Accelerator Physics Chromaticity and Hamiltonian Resonance Theory

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Lecture 15

Idea of Perturbation Theory

In normal betatron coordinates

$$w(s)=\frac{x(s)}{\beta(s)},\varphi=\int\frac{ds}{\nu_0\beta(s)},0\le\varphi\le2\pi$$

$$\frac{d^2w}{d\varphi^2}+\nu_0^2w=\nu_0^2\beta^2\Delta Kw$$

Write

$$w=w_0+w_1+w_2+\cdots$$

If $\left|\int_0^{2\pi}\beta^2\Delta K d\varphi\right|\ll1$

$$\frac{d^2w_0}{d\varphi^2}+\nu_0^2w_0=0$$

$$\frac{d^2w_1}{d\varphi^2}+\nu_0^2w_1=\nu_0^2\beta^2\Delta Kw_0$$

$$\vdots$$

$$\frac{d^2w_n}{d\varphi^2}+\nu_0^2w_n=\nu_0^2\beta^2\Delta Kw_{n-1}$$

will have

$$|w_n|\ll|w_{n-1}|\ll\cdots\ll|w_1|\ll|w_0|$$

except where a term blows up (resonance)

Chromaticity

Defined by

$$\xi = \frac{\Delta\mu}{2\pi(\delta p / p_0)}$$

Thin lens FODO system

$$\begin{pmatrix} 1 & 0 \\ -1/2f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/2f & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - L^2/f^2 & 2L + L^2/f \\ -\left(L - L^2/f\right)/2f & 1 - L^2/f^2 \end{pmatrix}$$

$\therefore$ μ for one period is

$$\cos\mu = 1 - L^2/f^2$$

Suppose particle has a momentum error $\delta p / p_0$

$$f(\delta p / p_0) = f_0 (1 + \delta p / p_0) \doteq \frac{f_0}{(1 - \delta p / p_0)}$$

Tune Shift

$$\cos(\mu_0 + \Delta\mu) = 1 - \frac{L^2}{2 f_0^{\,2}} (1 - \delta p / p_0)^2$$

$$\cos\mu_0 - \sin\mu_0 \sin\Delta\mu \doteq \cos\mu_0 + \frac{L^2}{f_0^{\,2}} \frac{\delta p}{p_0}$$

$$\sin\Delta\mu \doteq \Delta\mu = -\frac{2(1 - \cos\mu_0)}{\sin\mu_0} \frac{\delta p}{p_0} = -2\tan\frac{\mu_0}{2} \frac{\delta p}{p_0}$$

$$\xi = \frac{\Delta\mu}{2\pi(\delta p / p_0)} = \frac{-\tan(\mu_0/2)}{\pi}$$

This is per period. Total ring chromaticity "proportional" to the number of periods.

More Sophisticated

$$x'' + kx = kx(\delta p / p_0) - \frac{m}{2}(x^2 - y^2)$$

$$y'' - ky = -ky(\delta p / p_0) + mxy$$

$$x = x_\beta + D_x(\delta p / p_0) \qquad y = y_\beta$$

expand to second order

$$x_\beta'' + kx_\beta = kx_\beta(\delta p / p_0) - mx_\beta D_x(\delta p / p_0) - \frac{m}{2}(x_\beta^2 - y_\beta^2)$$

$$y_\beta'' - ky_\beta = -ky_\beta(\delta p / p_0) + mD_x(\delta p / p_0) y_\beta + mx_\beta y_\beta$$

Final terms give geometric aberations; ignor for now

tune change $\propto \sqrt{k}$

$$\Delta\mu_x = -\frac{\delta}{2}\oint \beta_x(k - mD_x)dz$$

$$\Delta\mu_y = \frac{\delta}{2}\oint \beta_y(k - mD_x)dz$$

General Formula for Chromaticity

$$\xi_x = -\frac{1}{4\pi}\oint \beta_x \left(k - mD_x\right) dz$$

$$\xi_y = \frac{1}{4\pi}\oint \beta_y \left(k - mD_x\right) dz$$

use sextupoles to zero out

for no sextupoles

$$\xi_x = -\frac{1}{4\pi}\oint \beta_x k dz$$

$$\xi_y = \frac{1}{4\pi}\oint \beta_y k dz$$

works for thin lens

$$\beta_+ = L\frac{\left(f/L\right)\left(\left(f/L\right)+1\right)}{\sqrt{\left(f/L\right)^2-1}} \qquad \beta_- = L\frac{\left(f/L\right)\left(\left(f/L\right)-1\right)}{\sqrt{\left(f/L\right)^2-1}}$$

