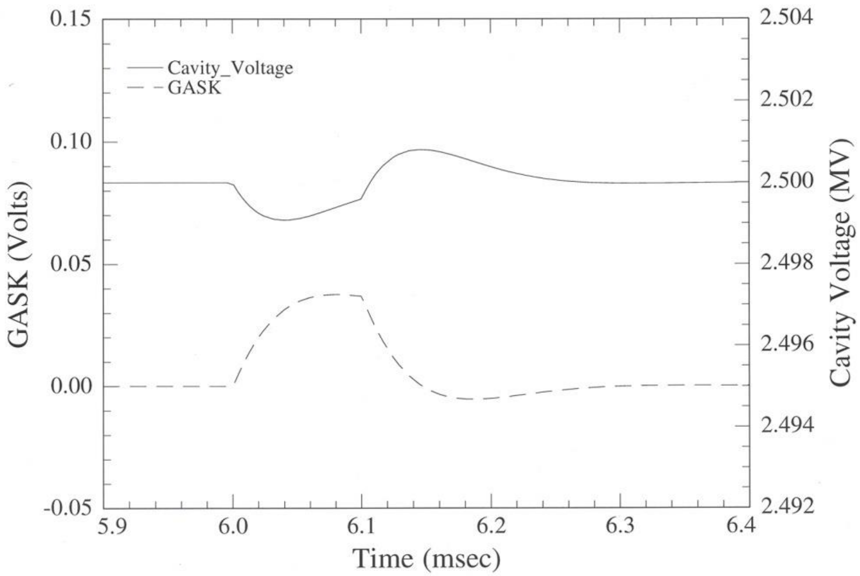
- To include amplitude and phase feedback, nonlinear effects from the klystron and be able to analyze transient response of the system, response to large parameter variations or beam current fluctuations
  - we developed a model of the cavity and low level controls using SIMULINK, a MATLAB-based program for simulating dynamic systems.
- Model describes the beam-cavity interaction, includes a realistic representation of low level controls, klystron characteristics, microphonic noise, Lorentz force detuning and coupling and excitation of mechanical resonances

RF System Model



RF Modeling: Simulations vs. Experimental Data





Measured and simulated cavity voltage and amplified gradient error signal (GASK) in one of CEBAF's cavities, when a 65 µA, 100 µsec beam pulse enters the cavity.

Conclusions

- We derived a differential equation that describes to a very good approximation the rf cavity and its interaction with beam.
- We derived useful relations among cavity's parameters and used phasor diagrams to analyze steady-state situations.
- We presented formula for the optimization of Q_{ext} under beam loading and microphonics.
- We showed an example of a Simulink model of the rf control system which can be useful when nonlinearities can not be ignored.

RF Focussing

In any RF cavity that accelerates longitudinally, because of Maxwell Equations there must be additional transverse electromagnetic fields. These fields will act to focus the beam and must be accounted properly in the beam optics, especially in the low energy regions of the accelerator. We will discuss this problem in greater depth in injector lectures. Let **A**(x,y,z) be the vector potential describing the longitudinal mode (Lorenz gauge)

$$\nabla \cdot \vec{A} = -\frac{1}{c}\frac{\partial \phi}{\partial t}$$

$$\nabla^2 \vec{A} = -\frac{\omega^2}{c^2}\vec{A} \qquad \nabla^2 \phi = -\frac{\omega^2}{c^2}\phi$$

For cylindrically symmetrical accelerating mode, functional form can only depend on r and z

$$A_z(r,z) = A_{z0}(z) + A_{z1}(z) r^2 + \dots$$

$$\phi(r,z) = \phi_0(z) + \phi_1(z) r^2 + \dots$$

Maxwell's Equations give recurrence formulas for succeeding approximations

$$(2n)^2 A_{zn} + \frac{d^2 A_{z,n-1}}{dz^2} = -\frac{\omega^2}{c^2} A_{z,n-1}$$

$$(2n)^2 \phi_n + \frac{d^2 \phi_{n-1}}{dz^2} = -\frac{\omega^2}{c^2} \phi_{n-1}$$

