Accelerator Physics
General Resonance Theory and BBU

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Lecture 16

Chromatic Errors and Correction

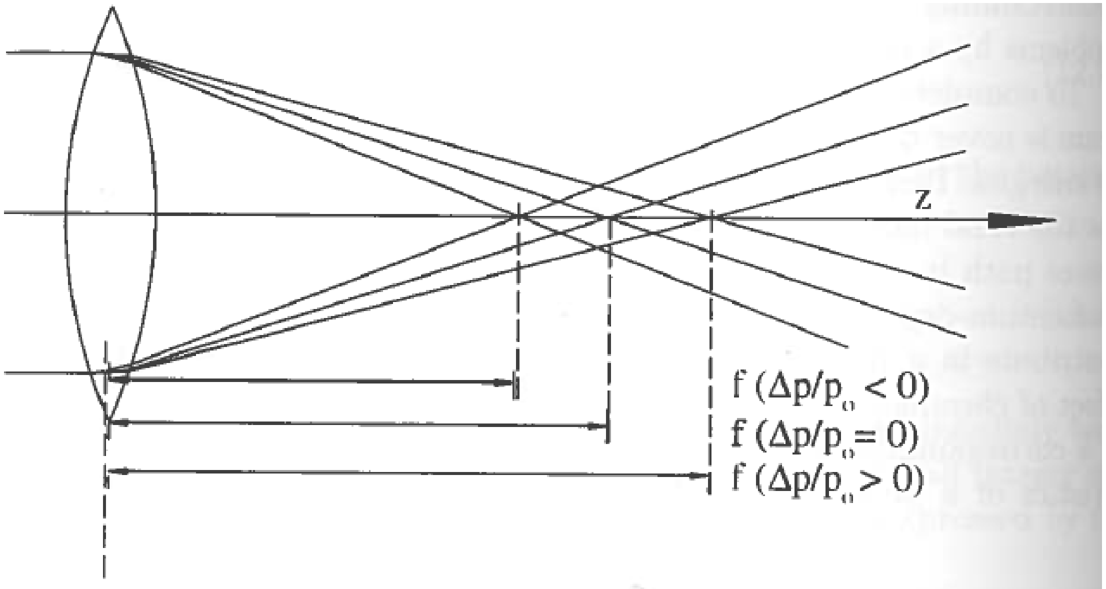


Fig. 12.5. Chromatic focusing errors

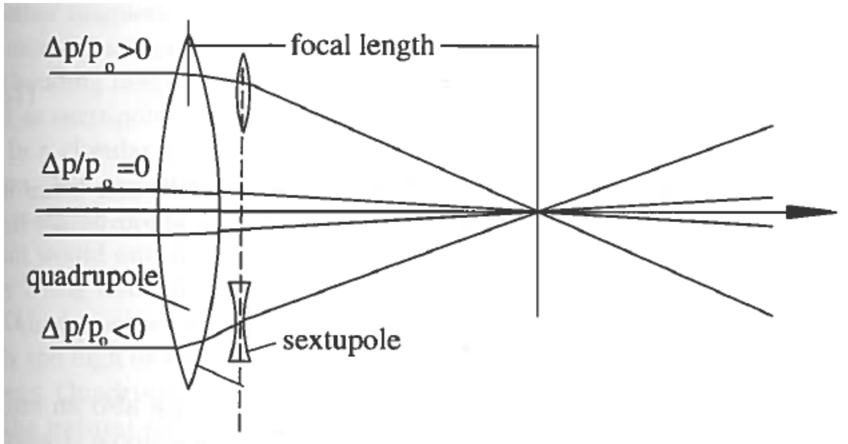


Fig. 12.6. Chromaticity correction with sextupoles

Superperiod Resonances

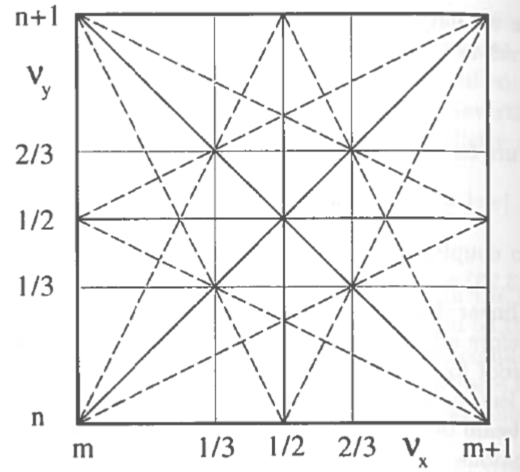


Fig. 13.1. Resonance diagram for a ring with superperiodicity 1, $N = 1$

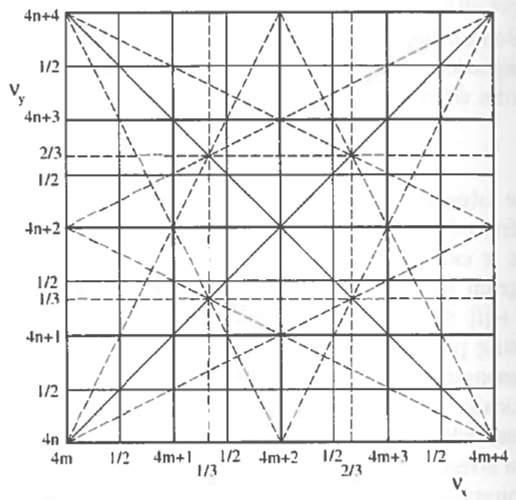


Fig. 13.2. Resonance diagram for a ring with superperiodicity 4, $N = 4$

Action-Angle Variables

Use coordinates in phase space analogous to polar coordinates, but including the non-linearities.

$$J = \frac{1}{2\pi}\oint p\,dq = \frac{1}{2}\frac{\dot{w}^2}{\nu_0} + \frac{1}{2}\nu_0 w^2$$

has units of action. For harmonic oscillator

$$H = \frac{\dot{w}^2}{2} + \frac{\nu_0^2 w^2}{2}$$

$$J = \frac{1}{2}\frac{\dot{w}^2}{\nu_0} + \frac{1}{2}\nu_0 w^2 = H/\nu_0$$

Hamilton equations of motion

$$\frac{dJ}{dt} = -\frac{\partial H}{\partial \psi} = 0 \rightarrow J \text{ is constant}$$

