Accelerator Physics
Closed Orbits and Chromaticity

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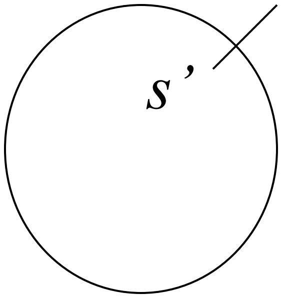
Lecture 14

Dipole Error

Kick at every turn. Solve a toy model:

$$\Delta\theta = \frac{\rho\Delta B}{(B\rho)} \qquad \left[\frac{d^2}{ds^2} + \frac{k_B}{Q}\frac{d}{ds} + k_B^2\right]x(s) = \Delta\theta\sum_{i=0}^{\infty}\delta(s - s' + iL)$$



$$x_{ih}(s) = \sum_{i=0}^{\infty}\exp\left(-k_B(s - s' + iL)/2Q\right)\sin\left(k_B'(s - s' + iL)\right)$$

$$k_B' = k_B\sqrt{1 - 1/4Q^2} \qquad q = \exp\left(-k_B(s - s')/2Q\right)$$

Geometric series summed

$$x_{ih}(s) \propto \exp\left(-k_B(s - s')/2Q\right)\frac{\sin\left(k_B'(s - s')\right)\left(1 - q\cos k_B'L\right) + \cos\left(k_B'(s - s')\right)q\sin k_B'L}{1 - 2q\cos k_B'L + q^2}$$

$$x_{ih}(s) \doteq \frac{\Delta\theta}{k_B}\frac{\cos\left(k_B(s - s') + k_B L/2\right)}{2\sin\left(k_B L/2\right)}$$

integer resonance

blows up when

$$k_B'L = n\pi$$

Closed Orbit Distortion

Perform summation over all kick sources

$$x_{co}(s) \doteq \sum_i \frac{\Delta\theta_i}{k_B} \frac{\cos\left(k_B\left(s-s_i\right)+k_B L/2\right)}{2\sin\left(k_B L/2\right)}$$

(bound) oscillation generated by error

Source (dipole powering, quad displacement, etc.)

Oscillation can be observed and corrected

Using the real betatron motion

$$\left(M_{s,s'}\right)_{12} = \sqrt{\beta(s)\beta(s')}\sin\left(\varphi(s)-\varphi(s')\right)$$

the proper result is

$$x_{co}(s) \doteq \sum_i \sqrt{\beta(z)\beta_i}\Delta\theta_i \frac{\cos\left(\varphi(s)-\varphi(s_i')+\nu\pi\right)}{2\sin(\nu\pi)}$$

Beta Measurement

If BPM close to steerer (there is little phase advance between them), and the tune has been measured, induce a closed orbit distortion to measure the β

$$x_{co}\left(s_{bpm}\right)=\frac{\beta_i\Delta\theta_i}{2}\cot\nu\pi$$

Dipole Error Distribution

$$u_{co}^2(s) = \beta(s)\sum_i \sum_j \sqrt{\beta(s_i)}\Delta\theta_i(s_i)\sqrt{\beta(s_j)}\Delta\theta_i(s_j)$$

$$\times \frac{\cos(\varphi(s)-\varphi(s_i)+\nu\pi)}{2\sin(\nu\pi)}\frac{\cos(\varphi(s)-\varphi(s_j)+\nu\pi)}{2\sin(\nu\pi)}ds_1 ds_2$$

Angular stuff averages assuming independence of error distributions

$$\langle\Delta\theta^2\rangle / 2$$

$$\langle u_{co}^2(s)\rangle = \frac{\beta(s)}{8\sin^2(\nu\pi)}\sum_i \beta_i \langle\Delta\theta\rangle_i^2$$

$$\sigma_u^2 = \frac{\beta(s)N}{8\sin^2(\nu\pi)}\langle\beta\rangle\sigma_{\Delta\theta}^2$$

For quad displacements replace

$$\sigma_{\Delta\theta} = \sigma_x / f$$

Closed Orbit Correction

Suppose orbit does not go through center of all BPMs. What do you do? (At CEBAF just steer to BPM centers!)