$$\frac{1}{f} = \int k dz = \frac{1}{\left(f/L\right)L}$$

$$\xi_x = -\frac{1}{4\pi}\oint \beta_x k dz$$

$$= -\frac{1}{4\pi}\left(\beta_+ \int k_+ dz + \beta_- \int k_- dz\right) = -\frac{\beta_+ - \beta_-}{4\pi}\int k dz$$

$$= -\frac{1}{2\pi}\frac{(f/L)}{\sqrt{(f/L)^2 - 1}} = -\frac{1}{2\pi}\tan\mu_0/2$$

Chromaticity Correction

$$\xi_x = \xi_{x0} + \frac{1}{4\pi}\oint \beta_x m D_x dz$$

$$\xi_y = \xi_{y0} - \frac{1}{4\pi}\oint \beta_y m D_x dz$$

Thin sextupoles

$$\xi_x = \xi_{x0} + \frac{1}{4\pi}\left(m_1 D_{x1}\beta_{x1} + m_2 D_{x2}\beta_{x2}\right) l_{sext} = 0$$

$$\xi_y = \xi_{y0} + \frac{1}{4\pi}\left(m_1 D_{x1}\beta_{y1} + m_2 D_{x2}\beta_{y2}\right) l_{sext} = 0$$

sextupole strength

$$m_1 l_s = -\frac{4\pi}{\eta_{x1}}\frac{\xi_{x0}\beta_{y2} - \xi_{x0}\beta_{x2}}{\beta_{x1}\beta_{y2} - \beta_{x2}\beta_{y1}}$$

$$m_2 l_s = -\frac{4\pi}{\eta_{x2}}\frac{\xi_{x0}\beta_{y1} - \xi_{y0}\beta_{x1}}{\beta_{x1}\beta_{y2} - \beta_{x2}\beta_{y1}}$$

Hamiltonian Resonance Theory

Use normal betatron coordinates: nth order multipole perturbation

$$\frac{d^2 w}{d\varphi^2} + \nu_0^2 w = p_n w^{n-1}$$

($n = 1$ is dipole error case and $n = 2$ quad error case we've already considered)

(general n used to evaluate rotated quads and/or multipole fields (e.g. sextupoles) and their effect on motion)

$$p_n = \sum_m p_{nm} e^{im\varphi}$$

$$w = w_0 + w_1 + w_2 + \cdots$$

$$w_0 = a e^{i\nu_0\varphi} + b e^{-i\nu_0\varphi}$$

$$w^{n-1} \approx w_0^{n-1}(\varphi) = \sum_{|q| \le n-1} W_q e^{-iq\nu_0\varphi}$$

$$\frac{d^2 w_1}{d\varphi^2} + \nu_0^2 w_1 = \sum_{q,m} W_q p_{nm} e^{-i(m+q\nu_0)\varphi}$$

general resonance condition

$$m + q\nu_0 = \pm\nu_0 \quad \text{with} \quad |q| \le n-1$$

Examples

$n = 1,\ q = 0$ (e.g. dipole error)

$m = \pm\nu_0$

integer resonance of before

$n = 2,\ q = -1, 0, 1$ (e.g. quadrupole error)

$m - \nu_0 = \pm\nu_0$

$m = \pm\nu_0$

$m + \nu_0 = \pm\nu_0$

$m = 0$ $\rightarrow$ tune shift

$m = \pm 2\nu_0$ $\rightarrow$ integer or 1/2 integer resonance

$m = \pm\nu_0$ no resonance of because $W_0 = 0$

Something new

$n = 3,\ q = -2, -1, 0, 1, 2$ (e.g. sextupole)

$m - 2\nu_0 = \nu_0$

$m = \pm\nu_0$

$m + 2\nu_0 = \nu_0$

$m = 3\nu_0$ $\rightarrow$ third order resonance

$m = \nu_0$ $\rightarrow$ integer resonance

$m = -\nu_0$ $\rightarrow$ integer resonance

$W_{\pm 1} = 0$ for sextupoles (Why?)