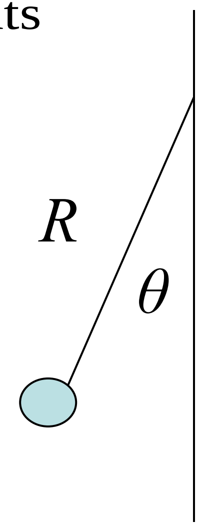
$$\frac{d\psi}{dt} = \frac{\partial H}{\partial J} = \nu_0$$

$$\dot{w} = \sqrt{2J}\sin\psi \qquad \nu_0 w = \sqrt{2J}\cos\psi$$

Real Pendulum

$$H = \frac{p_\theta^2}{2m} + mgR(1-\cos\theta) \approx \frac{p_\theta^2}{2m} + mgR\frac{\theta^2}{2}$$ for small displacements

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m} \qquad \frac{dp_\theta}{dt} = -\frac{\partial H}{\partial \theta} = mgR\sin\theta$$



$\ddot{\theta} + \frac{g}{R}\sin\theta = 0$ as should be

Original phase space coordinates p_θ, θ

Introduce action

$$J = 2\times\frac{1}{2\pi}\int_{-\theta_0}^{\theta_0} p_\theta d\theta = \frac{2\sqrt{2gR}}{\pi} m\int_0^{\theta_0}\sqrt{\cos\theta - \cos\theta_0}\,d\theta$$

where

$$H_0 = mgR(1-\cos\theta_0)$$

Solvable in elliptical integrals

$$J = 2m\sqrt{gR}\left[(1+\cos\theta_0)F\left(\frac{1}{2},\frac{1}{2};1;\frac{1-\cos\theta_0}{2}\right) - 2F\left(-\frac{1}{2},\frac{1}{2};1;\frac{1-\cos\theta_0}{2}\right)\right]$$

key point: frequency now depends on amplitude (action)

Nonlinear Hamiltonian

$$H_w = \frac{1}{2}\dot{w}^2 + \frac{1}{2}\nu_0^2 w^2 + \hat{p}_n \frac{\nu_0^{n/2}}{2^{n/2}} w^n$$

$$\hat{p}_n(\varphi) = -\frac{p_n(\varphi)}{n}\left(\frac{\nu_0}{2}\right)^{-n/2}$$

Introduce action-angle variables in unperturbed Hamiltonian

$$J = \frac{1}{2\pi}\oint pdq = \frac{1}{2}\frac{\dot{w}^2}{\nu_0} + \frac{1}{2}\nu_0 w^2$$

$$w = \sqrt{\frac{2J}{\nu_0}}\cos(\psi - \xi)$$

$$H = \nu_0 J + p_n(\varphi) J^{n/2}\cos^n(\psi - \xi)$$

ψ betatron phase ξ arbitrary phase

Action-Angle Coordinates

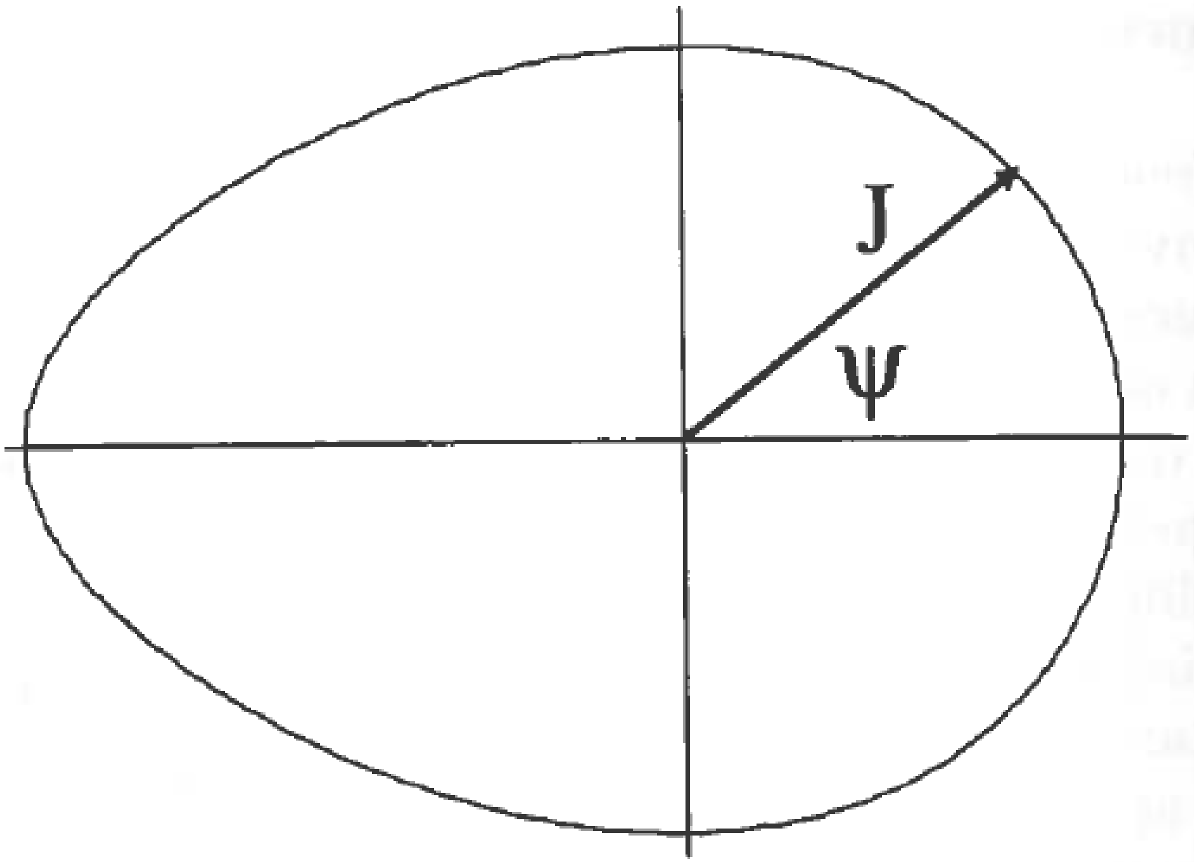


Fig. 13.3. Nonlinear perturbation of phase space motion

Definitions

nth order Perturbed Hamiltonian

$$H_1 = \Delta\nu_r + c_{n0} p_{n0} J_1^{n/2} + \tilde{p}_{nr} J_1^{n/2} \cos(m_r \psi_1)$$