Trim magnets added whose purpose is to bring CO as close to zero as possible.

$$u_{co}(s)=\sum_i \sqrt{\beta(s)\beta_i}\Delta\theta_i \frac{\cos\left(\varphi(s)-\varphi(s_i)+\nu\pi\right)}{2\sin(\nu\pi)}$$

At BPM j closed orbit reads

$$u_j=u_{co}(s_j)=\sum_i \sqrt{\beta(s_j)\beta_i}\Delta\theta_i \frac{\cos\left(\varphi(s_j)-\varphi(s_i)+\nu\pi\right)}{2\sin(\nu\pi)}$$

Measure response matrix as trim magnets (index k) varied

$$\Delta u_j=\sum_i \sqrt{\beta(s_j)\beta_k}\Delta\theta_k \frac{\cos\left(\varphi(s_j)-\varphi(s_i)-\nu\pi\right)}{2\sin(\nu\pi)}\equiv R_{jk}\Delta\theta_k$$

Correction Algorithm

Desire $\Delta\vec{u}_j = -u_j$. If have enough trims simply update

$$\Delta\vec{\theta}_k = -R_{kj}^{-1}\Delta\vec{u}_j$$

More sophisticated when less trims than BPMs, minimize

$$\sum_{i=1}^{N_{BPM}} x_{BPM}^2\left(\Delta\theta_1,\cdots,\Delta\theta_{N_{trims}}\right) = \left(\vec{u} - \left(R\cdot\vec{\theta}\right)\right)^2$$

analogous to "least squares fitting" and generally uses the same types of computer algorithms, including Singular Value Decomposition (SVD).

How many BPMs/trims?

Fourier Analyzing closed orbit equation

$$u_{co}(s) = \sqrt{\beta(s)}\sum_{l=-\infty}^{\infty}\frac{\nu^2 F_l e^{il\varphi(s)}}{\nu^2 - l^2} \qquad F_l = \int \beta^{3/2}(s)\Delta\theta(s)e^{-il\varphi(s)}ds$$

Need enough to resolve the betatron orbit and distribute uniformly in betatron phase

Quadrupole Field Errors

Error at location $s_0 = -u_j$; total strength $1/f = \int \Delta K dz$ focusing

$$M = \begin{pmatrix} 1 & 0 \\ -1/2f & 1 \end{pmatrix} \begin{pmatrix} \cos\mu + \alpha(s_0)\sin\mu & \beta(s_0)\sin\mu \\ -\dfrac{1-\alpha^2(s_0)}{\beta(s_0)}\sin\mu & \cos\mu - \alpha(s_0)\sin\mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/2f & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\mu + \alpha(s_0)\sin\mu - \dfrac{\beta(s_0)}{2f}\sin\mu & mess1 \\ mess2 & \cos\mu - \alpha(s_0)\sin\mu - \dfrac{\beta(s_0)}{2f}\sin\mu \end{pmatrix}$$

$$\cos(\mu + \Delta\mu) = \cos\mu - \frac{\beta(s_0)}{2f}\sin\mu$$

$$\Delta\nu = -\frac{\beta(s_0)}{4\pi f} = -\frac{\beta(s_0)}{4\pi f}\int (\Delta K) dz$$

Add to get total

More Generally

$$x'' + K(s)x = 0 \rightarrow x'' + K_0(s)x = -\Delta K x$$

Introduce normal betatron coordinates

$$w(s) = \frac{x(s)}{\beta(s)}, \varphi = \int \frac{ds}{\nu_0 \beta(s)}, 0 \le \varphi \le 2\pi$$

$$\frac{d^2 w}{d\varphi^2} + \nu_0^2 w = \nu_0^2 \beta^2 \Delta K w$$

Fourier expand rhs

$$F_0 = \frac{1}{2\pi} \int \nu_0 \beta^2 \Delta K d\varphi = \frac{1}{2\pi} \int \beta \Delta K ds$$

$$\frac{d^2 w}{d\varphi^2} + \left(\nu_0^2 - \nu_0 F_0\right) w = \frac{d^2 w}{d\varphi^2} + \nu^2 w$$

$$\nu = \nu_0 + \delta\nu \rightarrow \delta\nu = -\frac{F_0}{2} = -\frac{1}{4\pi} \oint \beta \Delta K ds$$