Gauge condition satisfied when

$$\frac{dA_{zn}}{dz} = -\frac{i\omega}{c}\phi_n$$

in the particular case $n = 0$

$$\frac{dA_{z0}}{dz} = -\frac{i\omega}{c}\phi_0$$

Electric field is

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \overleftarrow{A}}{\partial t}$$

And the potential and vector potential must satisfy

$$E_z(0,z) = -\frac{d\phi_0}{dz} - \frac{i\omega}{c} A_{z0}$$

$$\therefore \quad \frac{i\omega}{c} E_z(0,z) = \frac{d^2 A_{z0}}{dz^2} + \frac{\omega^2}{c^2} A_{z0} = -4A_{z1}$$

So the magnetic field off axis may be expressed directly in terms of the electric field on axis

$$\therefore \quad B_\theta \approx -2rA_{z1} = \frac{i}{2}\frac{\omega r}{c} E_z(0,z)$$

And likewise for the radial electric field (see also $\nabla \cdot \vec{E} = 0$)

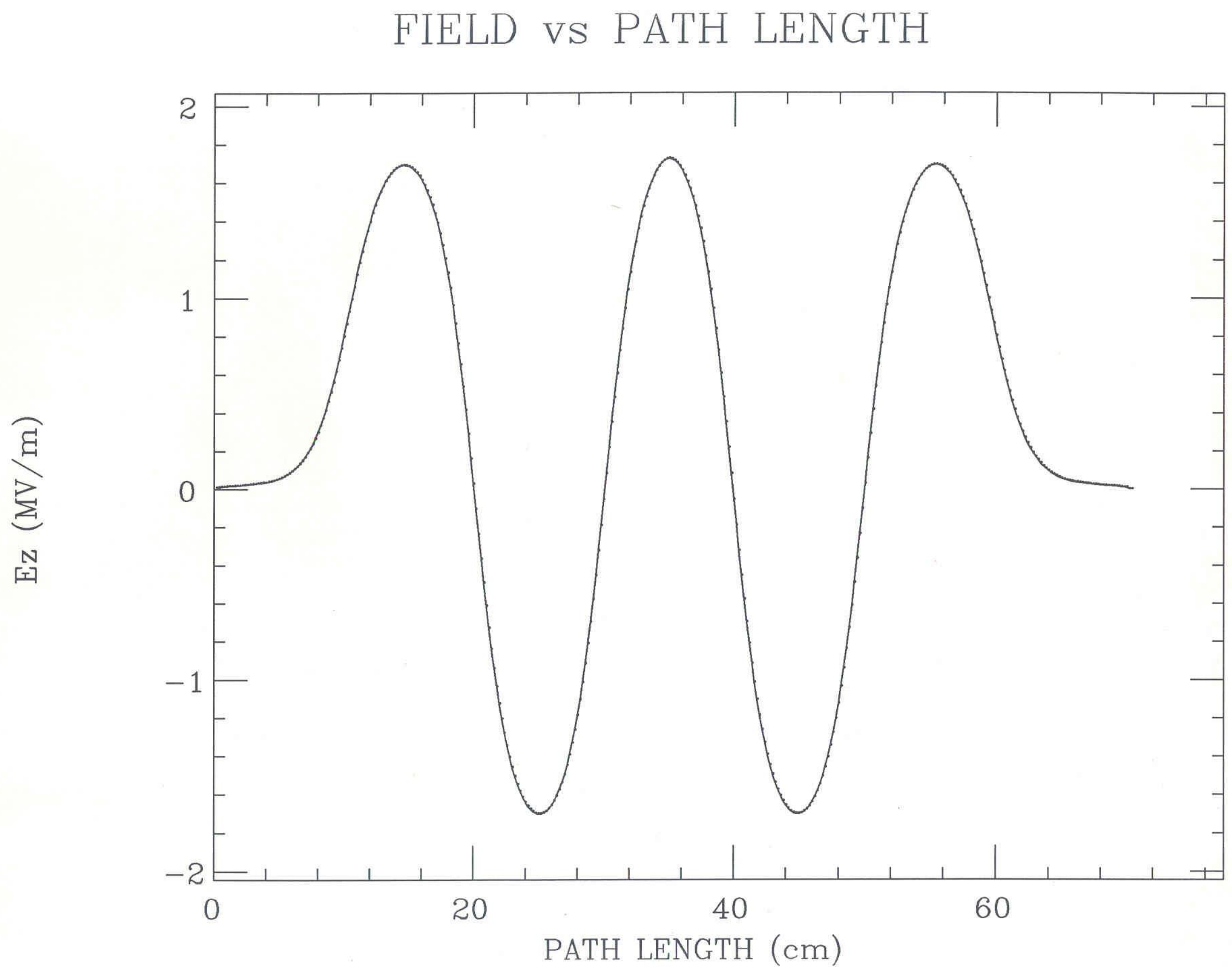
$$\therefore \quad E_r \approx -2r\phi_1(z) = -\frac{r}{2}\frac{dE_z(0,z)}{dz}$$