Amplitude Ratio

$$R = \frac{J}{J_0}$$

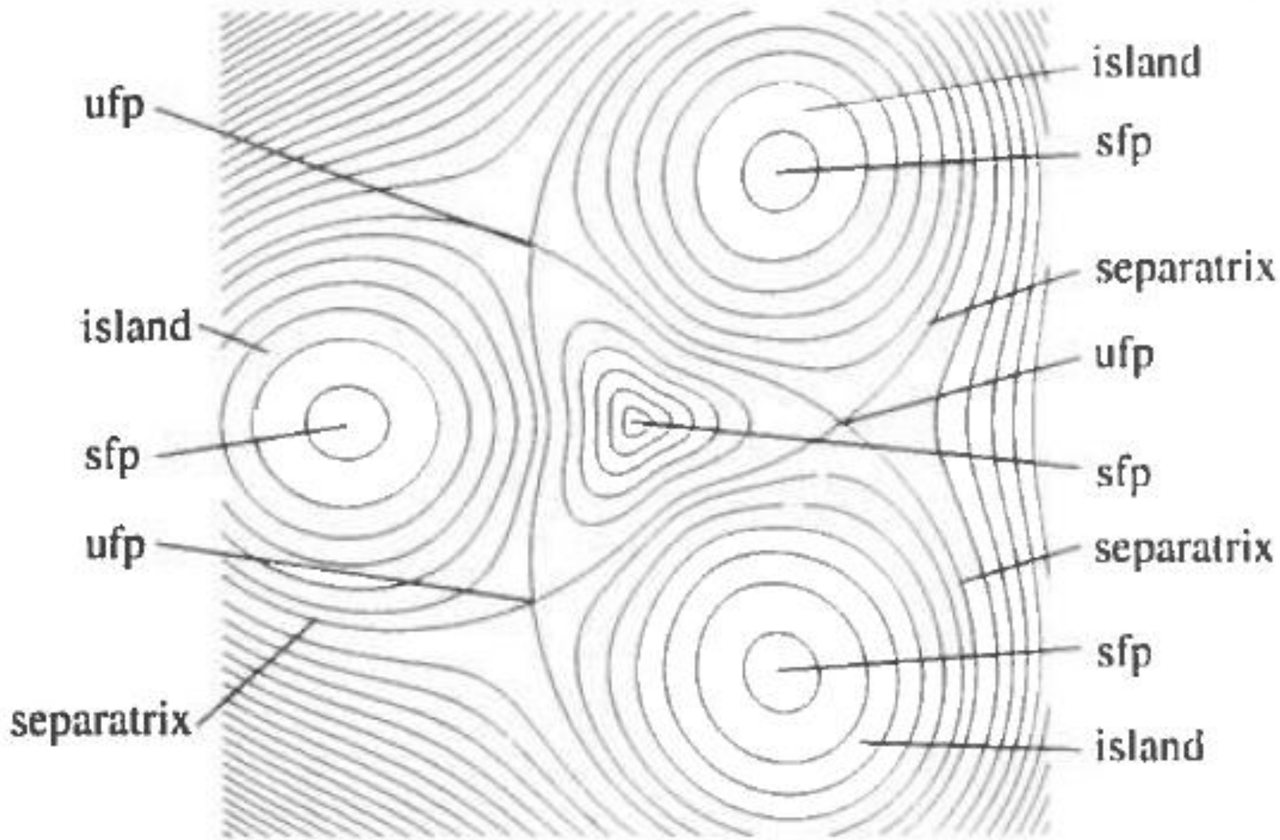
Detuning

$$\Delta = \frac{\Delta\nu_r}{\tilde{p}_{nr} J_0^{n/2-1}} = \frac{\nu_0 - \nu_r}{\tilde{p}_{nr} J_0^{n/2-1}}$$

Tune Spread Parameter

$$\Omega = \frac{c_{n0} p_{n0}}{\tilde{p}_{nr} J_0^{n/2-2}}$$

Phase Space Structure Near Resonance



sfp: stable fixed point
ufp: unstable fixed point

Fig. 13.6. Common features of resonance patterns

Resonance Structure

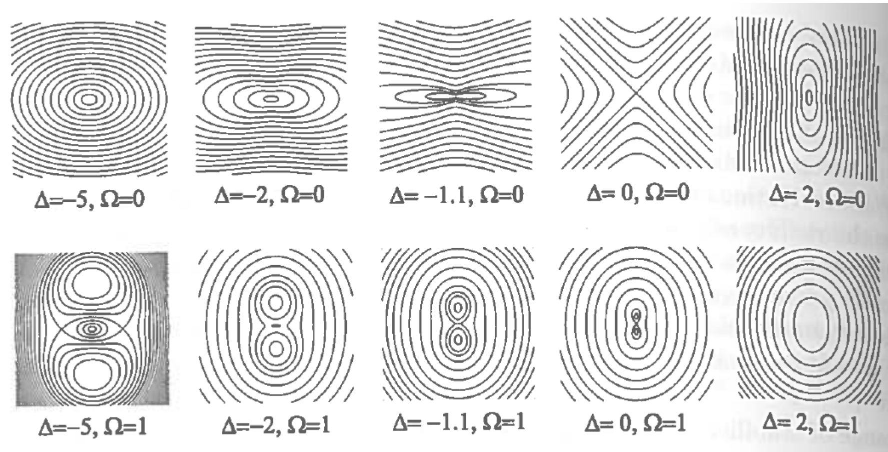


Fig. 13.4. (R, ψ_1) phase-space motion for a half-integer resonance. Top row from left to right: $\Omega = 0$ and $\Delta = (-5, -2, -1.1, 0, 2)$; bottom row: $\Omega = 1$ and $\Delta = (-5, -2, -1.1, 0, 2)$.

