Physics 319
Classical Mechanics

G. A. Krafft
Old Dominion University
Jefferson Lab

Lecture 16

Elliptical Motion

- First determine the constant Laplace-Runge-Lenz vector

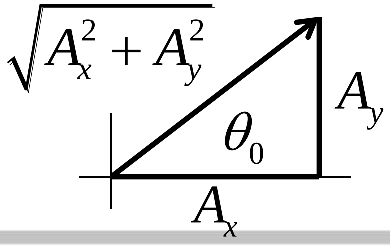
$$\frac{d}{dt}\cos\theta = -\dot{\theta}\sin\theta = -\frac{l}{\mu r^2}\frac{y}{r} = \frac{l}{Gm_1m_2}\frac{d^2}{dt^2}r\sin\theta$$

$$\frac{d}{dt}\sin\theta = \dot{\theta}\cos\theta = \frac{l}{\mu r^2}\frac{x}{r} = -\frac{l}{Gm_1m_2}\frac{d^2}{dt^2}r\cos\theta$$

$$-A_x = \cos\theta - \frac{l}{Gm_1m_2}\frac{d}{dt}\left(r\sin\theta\right) = \cos\theta - \frac{l}{Gm_1m_2}\left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)$$

$$-A_y = \sin\theta + \frac{l}{Gm_1m_2}\frac{d}{dt}\left(r\cos\theta\right) = \sin\theta + \frac{l}{Gm_1m_2}\left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)$$

$$A_x\hat{x} + A_y\hat{y} = \sqrt{A_x^2 + A_y^2}\cos\theta_0\hat{x} + \sqrt{A_x^2 + A_y^2}\sin\theta_0\hat{y}$$



Orbit as function of θ

- Determine the vector magnitude

$$A_x^2 + A_y^2 = 1 - 2\frac{l}{Gm_1m_2} r\dot{\theta} + \frac{l^2}{(Gm_1m_2)^2} r^2\dot{\theta}^2 + \frac{l^2}{(Gm_1m_2)^2}\dot{r}^2 = 1 + \frac{2El^2}{(Gm_1m_2)^2\mu}$$

- Equation for radius $r(\theta)$

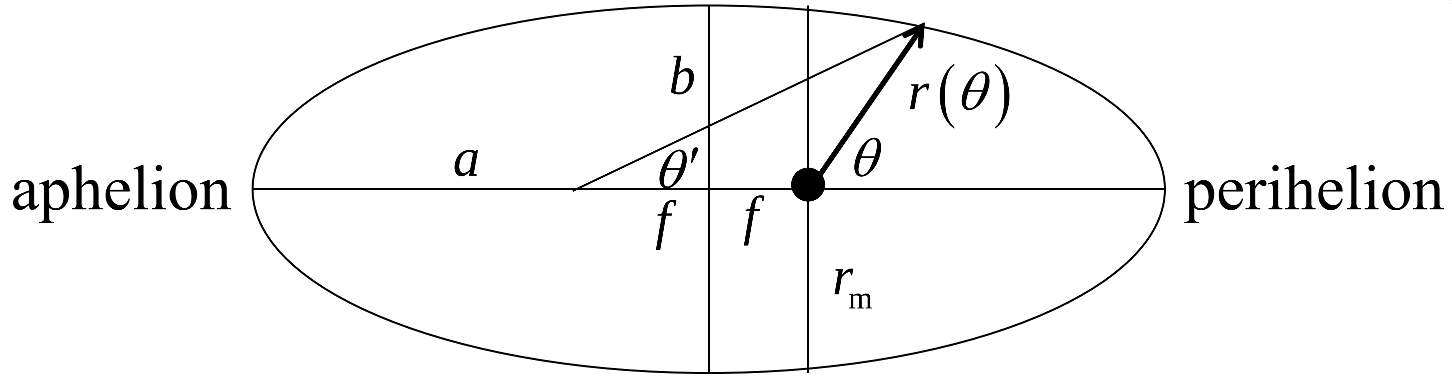
$$\vec{L} = l\hat{z} = \mu r^2\dot{\theta}\hat{z}$$