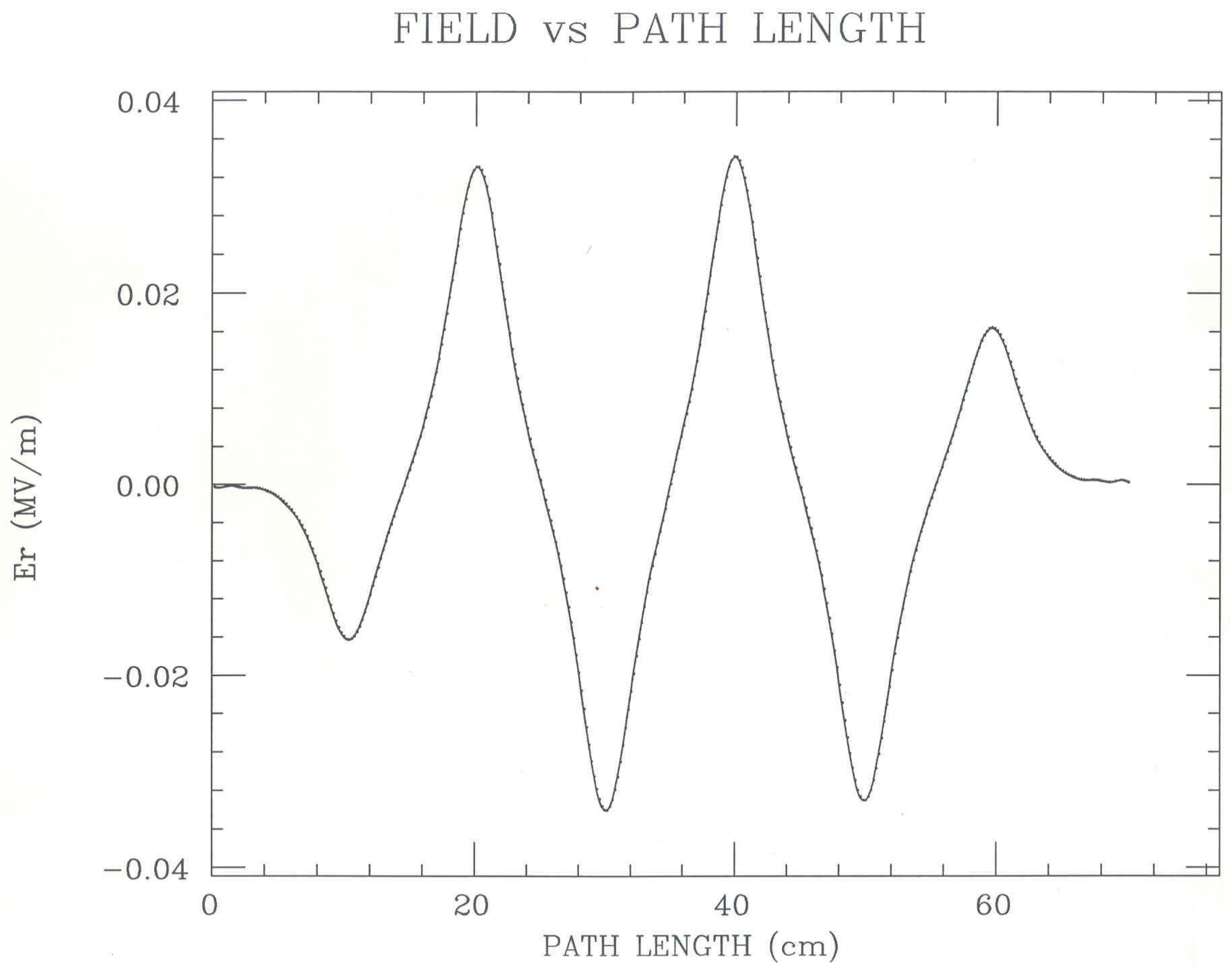
Explicitly, for the time dependence $\cos(\omega t + \delta)$

