Accelerator Physics
Particle Acceleration

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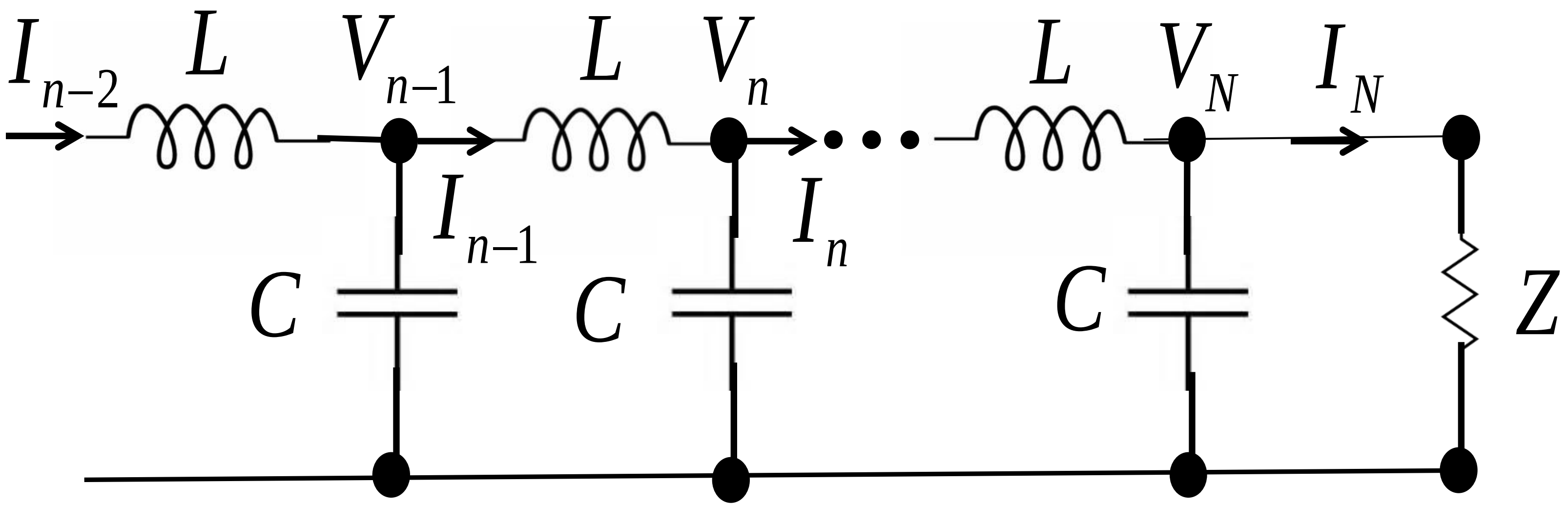
Lecture 3

Transmission Lines



Inductor Impedance and Current Conservation

$$V_{n-1} - V_n = i\omega L I_{n-1}$$

$$I_{n-1} - I_n = i\omega C V_n$$

Transmission Line Equations

- Standard Difference Equation with Solution

$$V_n = V_0 e^{-in\lambda}, I_n = I_0 e^{-in\lambda}$$

$$V_0\left(e^{i\lambda}-1\right) = i\omega L I_0 e^{i\lambda} \qquad I_0\left(e^{i\lambda}-1\right) = i\omega C V_0$$

$$2\sin\left(\lambda/2\right) = \pm\omega\sqrt{LC}$$

$$V^+ = \sqrt{L/C}\, I^+ e^{i\lambda/2} \qquad V^- = -\sqrt{L/C}\, I^- e^{i\lambda/2}$$

- Continuous Limit ($N \to \infty$)