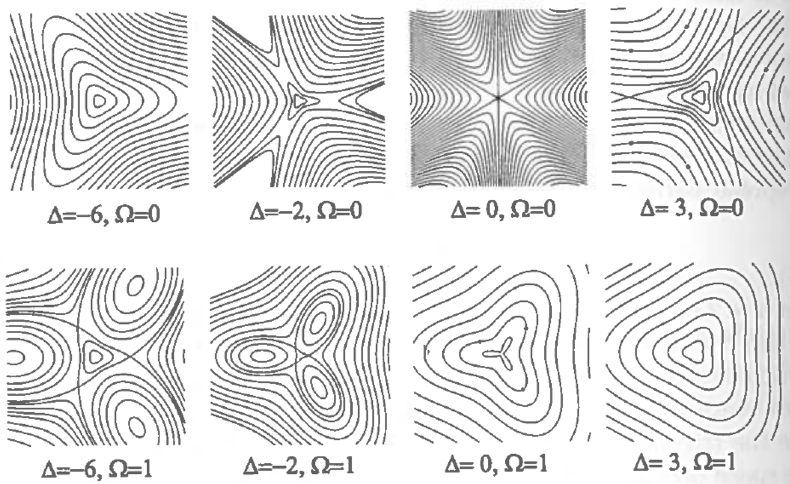


Fig. 13.5. (R, ψ_1) phase space motion for a third-order resonance. Top row from left to right: $\Omega = 0$, $\Delta = (-6, -2, 0, 3)$; bottom row: $\Omega = 1$, $\Delta = (-6, -2, 0, 3)$

Fourth Order Resonances

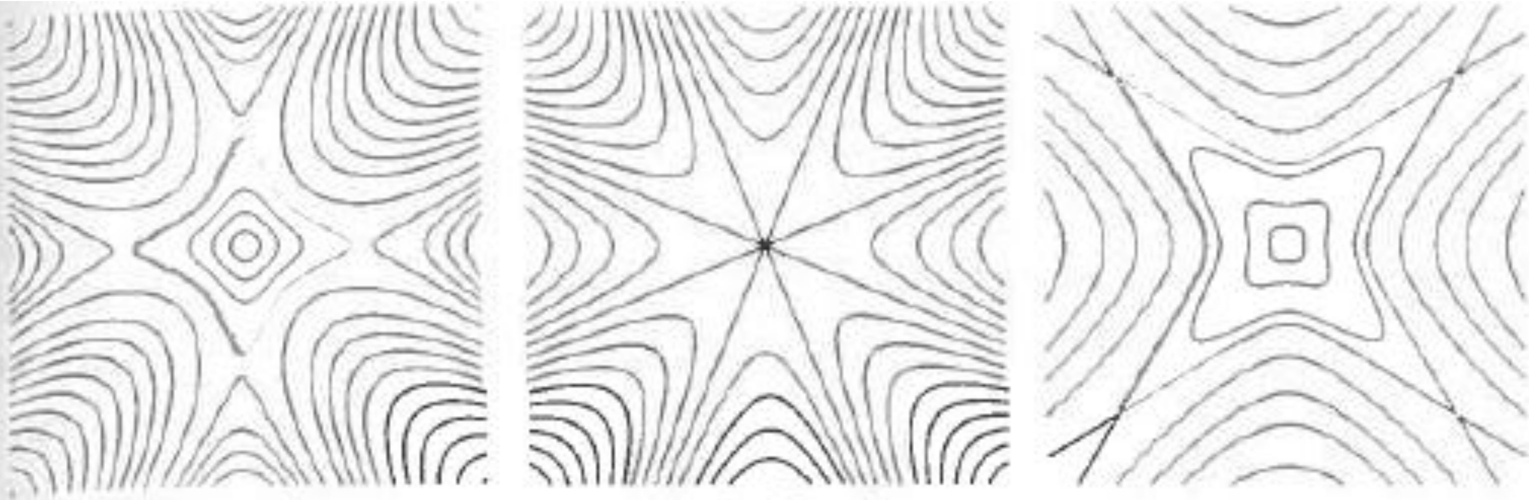


Fig. 13.7. Fourth-order resonance patterns. From left to right: $\Omega = 0$, $(\Delta < 0, \Delta = 0, \Delta > 0)$

Stop Bands

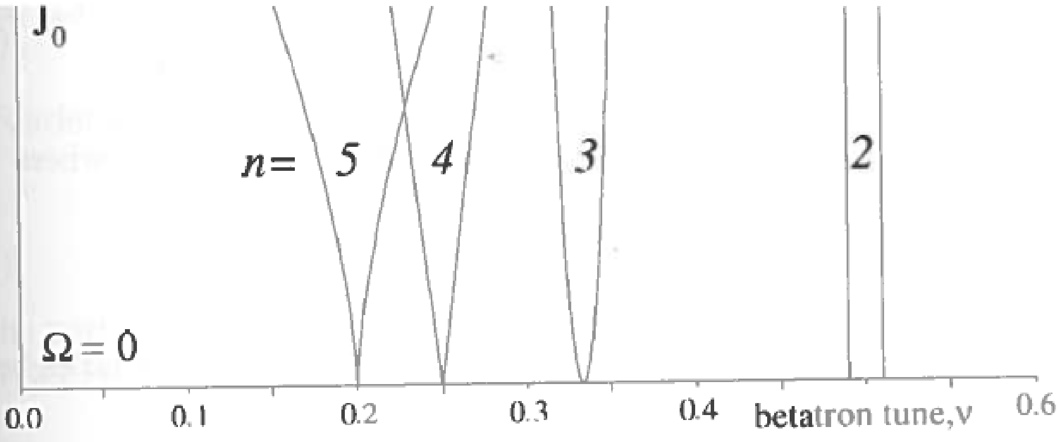


Fig. 13.8. Stop-band width as a function of the amplitude J_0 for resonances of order $n = 2, 3, 4, 5$ and detuning parameter $\Omega = 0$