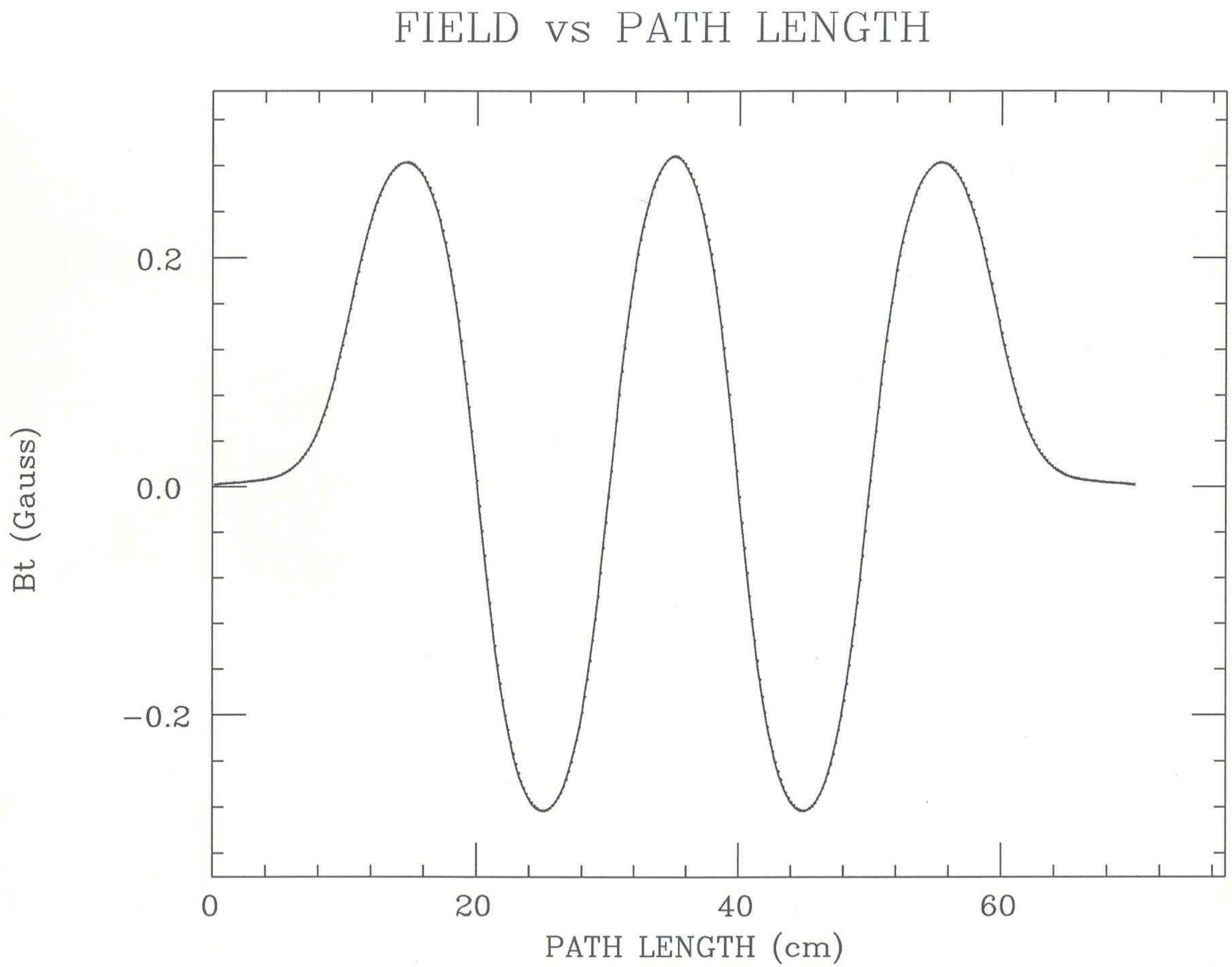
$$E_z(r,z,t) \approx E_z(0,z)\cos(\omega t + \delta)$$