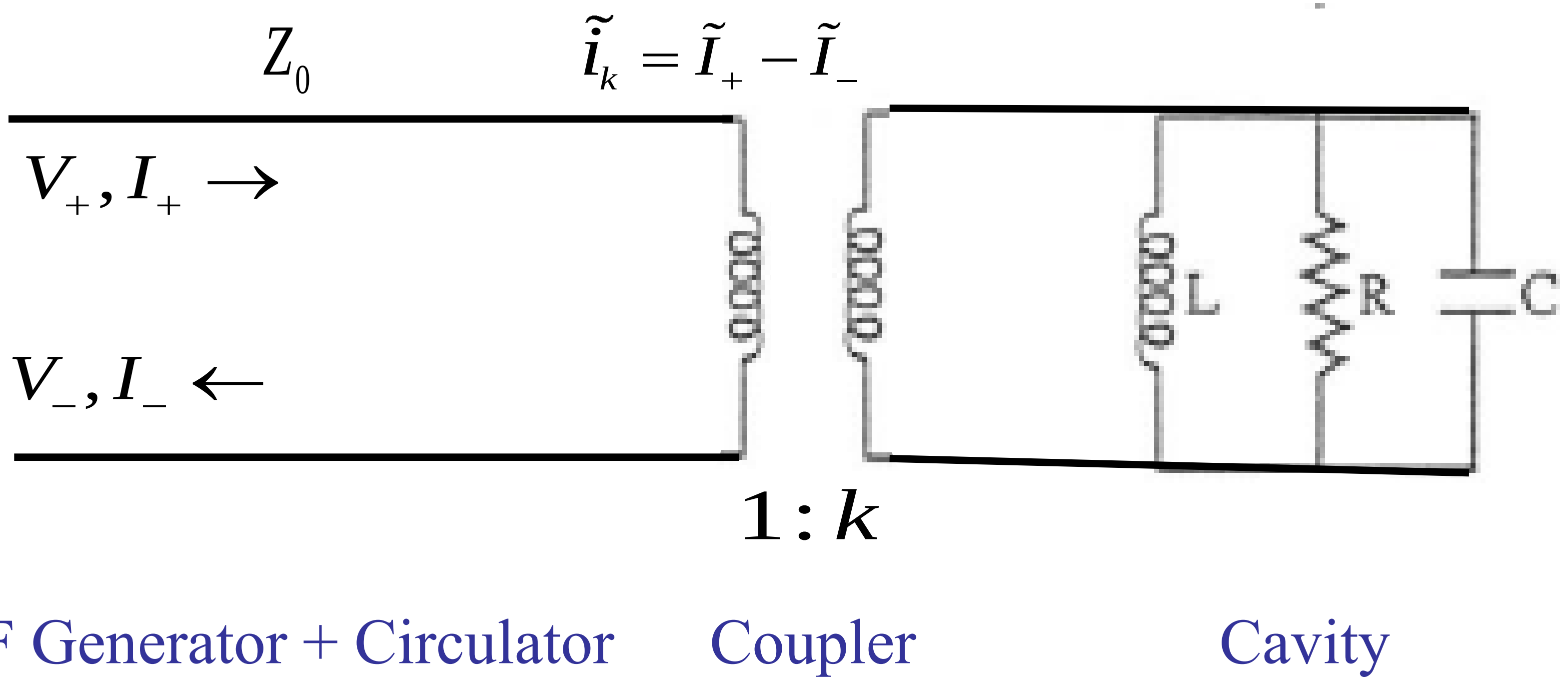
$$\lambda \to \pm\omega\sqrt{LC} \qquad k = \omega\sqrt{L'C'} \qquad v_\phi = 1/\sqrt{L'C'}$$

$$V^+\left(x,t\right) = V_0^+ e^{i\omega t - kx} = \sqrt{L'/C'}\, I_0^+ e^{i\omega t - kx} = Z_0 I_0^+ e^{i\omega t - kx}$$

$$V^-\left(x,t\right) = V_0^- e^{i\omega t + kx} = -\sqrt{L'/C'}\, I_0^- e^{i\omega t + kx} = -Z_0 I_0^- e^{i\omega t + kx}$$

Cavity Coupled to an RF Source

- Equivalent Circuit



- Coupling is represented by an ideal (lossless) transformer of turns ratio 1:*k*

Cavity Coupled to an RF Source

- From transmission line equations, forward power from RF source

$$|V_+|^2 / 2Z_0$$

- Reflected power to circulator

$$|V_-|^2 / 2Z_0$$

- Transformer relations

$$V_c = kV_k = k\left(V_+ + V_-\right)$$

$$i_c = i_k / k = \left(I_+ - I_-\right)/k = 2I_+ / k - V_c / \left(k^2 Z_0\right)$$