as above

Note: Method to measure β

Orbit Perturbation

Use Lagrange method of variation of parameters. If have

$$P''(z)+K(z)P(z)=p(z)$$

A solution to the inhomogeneous equation is

$$P(z)=\int^{z} p(z')G(z,z')\,dz'$$

$$G(z,z')=P_1(z)P_2(z')-P_2(z)P_1(z')$$

where P_1 and P_2 solve the homogeneous equation with Wronskian 1

$$P(z)=P_1(z)\int^{z} p(z')P_2(z')\,dz'-P_2(z)\int^{z} p(z')P_1(z')\,dz'$$

for normalized equation

$$\frac{d^2w}{d\varphi^2}+\nu_0^2 w=\nu_0^2\beta^2\Delta K w, \qquad P_1(\varphi)=\sin(\nu_0\varphi), P_2(\varphi)=\cos(\nu_0\varphi)$$

$$P(\varphi)=\int^{\varphi}\nu_0\beta^2(\varphi')\Delta K(\varphi')\sin(\nu_0(\varphi-\varphi'))\,d\varphi'$$

Specific Case

define location $\varphi = 0$ and suppose unperturbed orbit

displaced there with displacement a

$$w_{perturbed} = a\cos(\nu_0\varphi) + a\nu_0 \int_0^{\varphi} \beta^2(\varphi')\Delta K(\varphi')\cos(\nu_0\varphi')\sin(\nu_0(\varphi-\varphi'))\,d\varphi'$$

Also must equal

$$w_{perturbed} = a\cos(\nu\varphi)$$

Evaluate the total tune shift by going around 1 turn

$$\cos 2\pi(\nu_0 + \delta\nu) = \cos 2\pi\nu_0 + \nu_0 \int_0^{2\pi} \beta^2(\varphi')\Delta K(\varphi')\cos(\nu_0\varphi')\sin(\nu_0(2\pi-\varphi'))\,d\varphi'$$

$$-\sin(2\pi\nu_0)2\pi\delta\nu = \nu_0 \int_0^{2\pi} \beta^2(\varphi')\Delta K(\varphi')\cos(\nu_0\varphi')\sin(\nu_0(2\pi-\varphi'))\,d\varphi'$$

$$= \nu_0 \int_0^{2\pi} \beta^2(\varphi')\Delta K(\varphi')\cos(\nu_0\varphi')\left[\sin 2\pi\nu_0 \cos(-\nu_0\varphi') + \cos 2\pi\nu_0 \sin(-\nu_0\varphi')\right]d\varphi'$$

$$\delta\nu = -\frac{1}{4\pi}\oint \beta(z)\Delta K(z)\,dz - \frac{1}{4\pi\sin(2\pi\nu_0)}\oint \beta(z)\Delta K(z)\sin\left[2\pi\nu_0(\pi-\varphi(z))\right]dz$$

now must avoid 1/2 integers!

Stop Bands

If error too large cannot solve for $\delta\nu$.

Indicates breakdown of approximation and next level needed

$$w_{perturbed} = a\cos(\nu_0\varphi) + a\nu_0\int_0^{\varphi}\beta^2(\varphi')\Delta K(\varphi')\cos(\nu_0\varphi')\sin(\nu_0(\varphi-\varphi'))\,d\varphi'$$

is inserted in the equation for the perturbation

$$\cos 2\pi(\nu_0+\delta\nu) - \cos 2\pi\nu_0 = \nu_0\int_0^{\varphi}\beta^2(\varphi')\Delta K(\varphi')\cos(\nu_0\varphi')\sin(\nu_0(\varphi-\varphi'))\,d\varphi'$$

$$+\nu_0^2\int_0^{2\pi}\beta^2(\varphi')\Delta K(\varphi')\int_0^{\varphi'}\beta^2(\varphi'')\Delta K(\varphi'')\cos(\nu_0\varphi'')\sin(\nu_0(\varphi-\varphi''))\,d\varphi''\sin(\nu_0(\varphi-\varphi'))\,d\varphi'$$

$$I_1 = \pi F_0\sin(2\pi\nu_0) + \frac{\nu_0}{2}\int_0^{2\pi}\beta^2(\varphi')\Delta K(\varphi')\sin(2\nu_0(\pi-\varphi'))\,d\varphi'$$