$$E_r(r,z,t) \approx -\frac{r}{2}\frac{dE_z(0,z)}{dz}\cos(\omega t + \delta)$$

$$B_\theta(r,z,t) \approx -\frac{\omega r}{2c}E_z(0,z)\sin(\omega t + \delta)$$


FIELD vs PATH LENGTH
Ez (MV/m)
2
1
0
−1
−2
0
20
40
60
PATH LENGTH (cm)


FIELD vs PATH LENGTH
0.04
0.02
0.00
−0.02
−0.04
Er (MV/m)
0
20
40
60
PATH LENGTH (cm)


FIELD vs PATH LENGTH
0.2
0.0
−0.2
Bt (Gauss)
0
20
40
60
PATH LENGTH (cm)

Motion of a particle in this EM field

$$\frac{d\left(\gamma m\vec{V}\right)}{dt} = -e\left(\vec{E} + \frac{\vec{V}}{c} \times \vec{B}\right)$$

$$\gamma(z)\beta_x(z) = \gamma(-\infty)\beta_x(-\infty) + \int_{-\infty}^{z}\left[\begin{array}{l} -\dfrac{x(z')}{2}\dfrac{dG(z')}{dz'}\cos\left(\omega t(z') + \delta\right) \\ +\dfrac{\omega\beta_z(z')x(z')}{2c}G(z')\sin\left(\omega t(z') + \delta\right)\end{array}\right]\frac{dz'}{\beta_z(z')}$$

The normalized gradient is

$$G(z)=\frac{eE_z(z,0)}{mc^2}$$

and the other quantities are calculated with the integral equations

$$\gamma(z)=\gamma(-\infty)+\int_{-\infty}^{z}G(z')\cos(\omega t(z')+\delta)dz'$$

$$\gamma(z)\beta_z(z)=\gamma(-\infty)\beta_z(-\infty)+\int_{-\infty}^{z}\frac{G(z')}{\beta_z(z')}\cos(\omega t(z')+\delta)dz'$$

$$t(z)=\lim_{z_0\to-\infty}\frac{z_0}{\beta_z(-\infty)c}+\int_{-\infty}^{z}\frac{dz'}{\beta_z(z')c}$$

These equations may be integrated numerically using the cylindrically symmetric CEBAF field model to form the Douglas model of the cavity focussing. In the high energy limit the expressions simplify.