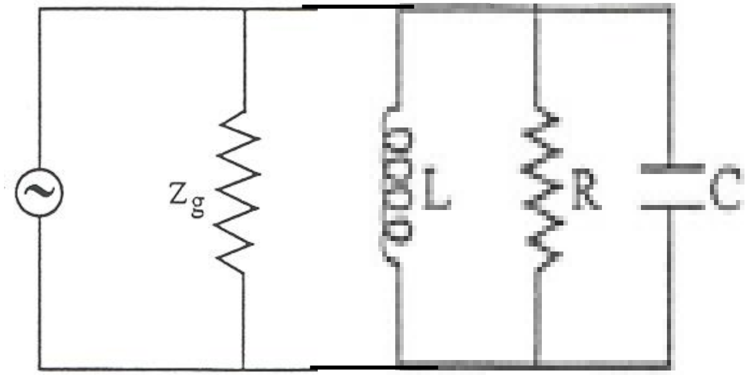
- Considering zero forward power case and definition of β

$$\beta = \left(|V_-|^2 / 2Z_0\right) / \left(|V_c|^2 / 2R\right) = R / \left(k^2 Z_0\right)$$

Cavity Coupled to an RF Source

- Loaded cavity looks like

$$i_g(t) = \frac{2I_+(t)}{k} = \mathrm{Re}\left(\tilde{i}_g e^{i\omega t}\right)$$



Wiedemann Fig. 16.1

$$\beta \equiv \frac{R}{Z_g} = \frac{R}{k^2 Z_0} \quad \therefore \quad Z_g = \frac{R}{\beta}$$

Wiedemann 16.1

- Effective and loaded resistance

$$\frac{1}{R_{eff}} = \frac{1}{R} + \frac{1}{Z_g} = \frac{1+\beta}{R} \qquad R_L = 2R_{eff} = \frac{R_a}{1+\beta}$$

- Solving transmission line equations

$$V_+ = \frac{1}{2}\left(\frac{V_c}{k} + kZ_0 i_c\right) \qquad V_- = \frac{1}{2}\left(\frac{V_c}{k} - kZ_0 i_c\right)$$

Powers Calculated

- Forward Power

$$P_g = \frac{V_c^2}{8Z_0k^2}\left|1+\frac{1}{\beta}\left[1+iQ_0\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\right]\right|^2 = \frac{V_c^2}{R_a}\frac{(1+\beta)^2}{4\beta}\left(1+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right)$$

- Reflected Power

$$P_{refl} = \frac{V_c^2}{8Z_0k^2}\left|1-\frac{1}{\beta}\left[1+iQ_0\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\right]\right|^2 = \frac{V_c^2}{R_a}\frac{(1+\beta)^2}{4\beta}\left(\frac{(\beta-1)^2}{(1+\beta)^2}+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right)$$

- Power delivered to cavity is

$$P_g - P_{refl} = \frac{V_c^2}{R_a}\frac{(1+\beta)^2}{4\beta}\left[1-\frac{(\beta-1)^2}{(1+\beta)^2}\right] = \frac{V_c^2}{R_a} = P_{diss}$$

as it must by energy conservation!

Some Useful Expressions

- Total energy W, in terms of cavity parameters

$$\frac{W}{P_g} = \frac{\frac{Q_0}{\omega_0} P_{diss}}{P_{diss}} \frac{4\beta}{(1+\beta)^2} \frac{1}{1+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}$$

$$\therefore \quad W = 4\beta \frac{Q_0}{\omega_0} \frac{1}{(1+\beta)^2 + Q_0^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2} P_g$$

$$\omega \simeq \omega_0 \quad \Rightarrow W \simeq \frac{4\beta}{(1+\beta)^2} \frac{Q_0}{\omega_0} \frac{1}{1+\left[2Q_L \frac{\omega-\omega_0}{\omega_0}\right]^2} P_g$$

- Total impedance

$$Z_{TOT} = \left[\frac{1}{Z_g} + \frac{1}{Z}\right]^{-1}$$

$$Z_{TOT} = \frac{R_a}{2}\left[(1+\beta) + iQ_0\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\right]^{-1}$$