Second term oscillatory and tends to average to zero

$$I_1 \approx \begin{cases} 2\pi^2 F_0\delta\nu & \text{for } \nu_0 = n+\delta\nu \\ -2\pi^2 F_0\delta\nu & \text{for } \nu_0 = n+1/2+\delta\nu \end{cases}$$

$$\int_0^{2\pi} d\varphi' \int_0^{\varphi'} d\varphi'' = \frac{1}{2}\int_0^{2\pi} d\varphi' \int_0^{\varphi'} d\varphi''$$

$$I_2 = -\frac{\nu_0^2}{16}\int_0^{2\pi} \beta^2(\varphi')\Delta k(\varphi')\int_0^{\varphi'} \beta^2(\varphi'')\Delta k(\varphi'')$$

$$\times\left\{\left(e^{2\pi i\nu_0} + e^{-2\pi i\nu_0}\right) - \left[e^{2i\nu_0(\pi-\varphi''+\varphi')} + e^{2i\nu_0(\pi-\varphi''+\varphi')}\right]\right\} d\varphi' d\varphi''$$

Integer Resonance

$$I_{2,n} = -\frac{\nu_0^2}{16}\int_0^{2\pi} \beta^2(\varphi')\Delta k(\varphi')\int_0^{\varphi'} \beta^2(\varphi'')\Delta k(\varphi'')$$

$$\times\left\{\left(e^{2\pi i\delta\nu} + e^{-2\pi i\delta\nu}\right) - \left[e^{2in(-\varphi''+\varphi')} + e^{-2in(-\varphi''+\varphi')}\right]\right\} d\varphi' d\varphi''$$

1/2 Integer Resonance

$$I_{2,n+1/2} = -\frac{\nu_0^2}{16}\int_0^{2\pi} \beta^2(\varphi')\Delta k(\varphi')\int_0^{\varphi'} \beta^2(\varphi'')\Delta k(\varphi'')$$

$$\times\left\{-\left(e^{2\pi i\delta\nu} + e^{-2\pi i\delta\nu}\right) + \left[e^{2i(n+1/2)(-\varphi''+\varphi')} + e^{-2i(n+1/2)(-\varphi''+\varphi')}\right]\right\} d\varphi' d\varphi''$$

$$\left|F_j\right|^2 = F_j F_j^* = \frac{\nu_0^2}{\pi^2}\int_0^{2\pi} \beta^2(\varphi')\Delta K(\varphi')e^{-ij\varphi'}d\varphi' \int_0^{2\pi} \beta^2(\varphi)\Delta K(\varphi'')e^{ij\varphi''}d\varphi''$$

$$I_{2,n} \approx \frac{\pi^2}{8}\left(\left|F_{2n}\right|^2 - 4\left|F_0\right|^2\right)$$

$$I_{2,n} \approx -\frac{\pi^2}{8}\left(\left|F_{2n+1}\right|^2 - 4\left|F_0\right|^2\right)$$

$$\cos 2\pi(\nu_0 + \delta\nu) - 1 = -2\pi^2\delta\nu^2 + 2\pi^2 F_0\delta\nu + \frac{\pi^2}{8}\left(\left|F_{2n}\right|^2 - 4\left|F_0\right|^2\right)$$

$$2\pi^2\delta\nu^2 - 2\pi^2 F_0\delta\nu = \frac{\pi^2}{8}\left(\left|F_{2n}\right|^2 - 4\left|F_0\right|^2\right)$$

$$\delta\nu = F_0/2 \pm \left|F_{2n}\right|/4$$

$$\Delta\nu = \frac{1}{2}\left|F_{2n}\right| = \frac{1}{2\pi}\left|\oint \beta(z)\Delta K(z)e^{-i2n\varphi(z)}dz\right|$$

similarly

$$\Delta\nu = \frac{1}{2}\left|F_{2n}\right| = \frac{1}{2\pi}\left|\oint \beta(z)\Delta K(z)e^{-i2n\varphi(z)}dz\right|$$

Chromaticity

Defined by

$$\xi = \frac{\Delta\mu}{2\pi(\delta p / p_0)}$$

Thin lens FODO system

$$\begin{pmatrix} 1 & 0 \\ -1/2f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/2f & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - L^2/f^2 & 2L + L^2/f \\ -\left(L - L^2/f\right)/2f & 1 - L^2/f^2 \end{pmatrix}$$

$\therefore\ \mu$ for one period is

$$\cos\mu = 1 - L^2/f^2$$

Suppose particle has a momentum error $\delta p / p_0$

$$f(\delta p / p_0) = f_0(1 + \delta p / p_0) \doteq \frac{f_0}{(1 - \delta p / p_0)}$$