Octupoles

$$n = 4,\ q = -3, -1, 1, 3$$

$$m - 3\nu_0 = \nu_0$$

$$m - 1\nu_0 = \nu_0$$

$$m + 1\nu_0 = \nu_0$$

$$m + 3\nu_0 = \nu_0$$

$$m = 4\nu_0 \quad \rightarrow \text{quarter integer resonance}$$

$$m = 2\nu_0 \quad \rightarrow \text{half integer resonance}$$

$$m = 0 \quad \rightarrow \text{tune shift}$$

$$m = -2\nu_0 \quad \rightarrow \text{half integer resonance}$$

General resonance condition re-expressed

$$|m| = \left(|q| \pm 1\right)\nu_0$$

with periodicity N

$$|j|\,N = \left(|q| \pm 1\right)\nu_0$$

eliminates possibilities!

Coupling Resonances

$$\frac{d^2 w}{d\varphi^2} + \nu_{0x}^2 w = p_{nr} w^{n-1} v^{r-1}$$

$$p_{nr} = \sum_m p_{nrm} e^{im\varphi}$$

$$w^{n-1} = \sum_{|l| \le n-1} W_q e^{il\nu_{0x}\varphi}$$

$$v^{r-1} = \sum_{|q| \le r-1} W_q e^{iq\nu_{0y}\varphi}$$

$$\frac{d^2 w_1}{d\varphi^2} + \nu_{0x}^2 w_1 = \sum_{l,q,m} p_{nrm} W_l V_q e^{i\left(m + l\nu_{0x} + q\nu_{0y}\right)\varphi}$$

general resonance condition

$$m + l\nu_{0x} + q\nu_{0y} = \nu_{0x}$$

$|l| + |q| + 1$ is called the order of coupling resonance

Example: Rotated quad

$$l = 0, q = \pm 1$$

$$m \pm q\nu_{0y} = \nu_{0x}$$

resonance occurs for

$|m| = \nu_{0x} + \nu_{0y}$ and $|m| = \nu_{0x} - \nu_{0y}$

"Sum" and "difference" resonances

Resonance diagram

All potential resonances summarized in

$$k\nu_{0x} - l\nu_{0y} = iN$$

resonance order $|l| + |k| + 1$

Perturbation Terms in Normalized Coordinates

Order n	r	$\hat{p}_{nx}(\varphi) w^{n-1} v^{r-1}$	$\hat{p}_{ny}(\varphi) w^{n-1} v^{r-1}$
1	2	$-\nu_{x0}^2 \beta_x^{3/2} \beta_y^{1/2} \bar{k} v$	
2	1		$-\nu_{y0}^2 \beta_x^{1/2} \beta_y^{3/2} \bar{k} w$
3	1	$-\nu_{x0}^2 \beta_x^{5/2} \frac{1}{2} m w^2$	
2	2		$-\nu_{y0}^2 \beta_x^{1/2} \beta_y^2 m v w$
1	3	$\nu_{x0}^2 \beta_x^{3/2} \beta_y \frac{1}{2} m v^2$	
4	1	$-\nu_{x0}^2 \beta_x^3 \frac{1}{6} r w^3$	
3	2		$\nu_{y0}^2 \beta_x \beta_y^2 \frac{1}{2} r w^2 v$
2	3	$\nu_{x0}^2 \beta_x^2 \beta_y \frac{1}{2} r w v^2$	
1	4		$-\nu_{y0}^2 \beta_y^3 \frac{1}{6} r w^3$

Nonlinear Hamiltonian

$$H_w = \frac{1}{2}\dot{w}^2 + \frac{1}{2}\nu_0^2 w^2 + \hat{p}_n \frac{\nu_0^{n/2}}{2^{n/2}} w^n$$

$$\hat{p}_n(\varphi) = -\frac{p_n(\varphi)}{n}\left(\frac{\nu_0}{2}\right)^{-n/2}$$

Introduce action-angle variables in unperturbed Hamiltonian

$$J = \frac{1}{2\pi}\oint p\,dq = \frac{1}{2}\frac{\dot{w}^2}{\nu_0} + \frac{1}{2}\nu_0 w^2$$

$$w = \sqrt{\frac{2J}{\nu_0}}\cos(\psi - \xi)$$

$$H = \nu_0 J + p_n(\varphi) J^{n/2} \cos^n(\psi - \xi)$$

ψ betatron phase ξ arbitrary phase