When Cavity is Detuned

- Define "Tuning angle" Ψ:

$$\tan\Psi \equiv -Q_L\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\approx -2Q_L\frac{\omega-\omega_0}{\omega_0}\quad \text{for}\quad \omega\approx\omega_0$$

$$\therefore$$

$$W=\frac{4\beta}{(1+\beta)^2}\frac{Q_0}{\omega_0}\frac{1}{1+\tan^2\Psi}P_g$$

Wiedemann 16.12

- Note that:

$$P_{diss}=\frac{4\beta}{(1+\beta)^2}\frac{1}{1+\tan^2\Psi}P_g$$

Optimal β Without Beam

- Optimal coupling: W/P_g maximum or $P_{diss} = P_g$
  which implies for $\Delta\omega = 0$, $\beta = 1$
  This is the case called "critical" coupling
- Reflected power is consistent with energy conservation:

$$P_{refl} = P_g - P_{diss}$$

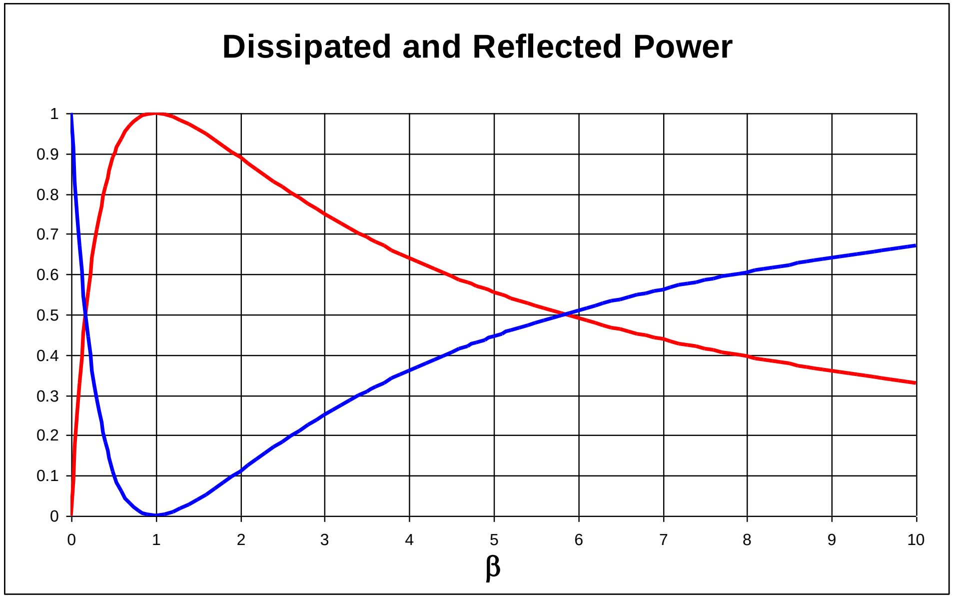
$$P_{refl} = P_g \left[1 - \frac{4\beta}{(1+\beta)^2} \frac{1}{1+\tan^2 \Psi} \right]$$

- On resonance:

$$W = \frac{4\beta}{(1+\beta)^2} \frac{Q_0}{\omega_0} P_g$$

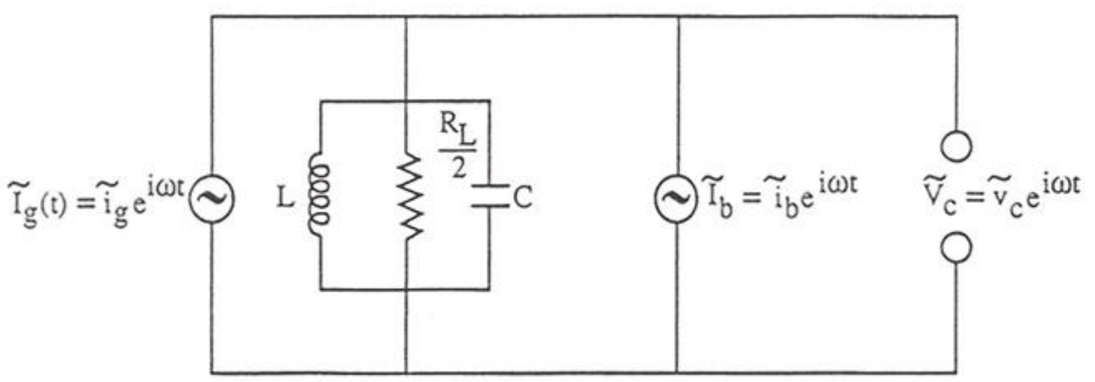
$$P_{diss} = \frac{4\beta}{(1+\beta)^2} P_g$$