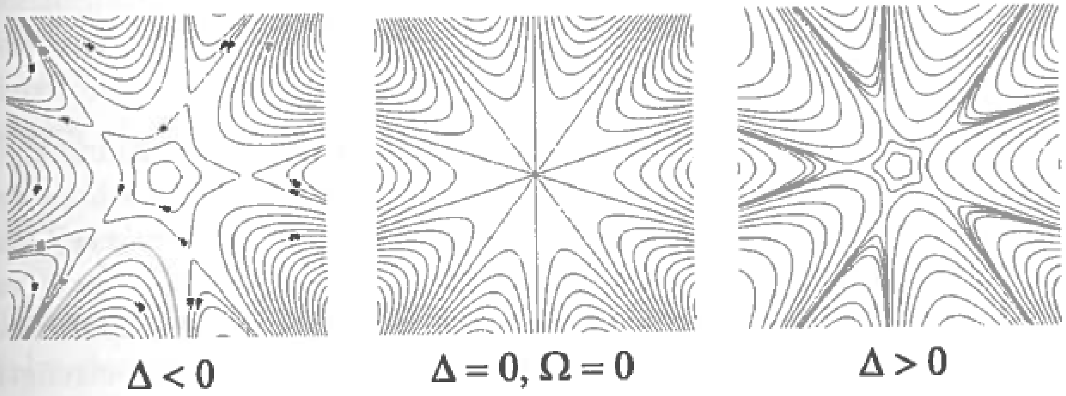


Fig. 13.9. Fifth-order resonance patterns. From left to right: $\Omega = 0$, $(\Delta < 0, \Delta = 0, \Delta > 0)$

Resonant Extraction

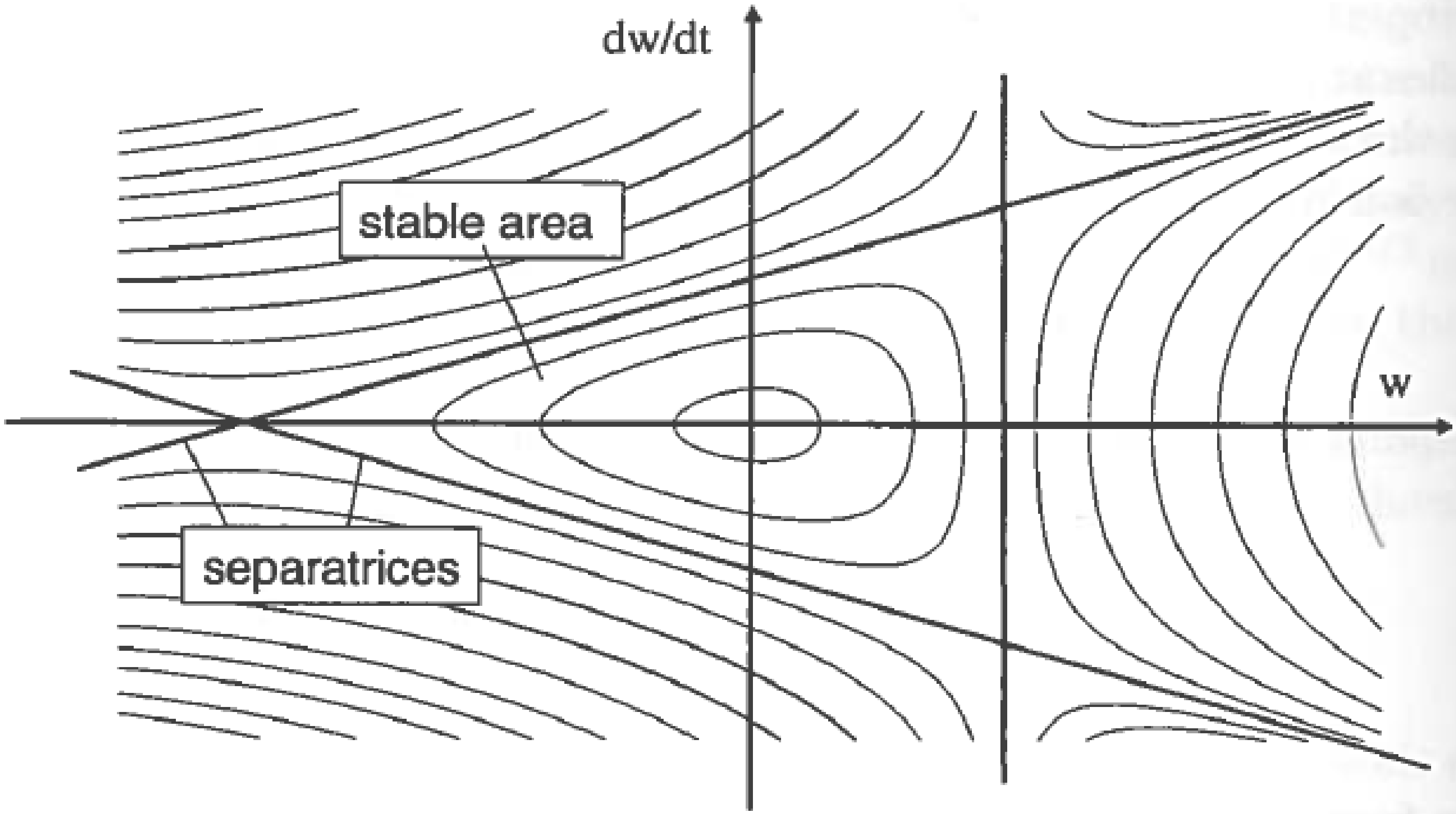
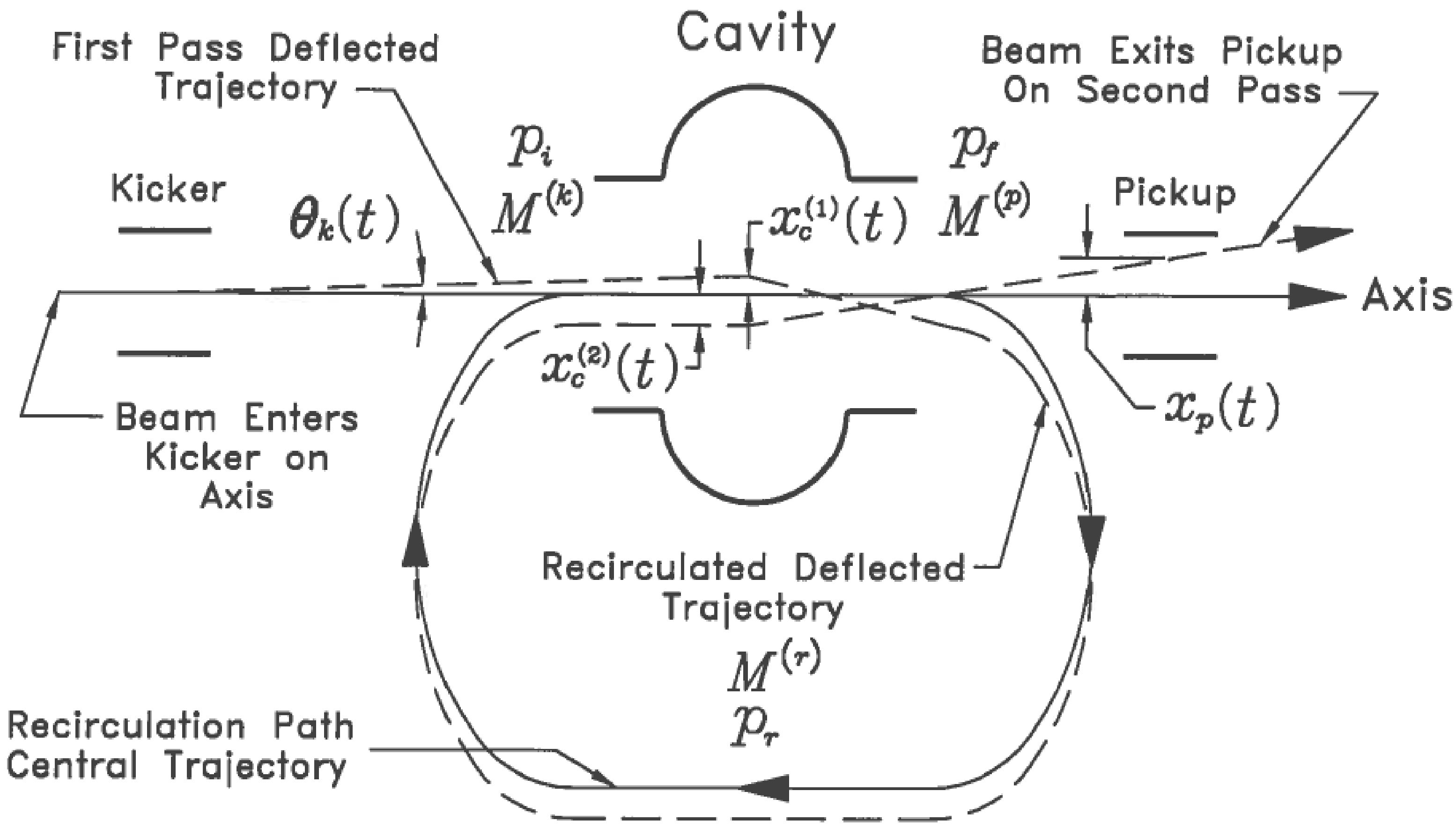