$$x(z) = x(a) + \int_a^z \frac{\gamma(z')\beta_x(z')}{\gamma(z')\beta_z(z')} dz'$$

$$\approx x(a) + \frac{\beta_x(-\infty)}{\beta_z(-\infty)}(z-a) - \int_a^z \frac{x(z')}{2} \frac{G(z')}{\gamma(z')\beta_z^2(z')} \cos(\omega t(z') + \delta) dz'$$

Transfer Matrix

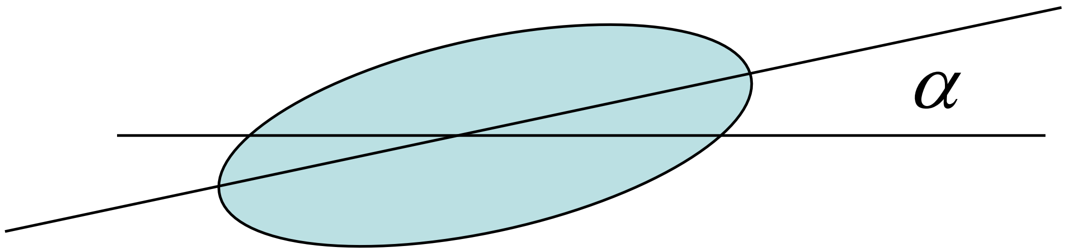
For position-momentum transfer matrix

$$T = \begin{pmatrix} 1-\dfrac{E_G}{2E} & \dfrac{L}{\gamma} \\ -\dfrac{I}{4\gamma} & 1+\dfrac{E_G}{2E} \end{pmatrix}$$

$$I = \cos^2(\delta)\int_{-\infty}^{\infty} G^2(z)\cos^2(\omega z/c)dz$$

$$+\sin^2(\delta)\int_{-\infty}^{\infty} G^2(z)\sin^2(\omega z/c)dz$$

Kick Generated by mis-alignment



$$\Delta\gamma\beta = \frac{E_G \alpha}{2E}$$

Damping and Antidamping

By symmetry, if electron traverses the cavity exactly on axis, there is no transverse deflection of the particle, but there is an energy increase. By conservation of transverse momentum, there must be a decrease of the phase space area. For linacs NEVER use the word "adiabatic"

$$\frac{d\left(\gamma m\vec{\mathrm{V}}_{\mathrm{transverse}}\right)}{dt}=0$$

$$\gamma\left(z\right)\beta_x\left(z\right)=\gamma\left(-\infty\right)\beta_x\left(-\infty\right)$$

Conservation law applied to angles

$$\beta_x, \beta_y << \beta_z \approx 1$$

$$\theta_x = \beta_x / \beta_z \sim \beta_x \quad \theta_y = \beta_y / \beta_z \sim \beta_y$$

$$\theta_x\left(z\right) = \frac{\gamma\left(-\infty\right)\beta_z\left(-\infty\right)}{\gamma\left(z\right)\beta_z\left(z\right)}\theta_x\left(-\infty\right)$$

$$\theta_y\left(z\right) = \frac{\gamma\left(-\infty\right)\beta_z\left(-\infty\right)}{\gamma\left(z\right)\beta_z\left(z\right)}\theta_y\left(-\infty\right)$$

Phase space area transformation

$$dx \wedge d\theta_x(z) = \frac{\gamma(-\infty)\beta_z(-\infty)}{\gamma(z)\beta_z(z)} dx \wedge d\theta_x(-\infty)$$

$$dy \wedge d\theta_y(z) = \frac{\gamma(-\infty)\beta_z(-\infty)}{\gamma(z)\beta_z(z)} dy \wedge d\theta_y(-\infty)$$

Therefore, if the beam is accelerating, the phase space area after the cavity is less than that before the cavity and if the beam is decelerating the phase space area is greater than the area before the cavity. The determinate of the transformation carrying the phase space through the cavity has determinate equal to

$$\mathrm{Det}\left(M_{cavity}\right) = \frac{\gamma(-\infty)\beta_z(-\infty)}{\gamma(z)\beta_z(z)}$$