$$P_{refl} = \left(\frac{1-\beta}{1+\beta} \right)^2 P_g$$



Equivalent Circuit: Cavity with Beam

- Beam through the RF cavity is represented by a current generator that interacts with the total impedance (including circulator).
- Equivalent circuit:



$$i_C = C\frac{dV_c}{dt}, \quad i_R = \frac{V_c}{R_L/2}, \quad V_c = L\frac{di_L}{dt}$$

i_g the current induced by generator, i_b beam current

- Differential equation that describes the dynamics of the system:

$$\frac{d^2V_c}{dt^2} + \frac{\omega_0}{Q_L}\frac{dV_c}{dt} + \omega_0^2 V_c = \frac{\omega_0 R_L}{2Q_L}\frac{d}{dt}\left(i_g - i_b\right)$$

Cavity with Beam

- Kirchoff's law:

$$i_L + i_R + i_C = i_g - i_b$$

- Total current is a superposition of generator current and beam current and beam current opposes the generator current.

- Assume that voltages and currents are represented by complex phasors

$$V_c(t) = \mathrm{Re}\left(\tilde{V}_c e^{i\omega t}\right)$$
$$i_g(t) = \mathrm{Re}\left(\tilde{i}_g e^{i\omega t}\right)$$
$$i_b(t) = \mathrm{Re}\left(\tilde{i}_b e^{i\omega t}\right)$$

where ω is the generator angular frequency and $\tilde{V}_c, \tilde{i}_g, \tilde{i}_b$ are complex quantities.

Voltage for a Cavity with Beam

- Steady state solution

$$(1 - i\tan\Psi)\tilde{V}_c = \frac{R_L}{2}(\tilde{i}_g - \tilde{i}_b)$$

where Ψ is the tuning angle.

- Generator current

$$|\tilde{i}_g| = 2I^+ = \frac{2}{k}\sqrt{\frac{2P_g}{Z_0}} = 2\sqrt{\beta}\sqrt{\frac{2P_g}{R}} = 4\sqrt{\beta}\sqrt{\frac{P_g}{R_a}}$$

- For short bunches: $|\tilde{i}_b| \approx 2I_0$ where I_0 is the average beam current.

Wiedemann 16.19

Voltage for a Cavity with Beam

- At steady-state: $$\tilde{V}_c = \frac{R_L/2}{(1-i\tan\Psi)}\tilde{i}_g - \frac{R_L/2}{(1-i\tan\Psi)}\tilde{i}_b$$

or $$\tilde{V}_c = \frac{R_L}{2}\tilde{i}_g \cos\Psi e^{i\Psi} - \frac{R_L}{2}\tilde{i}_b \cos\Psi e^{i\Psi}$$

or $$\tilde{V}_c = \boxed{\tilde{V}_{gr}\cos\Psi e^{i\Psi}} + \boxed{\tilde{V}_{br}\cos\Psi e^{i\Psi}}$$

or $$\tilde{V}_c = \tilde{V}_g + \tilde{V}_b$$

$$\left\{ \begin{array}{l} \tilde{V}_{gr} = \frac{R_L}{2}\tilde{i}_g \\ \tilde{V}_{br} = -\frac{R_L}{2}\tilde{i}_b \end{array} \right\}$$ are the generator and beam-loading voltages on resonance

and $$\left\{ \begin{array}{l} \tilde{V}_g \\ \tilde{V}_b \end{array} \right\}$$ are the generator and beam-loading voltages.