Fig. 13.10. Third-order resonance

Multipass BBU Instability



BBU Theory

following Krafft, Laubach, and Bisognano

$$W_{transverse}(\tau) = \left(\frac{R}{Q}\right)_{HOM} \frac{k_{HOM}\,\omega_{HOM}}{2} e^{-\omega_{HOM}\tau/2Q_{HOM}} \sin\omega_{HOM}\tau$$

Single cavity/Single HOM case

$$V(t) = \int_{-\infty}^{t} W_{transverse}(t-t')I(t')\,d(t')\,dt'$$

On the second pass

$$d(t) = \frac{T_{12}eV(t-t_r)}{c}$$

With no initial displacement

$$V(t) = \frac{T_{12}e}{c}\int_{-\infty}^{t} W_{transverse}(t-t')I(t')V(t'-t_r)\,dt'$$

Delay differential (integral) equation

Beam current

$$I(t) = \sum_{m=-\infty}^{\infty} I_0 t_0 \delta(t - m t_0)$$

Normal mode

$$V(n t_0) = V_0 e^{-i\omega n t_0}$$

Sum the geometric series for eigenvalue equation

$$1 = K e^{i\omega t_r} \frac{e^{i\omega t_0} e^{-\omega_{HOM} t_0 / 2Q_{HOM}} \sin \omega_{HOM} t_0}{1 + \left(e^{i\omega t_0} e^{-\omega_{HOM} t_0 / 2Q_{HOM}}\right)^2 - 2 e^{i\omega t_0} e^{-\omega_{HOM} t_0 / 2Q_{HOM}} \cos \omega_{HOM} t_0}$$

$$K = (R/Q)_{HOM} \, k_{HOM}^2 e T_{12} I_0 t_0 / 2$$

For $T_{12} \sin \omega_{HOM} t_r < 0 \rightarrow K \ll 1$ at threshold

Perturbation Theory Works

$$e^{i\omega t_0} e^{-\omega_{HOM} t_0 / 2Q_{HOM}} \doteq \left[1 \mp \frac{iK}{2} e^{i\omega t_r} \right] e^{\pm i\omega_{HOM} t_0}$$