$$r\cos\theta A_x + r\sin\theta A_y = -r + \frac{l^2}{Gm_1m_2\mu}$$

$$= r\sqrt{A_x^2 + A_y^2}\cos(\theta - \theta_0)$$

$$\therefore r(\theta) = \frac{\left(l^2/(Gm_1m_2\mu)\right)}{1 + \sqrt{1 + \frac{2El^2}{(Gm_1m_2)^2\mu}}\cos(\theta - \theta_0)} = \frac{r_{\mathrm{m}}}{1 + \sqrt{1 - (E/E_{\mathrm{min}})}\cos(\theta - \theta_0)}$$

Parametric Equation for Ellipse



- Sum of the lengths to the two foci constant $2L$

$$r\cos\theta + 2f = (2L - r)\cos\theta'$$

$$r\sin\theta = (2L - r)\sin\theta'$$

$$r^2 + 4r\cos\theta f + 4f^2 = 4L^2 - 4Lr + r^2$$

$$r(\theta) = \frac{L^2 - f^2}{L + f\cos\theta} = \frac{b^2 / L}{1 + (f/L)\cos\theta}$$

$$r_{max} = L + f \qquad r_{min} = L - f \qquad L = a$$

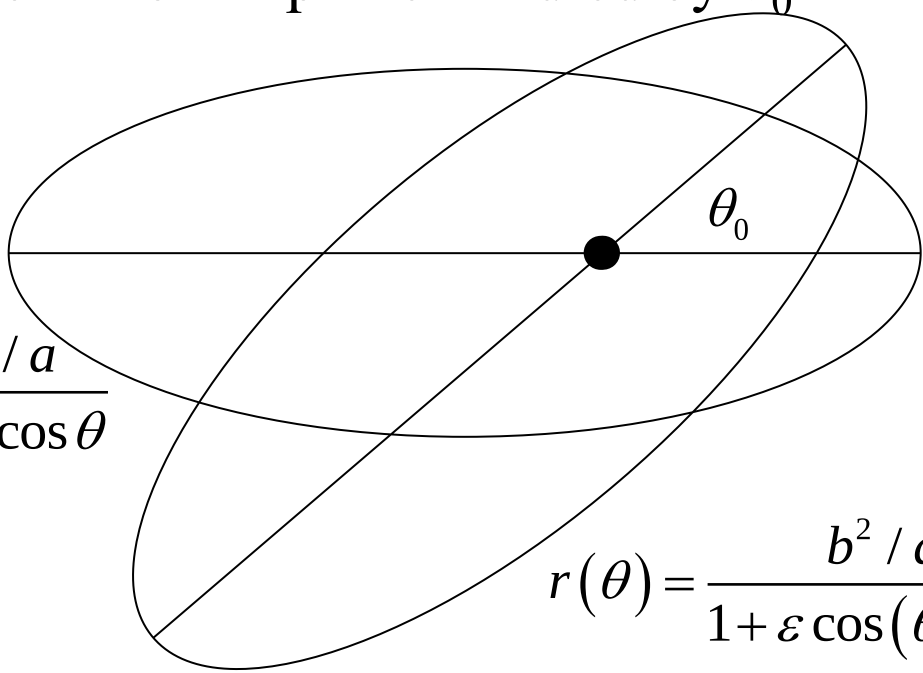
- Eccentricity is $\varepsilon = f/L = \sqrt{1 - b^2/a^2}$

More on Ellipse

- In Homework show using Cartesian coordinates centered on dot get the ellipse equation (Kepler's First Law)

$$\frac{(x+f)^2}{a^2}+\frac{y^2}{b^2}=1 \qquad f=a\varepsilon$$

- Ellipse orientation in plane handled by θ_0



$$r(\theta)=\frac{b^2/a}{1+\varepsilon\cos\theta}$$

$$r(\theta)=\frac{b^2/a}{1+\varepsilon\cos(\theta-\theta_0)}$$

Method Two

- Most common analysis, integrate the energy function

$$\frac{dr}{d\theta}=\frac{dr}{dt}\frac{dt}{d\theta}=\sqrt{2E/\mu-2U(r)/\mu-l^2/\mu^2r^2}\,\frac{\mu r^2}{l}$$