Voltage for a Cavity with Beam

- Note that:

$$|\tilde{V}_{gr}| = \frac{2\sqrt{\beta}}{\sqrt{1+\beta}}\sqrt{P_g R_L} \approx 2\sqrt{P_g R_L} \quad \text{for large } \beta$$

Wiedemann 16.16

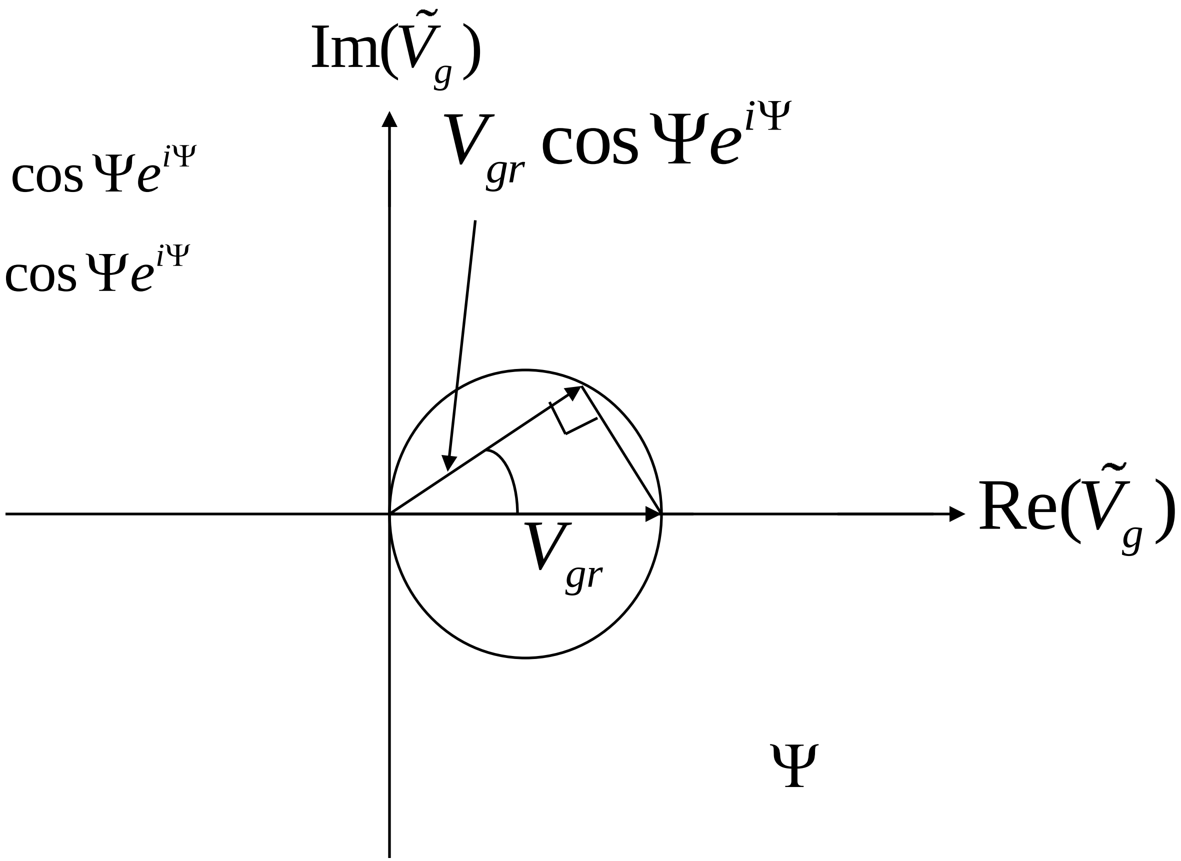
$$|\tilde{V}_{br}| = R_L I_0$$

Wiedemann 16.20

Voltage for a Cavity with Beam

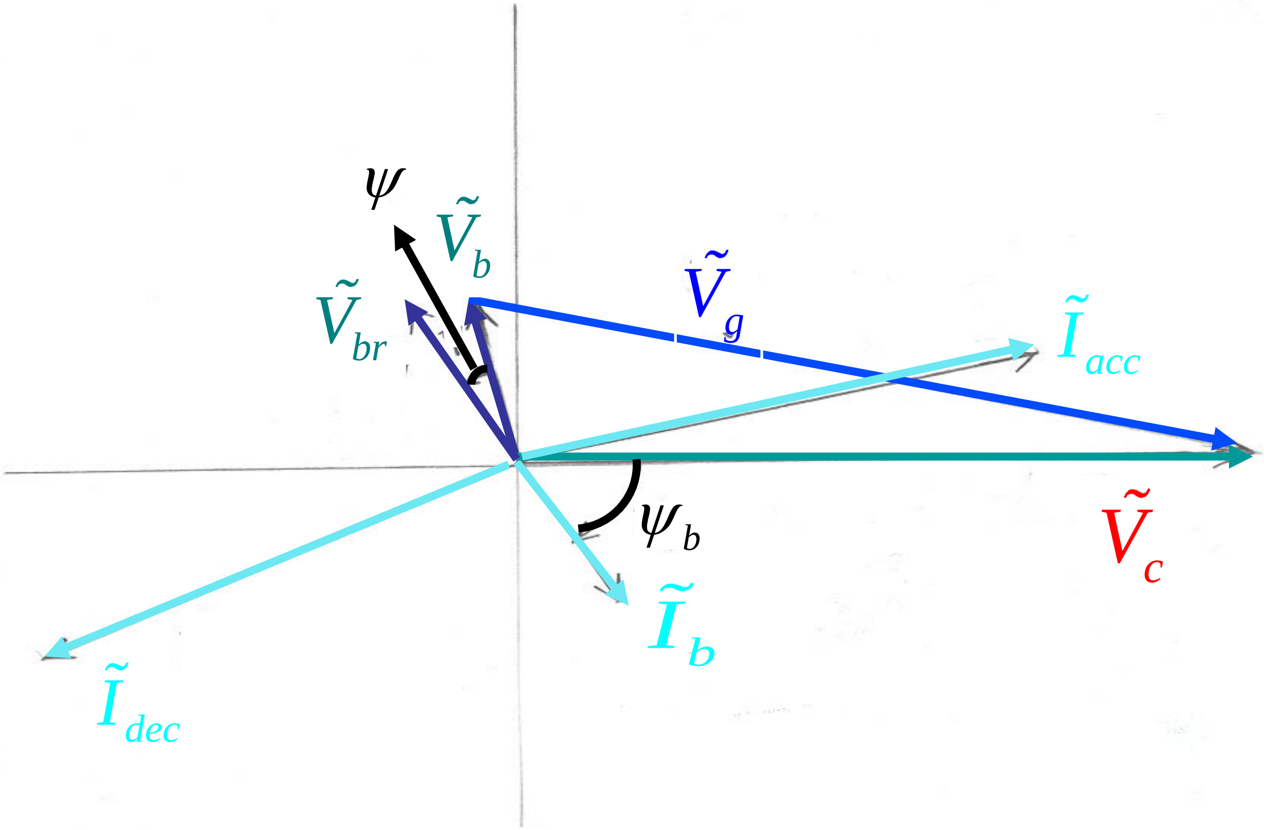
$$\tilde{V}_g = \tilde{V}_{gr} \cos \Psi e^{i\Psi}$$

$$\tilde{V}_b = \tilde{V}_{br} \cos \Psi e^{i\Psi}$$