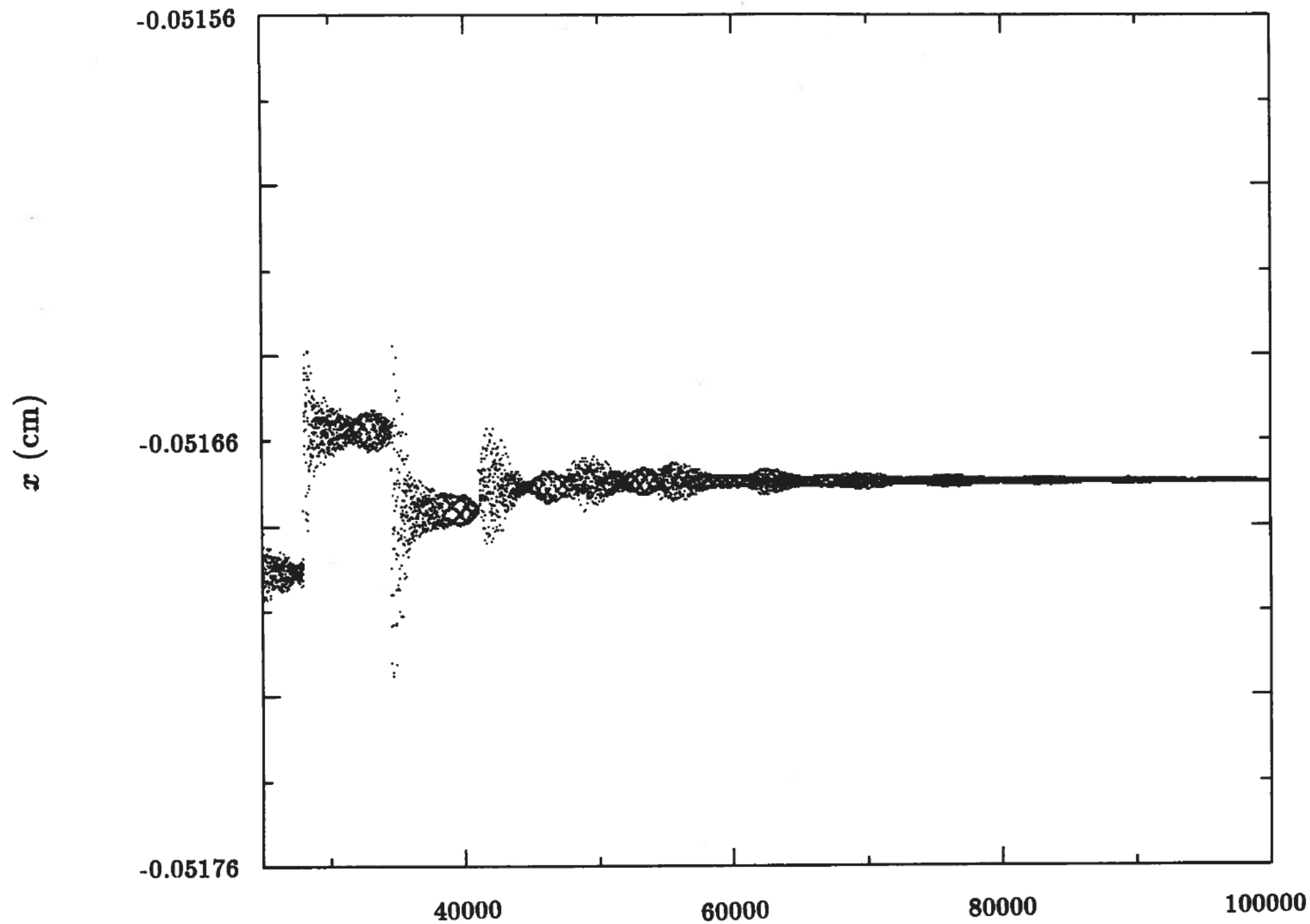
Growth rate

$$\mathrm{Im}(\omega) \doteq -\frac{K \sin(\omega_{HOM} t_r)}{2t_0} - \frac{\omega_{HOM}}{2Q_{HOM}}$$

Threshold current

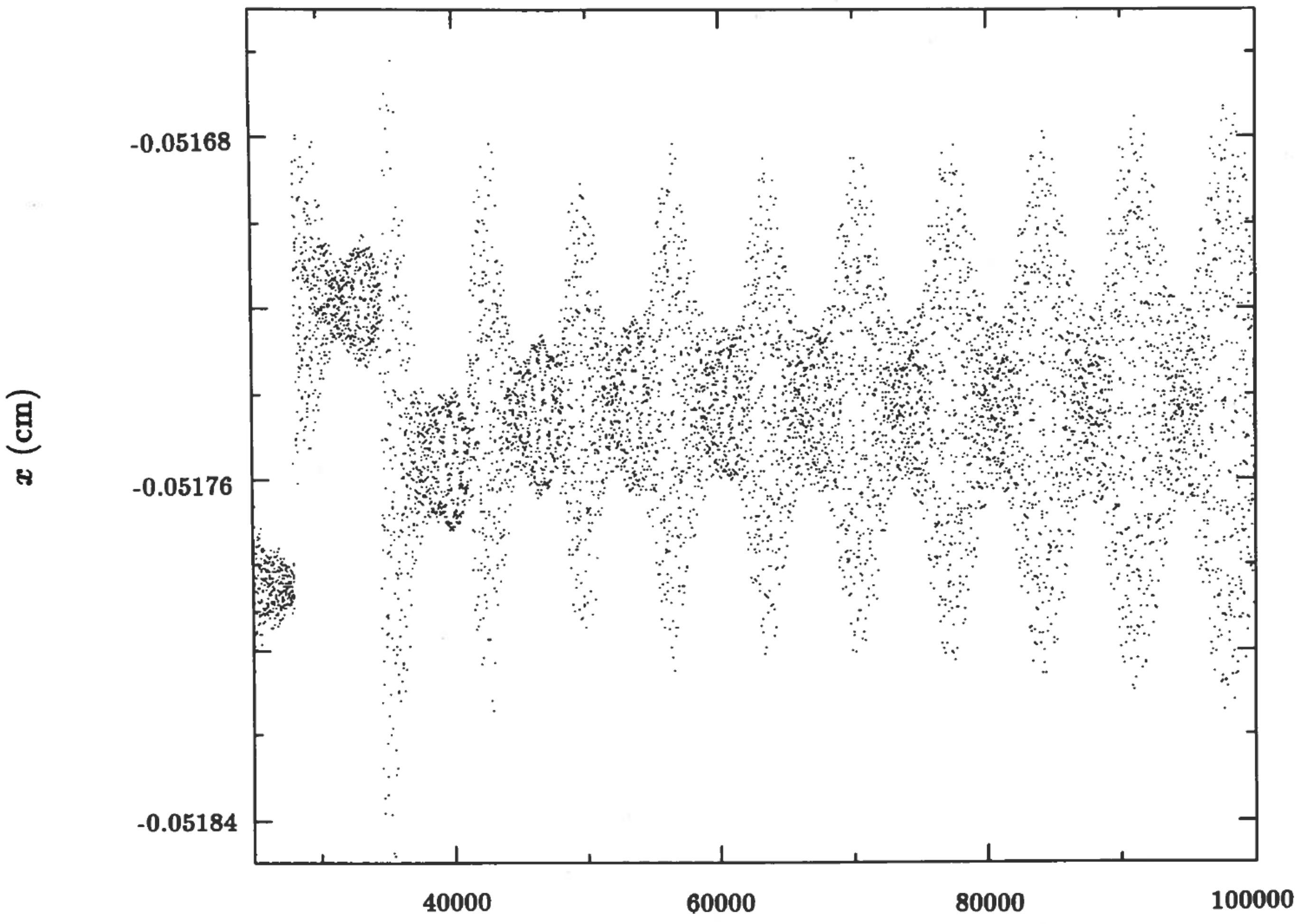
$$I_{th} = \frac{1}{e} \frac{2\omega_{HOM}}{(R/Q)_{HOM} \, Q_{HOM} k_{HOM}^2 \left| T_{12} \sin(\omega_{HOM} t_r) \right|}$$