$$\frac{dr}{r^2\sqrt{2E\mu/l^2-2\mu U(r)/l^2-1/r^2}}=d\theta$$

$$\int\frac{dr}{r^2\sqrt{2E\mu/l^2-2\mu U(r)/l^2-1/r^2}}=\theta-\theta_0$$

- Check: 2D harmonic oscillator

$$U(r)=kr^2/2$$

$$\int\frac{dr}{r^2\sqrt{2E\mu/l^2-\mu kr^2/l^2-1/r^2}}=\theta-\theta_0$$

$$x=1/r^2-E\mu/l^2\qquad dx=-2dr/r^3$$

Linear Restoring Force

$$-\int \frac{dx}{\sqrt{\left(E\mu/l^2\right)^2 - \mu k/l^2 - x^2}} = 2\left(\theta-\theta_0\right)$$

$$\cos^{-1}\frac{1/r^2 - E\mu/l^2}{\sqrt{E^2\mu^2/l^4 - \mu k/l^2}} = 2\left(\theta-\theta_0\right)$$

$$\frac{1}{r^2} - E\mu/l^2 = \sqrt{E^2\mu^2/l^4 - \mu k/l^2}\cos 2\left(\theta-\theta_0\right)$$

$$E\mu/l^2\left(r^2\cos^2\left(\theta-\theta_0\right) + r^2\sin^2\left(\theta-\theta_0\right)\right)$$

$$+\sqrt{E^2\mu^2/l^4 - \mu k/l^2}\left(r^2\cos^2\left(\theta-\theta_0\right) - r^2\sin^2\left(\theta-\theta_0\right)\right) = 1$$

- Equation for ellipse centered at origin

semi-major axis

$$1/\left(E\mu/l^2 - \sqrt{E^2\mu^2/l^4 - \mu k/l^2}\right)^{1/2}$$

semi-minor axis

$$1/\left(E\mu/l^2 + \sqrt{E^2\mu^2/l^4 - \mu k/l^2}\right)^{1/2}$$

Period

- Use area result. Total area of orbit ellipse

$$\pi ab = \frac{\pi}{\left(E\mu/l^2 - \sqrt{E^2\mu^2/l^4 - \mu k/l^2}\right)^{1/2}\left(E\mu/l^2 + \sqrt{E^2\mu^2/l^4 - \mu k/l^2}\right)^{1/2}}$$

- Time derivative of area swept out by radius vector

$$\frac{dA}{dt} = \frac{l}{2\mu}$$

- Period

$$T = \frac{\pi ab}{l/2\mu} = \frac{2\pi\mu}{\sqrt{\mu k/l^2}\,l} = \frac{2\pi}{\sqrt{k/\mu}}$$

- Works!

$$\omega = \sqrt{\frac{k}{\mu}}$$

Kepler Problem

- Case of gravitational attraction

$$U(r) = -Gm_1m_2/r$$

$$\int \frac{dr}{r^2\sqrt{2E\mu/l^2 + 2Gm_1m_2\mu/rl^2 - 1/r^2}} = \theta - \theta_0$$

$$x = 1/r - Gm_1m_2\mu/l^2 \qquad dx = -dr/r^2$$

$$-\int \frac{dx}{\sqrt{2E\mu/l^2 + \left(Gm_1m_2\mu/l^2\right)^2 - x^2}} = \theta - \theta_0$$

$$\frac{1/r - Gm_1m_2\mu/l^2}{\sqrt{2E\mu/l^2 + \left(Gm_1m_2\mu/l^2\right)^2}} = \cos(\theta - \theta_0)$$

$$r(\theta) = \frac{l^2/(Gm_1m_2\mu)}{1 + \sqrt{1 + \frac{2El^2}{\mu(Gm_1m_2)^2}}\cos(\theta - \theta_0)} = \frac{r_{\mathrm{m}}}{1 + \sqrt{1 - (E/E_{\mathrm{min}})}\cos(\theta - \theta_0)}$$