As Ψ increases, the magnitudes of both V_g and V_b decrease while their phases rotate by Ψ.

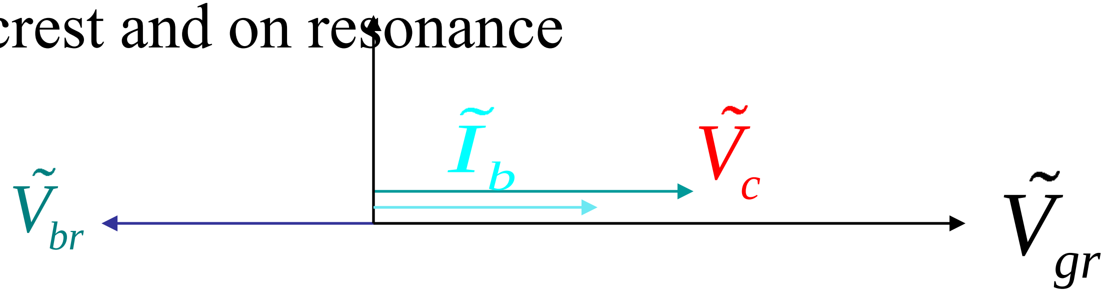
Example of a Phasor Diagram



Wiedemann
Fig. 16.3

On Crest/On Resonance Operation

- Typically linacs operate on resonance and on crest in order to receive maximum acceleration.
- On crest and on resonance



$$\Rightarrow \quad V_c = V_{gr} - V_{br}$$

where V_c is the accelerating voltage.