Tune Shift

$$\cos(\mu_0+\Delta\mu)=1-\frac{L^2}{2f_0^{\,2}}\left(1-\delta p/p_0\right)^2$$

$$\cos\mu_0-\sin\mu_0\sin\Delta\mu\doteq\cos\mu_0+\frac{L^2}{f_0^{\,2}}\frac{\delta p}{p_0}$$

$$\sin\Delta\mu\doteq\Delta\mu=-\frac{2\left(1-\cos\mu_0\right)}{\sin\mu_0}\frac{\delta p}{p_0}=-2\tan\frac{\mu_0}{2}\frac{\delta p}{p_0}$$

$$\xi=\frac{\Delta\mu}{2\pi\left(\delta p/p_0\right)}=\frac{-\tan\left(\mu_0/2\right)}{\pi}$$

This is per period. Total ring chromaticity "proportional" to the number of periods.

More Sophisticated

$$x'' + kx = kx(\delta p / p_0) - \frac{m}{2}(x^2 - y^2)$$

$$y'' - ky = -ky(\delta p / p_0) + mxy$$

$$x = x_\beta + D_x(\delta p / p_0) \qquad y = y_\beta$$

expand to second order

$$x''_\beta + kx_\beta = kx_\beta(\delta p / p_0) - mx_\beta D_x(\delta p / p_0) - \frac{m}{2}(x_\beta^2 - y_\beta^2)$$

$$y''_\beta - ky_\beta = -ky_\beta(\delta p / p_0) + mD_x(\delta p / p_0) y_\beta + mx_\beta y_\beta$$

Final terms give geometric aberations; ignor for now

tune change $\propto \sqrt{k}$

$$\Delta\mu_x = -\frac{\delta}{2}\oint \beta_x(k - mD_x)dz$$

$$\Delta\mu_y = \frac{\delta}{2}\oint \beta_y(k - mD_x)dz$$

General Formula for Chromaticity

$$\xi_x = -\frac{1}{4\pi}\oint \beta_x \left(k - mD_x\right)dz$$

$$\xi_y = \frac{1}{4\pi}\oint \beta_y \left(k - mD_x\right)dz$$

use sextupoles to zero out

for no sextupoles

$$\xi_x = -\frac{1}{4\pi}\oint \beta_x k dz$$

$$\xi_y = \frac{1}{4\pi}\oint \beta_y k dz$$

works for thin lens

$$\beta_+ = L\frac{\left(f/L\right)\left(\left(f/L\right)+1\right)}{\sqrt{\left(f/L\right)^2 - 1}} \qquad \beta_- = L\frac{\left(f/L\right)\left(\left(f/L\right)-1\right)}{\sqrt{\left(f/L\right)^2 - 1}}$$

$$\frac{1}{f} = \int k dz = \frac{1}{\left(f/L\right)L}$$

$$\xi_x = -\frac{1}{4\pi}\oint \beta_x k dz$$

$$= -\frac{1}{4\pi}\left(\beta_+ \int k_+ dz + \beta_- \int k\_ dz\right) = -\frac{\beta_+ - \beta_-}{4\pi}\int k dz$$

$$= -\frac{1}{2\pi}\frac{(f/L)}{\sqrt{(f/L)^2 - 1}} = -\frac{1}{2\pi}\tan\mu_0/2$$

Chromaticity Correction

$$\xi_x = \xi_{x0} + \frac{1}{4\pi}\oint \beta_x m D_x dz$$

$$\xi_y = \xi_{y0} - \frac{1}{4\pi}\oint \beta_y m D_x dz$$

Thin sextupoles

$$\xi_x = \xi_{x0} + \frac{1}{4\pi}\left(m_1 D_{x1}\beta_{x1} + m_2 D_{x2}\beta_{x2}\right) l_{sext} = 0$$

$$\xi_y = \xi_{y0} + \frac{1}{4\pi}\left(m_1 D_{x1}\beta_{y1} + m_2 D_{x2}\beta_{y2}\right) l_{sext} = 0$$

sextupole strength

$$m_1 l_s = -\frac{4\pi}{\eta_{x1}}\frac{\xi_{x0}\beta_{y2} - \xi_{x0}\beta_{x2}}{\beta_{x1}\beta_{y2} - \beta_{x2}\beta_{y1}}$$

$$m_2 l_s = -\frac{4\pi}{\eta_{x2}}\frac{\xi_{x0}\beta_{y1} - \xi_{y0}\beta_{x1}}{\beta_{x1}\beta_{y2} - \beta_{x2}\beta_{y1}}$$