By concatenation of the transfer matrices of all the accelerating or decelerating cavities in the recirculated linac, and by the fact that the determinate of the product of two matrices is the product of the determinates, the phase space area at each location in the linac is

$$dx \wedge d\theta_x(z) = \frac{\gamma(0)\beta_z(0)}{\gamma(z)\beta_z(z)} dx \wedge d\theta_x(0)$$

$$dy \wedge d\theta_y(z) = \frac{\gamma(0)\beta_z(0)}{\gamma(z)\beta_z(z)} dy \wedge d\theta_y(0)$$

Same type of argument shows that things like orbit fluctuations are damped/amplified by acceleration/deceleration.

Transfer Matrix Non-Unimodular

$$M_{tot} = M_1 \cdot M_2$$

$$P(M) \equiv \frac{M}{\det M}$$

$$P(M) \qquad \text{unimodular!}$$

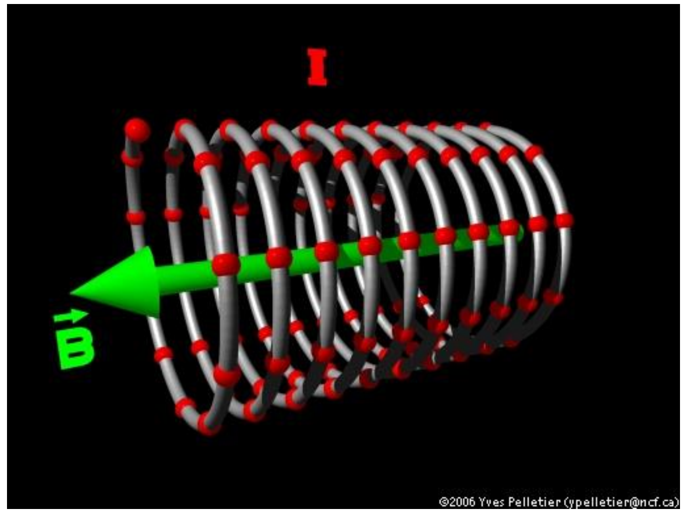
$$P(M_{tot}) = \frac{M_{tot}}{\det M_{tot}} = \frac{M_1}{\det M_1} \frac{M_2}{\det M_2} = P(M_1) \cdot P(M_2)$$

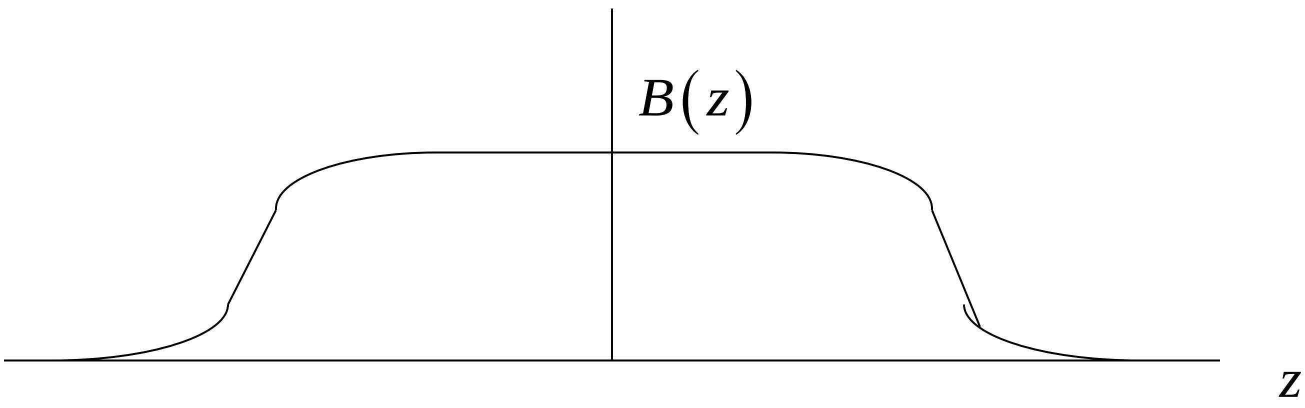
$\therefore$ can separately track the "unimodular part" (as before!)