Method Three (Taylor)

- Write a differential equation for $u = 1/r$

$$\mu\ddot{r} = -\frac{\partial U}{\partial r} + \frac{l^2}{\mu r^3} \qquad r = r\left(\theta(t)\right), \frac{dr}{dt}\left(\theta(t)\right) = \frac{dr}{d\theta}\dot{\theta} = \frac{l}{\mu r^2}\frac{dr}{d\theta}$$

$$u = \frac{1}{r} \rightarrow \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{\mu}{l}\dot{r}$$

$$\frac{d^2u}{d\theta^2}\dot{\theta} = -\frac{\mu}{l}\ddot{r} = -\frac{1}{l}\left[-\frac{\partial U}{\partial r} + \frac{l^2}{\mu r^3}\right]$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{l^2u^2}\frac{\partial U}{\partial r}\left(r = 1/u\right)$$

- For Newton gravity

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu G m_1 m_2}{l^2}$$

- Not so "attractive" because need to write a differential equation for $1/r^2$ for linear restoring force

Relations Among Parameters

- For gravitational attraction and bound orbit

$$r(\theta) = \frac{r_{\mathrm{m}}}{1 + \sqrt{1 - (E/E_{\min})}\cos(\theta - \theta_0)}$$

$$\varepsilon = \sqrt{1 - (E/E_{\min})} \qquad (E/E_{\min} > 0)$$

$$a = \frac{r_{\mathrm{m}}}{1-\varepsilon^2} = \frac{l^2/(Gm_1 m_2 \mu)}{1-\varepsilon^2} = \frac{Gm_1 m_2}{2(-E)}$$

$$\frac{l^2}{\mu} = a(1-\varepsilon^2) G m_1 m_2$$

$$b = \frac{r_{\mathrm{m}}}{\sqrt{1-\varepsilon^2}}$$

Kepler's Third Law

- Revolution period T is

$$T = \frac{2\pi ab\mu}{l} \rightarrow T^2 = \frac{4\pi^2 a^4 \left(1-\varepsilon^2\right)\mu^2}{l^2}$$

$$\frac{l^2}{\mu} = a\left(1-\varepsilon^2\right)Gm_1m_2$$

$$T^2 = \frac{4\pi^2 a^3 \mu}{Gm_1m_2} = \frac{4\pi^2 a^3}{G\left(m_1+m_2\right)}$$

- Only power laws for the potential with closed (elliptical) orbits are linear restoring force (kr^2) and gravitation/Coulomb force (k/r)

Unbound Motion

- $E = 0$ means the motion is a parabola with focus at sun. Minimum radius is $r_m/2$
- For $E > 0$, motion is a hyperbola, $\varepsilon > 1$. Orbit asymptotes are

$$\cos\left(\theta_{asymp} - \theta_0\right) = -\frac{1}{\varepsilon}$$

$$\theta_{asymp} = \theta_0 \pm \cos^{-1}\left(-1/\varepsilon\right)$$

$$\pi/2 < \theta_{asymp} - \theta_0 < \pi$$

- Minimum radius

$$r_{\min} = \frac{r_m}{1+\varepsilon}$$

Impact Parameter

- Distance of closest approach for particle without interacting is called the impact parameter b_{im}
- Equation for hyperbola

$$y^2 \frac{\varepsilon^2 - 1}{r_m^2} = \frac{\left(\varepsilon^2 - 1\right)^2}{r_m^2} \left(x - \frac{r_m \varepsilon}{\varepsilon^2 - 1} \right)^2 - 1$$

- Equation for asymptotes

$$y \doteq \pm \sqrt{\left(\varepsilon^2 - 1\right)} \left(\frac{r_m \varepsilon}{\varepsilon^2 - 1} - x \right)$$

- Distance of closest approach and eccentricity

$$b_{im} = \frac{r_m}{\sqrt{\varepsilon^2 - 1}} \qquad \varepsilon = \sqrt{1 + \left(\frac{r_m}{b_{im}} \right)^2} = \sqrt{1 + \left(\frac{2Eb_{im}}{Gm_1 m_2} \right)^2}$$