CEBAF (Design) Simulations



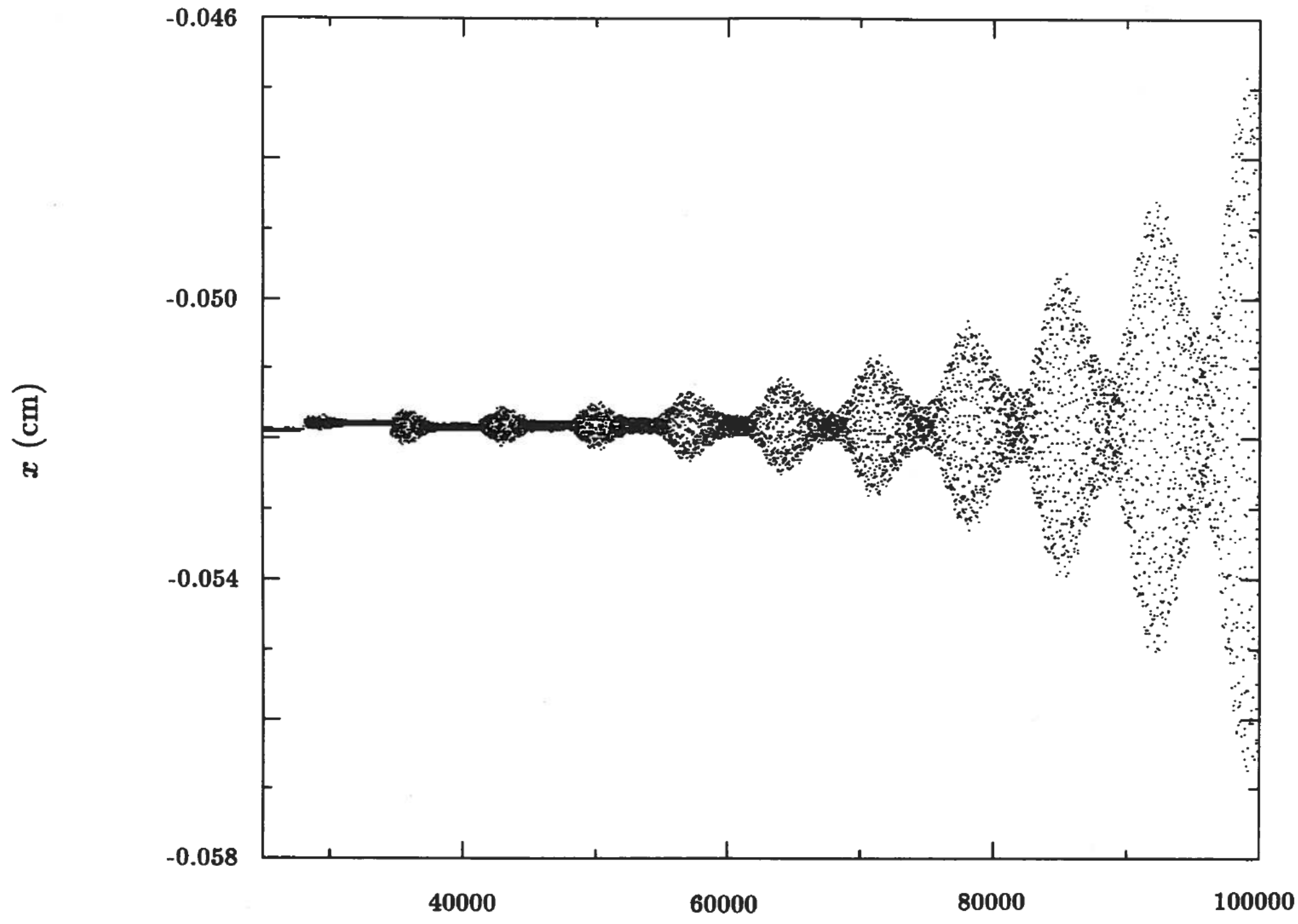
Bunch Number

Stable



Bunch Number

Close to threshold



Bunch Number

Unstable

Chromatic (Landau) Damping

When T_{12} depends on energy offset δ, threshold current is modified to

$$I_{th} = \frac{1}{e}\frac{2\omega_{HOM}}{(R/Q)_{HOM}\, Q_{HOM} k_{HOM}^2 \left| T_{12,eff} \sin\left(\omega_{HOM} t_r\right)\right|}$$

$$T_{12,eff} = \frac{\int T_{12}(\delta)\, f(\delta)\, d\delta}{\int f(\delta)\, d\delta}$$

If $\xi\delta \gg 1,\ T_{12,eff} \rightarrow 0$