and normalize by accumulated determinate

Solenoid Focussing

Can also have continuous focusing in both transverse directions by applying solenoid magnets:





Busch's Theorem

For cylindrical symmetry magnetic field described by a vector potential:

$$\vec{A} = A_\theta(z,r)\hat{\theta} \qquad B_z = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\theta(z,r)\right) \quad \text{is nearly constant}$$

$$\therefore A_\theta(z,r) \doteq \frac{B_z(r=0,z)r}{2} \qquad B_r = -\frac{B_z'(r=0,z)r}{2}$$

Conservation of Canonical Momentum gives Busch's Theorem:

$$P_\theta = \gamma m r^2 \dot{\theta} + qrA_\theta = const$$

for particle with $\dot{\theta} = 0$ where $B_z = 0,\ P_\theta = 0$

$$\gamma m r^2 \dot{\theta} = -\frac{qr^2 B_z}{2} \rightarrow \dot{\theta} = -\frac{\Omega_c}{2} = -\omega_{Larmor}$$

Beam rotates at the Larmor frequency which implies coupling

Radial Equation

$$\frac{d}{dt}(\gamma m\dot{r}) - \gamma m r\omega_L^2 = qr\dot{\theta}B_z = -2\gamma m r\omega_L^2$$

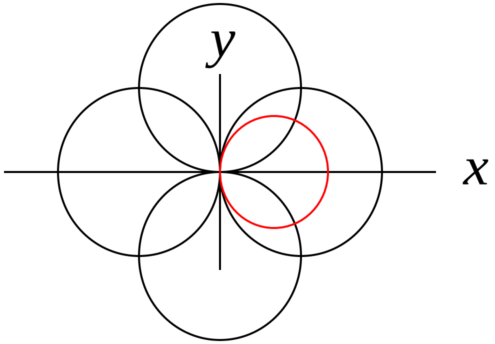
$$\therefore k = \frac{\omega_L^2}{\beta_z^2 c^2}$$

thin lens focal length

$$\frac{1}{f} = \frac{e^2 \int_{-\infty}^{\infty} B_z^2 dz}{4\beta_z^2 \gamma^2 m^2 c^2}$$

weak compared to quadrupole for high γ

If go to full ¼ oscillation inside the magnetic field in the "thick" lens case, all particles end up at r = 0! Non-zero emittance spreads out perfect focusing!



Larmor's Theorem

This result is a special case of a more general result. If go to frame that rotates with the local value of Larmor's frequency, then the transverse dynamics including the magnetic field are simply those of a harmonic oscillator with frequency equal to the Larmor frequency. Any force from the magnetic field linear in the field strength is "transformed away" in the Larmor frame. And the motion in the two transverse degrees of freedom are now decoupled. Pf: The equations of motion are

$$\frac{d}{dt}(\gamma m\dot{r}) - \gamma m r\dot{\theta}^2 = qr\dot{\theta}B_z$$

$$\gamma m r^2\dot{\theta} + qA_\theta = cons = P_\theta$$

$$\frac{d}{dt}(\gamma m\dot{r}) - \gamma m r\dot{\theta}'^2 + 2\gamma m r\theta'\omega_L - \gamma m r\omega_L^2 = qr\dot{\theta}'B_z - qr\omega_L B_z$$

$$\gamma m r^2\dot{\theta}' = P_\theta$$

$$\left.\begin{aligned} \frac{d}{dt}(\gamma m\dot{r}) - \gamma m r\dot{\theta}'^2 &= -\gamma m r\omega_L^2 \\ \gamma m r^2\dot{\theta}' &= P_\theta \end{aligned}\right\} \text{2-D Harmonic Oscillator}$$