More Useful Equations

- We derive expressions for W, V_a, P_{diss}, P_{refl} in terms of β and the loading parameter K, defined by: $K=I_0\sqrt{R_a}/(2\sqrt{P_g})$

From:

$$|\tilde{V}_{gr}| = \frac{2\sqrt{\beta}}{\sqrt{1+\beta}}\sqrt{P_g R_L}$$

$$|\tilde{V}_{br}| = R_L I_0$$

$$V_c = V_{gr} - V_{br}$$

$$\Rightarrow$$

$$V_c = \sqrt{P_g R_L}\left\{\frac{2\sqrt{\beta}}{\sqrt{1+\beta}}\left(1-\frac{K}{\sqrt{\beta}}\right)\right\}$$

$$W = \frac{4\beta}{(1+\beta)^2}\frac{Q_0}{\omega_0}\left(1-\frac{K}{\sqrt{\beta}}\right)^2 P_g$$

$$P_{diss} = \frac{4\beta}{(1+\beta)^2}\left(1-\frac{K}{\sqrt{\beta}}\right)^2 P_g$$

$$I_0 V_a = I_0\sqrt{R_a P_{diss}}$$

$$\eta \equiv \frac{I_0 V_c}{P_g} = \frac{2\sqrt{\beta}}{\sqrt{1+\beta}} 2K\left(1-\frac{K}{\sqrt{\beta}}\right)$$

$$P_{refl} = P_g - P_{diss} - I_0 V_a \quad \Rightarrow \quad P_{refl} = \frac{\left[(\beta-1)-2K\sqrt{\beta}\right]^2}{(\beta+1)^2} P_g$$

More Useful Equations

- For β large,

$$P_g \simeq \frac{1}{4R_L}(V_c + I_0 R_L)^2$$

$$P_{refl} \simeq \frac{1}{4R_L}(V_c - I_0 R_L)^2$$

- For P_{refl}=0 (condition for matching) $\Rightarrow$

$$R_L = \frac{V_c^M}{I_0^M}$$

and

$$P_g \simeq \frac{I_0^M V_c^M}{4}\left(\frac{V_c}{V_c^M} + \frac{I_0}{I_0^M}\right)^2$$

Example

- For V_c=14 MV, L=0.7 m, Q_L=2x10^7 , Q_0=1x10^{10} :

Power	I_0 = 0	I_0 = 100 μA	I_0 = 1 mA
P_g	3.65 kW	4.38 kW	14.033 kW
P_{diss}	29 W	29 W	29 W
I_0V_c	0 W	1.4 kW	14 kW
P_{refl}	3.62 kW	2.951 kW	~ 4.4 W

Off Crest/Off Resonance Operation

- Typically electron storage rings operate off crest in order to ensure stability against phase oscillations.
- As a consequence, the rf cavities must be detuned off resonance in order to minimize the reflected power and the required generator power.
- Longitudinal gymnastics may also impose off crest operation in recirculating linacs.
- We write the beam current and the cavity voltage as

$$\tilde{I}_b = 2I_0 e^{i\psi_b}$$

$$\tilde{V}_c = V_c e^{i\psi_c} \quad \text{and set } \psi_c = 0$$

- The generator power can then be expressed as:

$$P_g = \frac{V_c^2}{R_L}\frac{(1+\beta)}{4\beta}\left\{\left[1+\frac{I_0 R_L}{V_c}\cos\psi_b\right]^2+\left[\tan\Psi-\frac{I_0 R_L}{V_c}\sin\psi_b\right]^2\right\}$$

Wiedemann 16.31

Off Crest/Off Resonance Operation

- Condition for optimum tuning:

$$\tan\Psi = \frac{I_0 R_L}{V_c}\sin\psi_b$$

- Condition for optimum coupling:

$$\beta_{\text{opt}} = 1 + \frac{I_0 R_a}{V_c}\cos\psi_b$$

- Minimum generator power:

$$P_{g,\min} = \frac{V_c^2 \beta_{\text{opt}}}{R_a}$$

Wiedemann 16.36