Accelerator Physics
Multipoles and Closed Orbits

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Lecture 13

Oscillators Similtaneously Excited

$$u_i(t) = 1$$

$$\ddot{u} + \Omega^2 u = Fe^{-i\omega t}$$

$$u(t) = \frac{Fe^{-i\omega t}}{2\omega}\left(\frac{1}{\Omega - \omega} - \frac{1}{\Omega + \omega}\right)$$

Many oscillators distributed in frequency

$$\psi(\Omega) = \frac{1}{N}\frac{dN}{d\Omega}$$

$$U = \frac{\sum_{i=1}^{N} u_i}{N}$$

$$U = \frac{Fe^{-i\omega t}}{2\omega}\int\left[\frac{1}{\Omega - \omega} - \frac{1}{\Omega + \omega}\right]d\Omega$$

for $\psi(\Omega) = \psi(-\Omega)$

$$U = \frac{Fe^{-i\omega t}}{\omega}\int\frac{\psi(\Omega)}{\Omega - \omega}d\Omega$$

Resonance Effect

ODU

$$U = \frac{Fe^{-i\omega t}}{\omega}\left[+i\pi\psi(\omega) + P.V.\int \frac{\psi(\Omega)}{\Omega - \omega} d\Omega \right]$$

$$\dot{U} = Fe^{-i\omega t}\left[\pi\psi(\omega) - iP.V.\int \frac{\psi(\Omega)}{\Omega - \omega} d\Omega \right]$$

For our analytic Lorentzian

$$\psi(\Omega) = \frac{\Delta}{\pi\left(\Delta^2 + \Omega^2\right)}$$

$$\dot{U} = \frac{Fe^{-i\omega t}}{-i\Delta - \omega} = \frac{Fe^{-i\omega t}}{\Delta^2 + \omega^2}(\Delta - i\omega)$$

Energy goes in!

Where does it go?

Physical interpretation of $\nabla \times \vec{H} = \vec{J}$

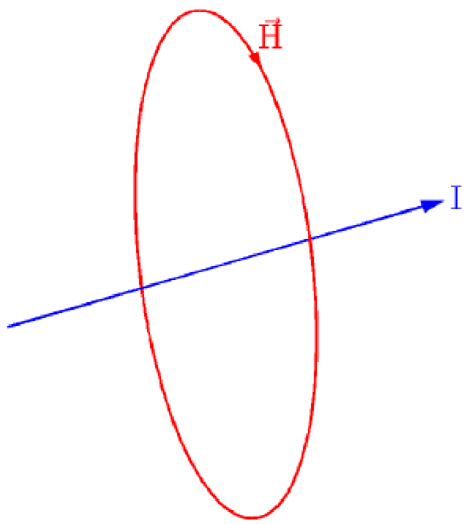
Stokes' theorem tells us that for *any* smooth vector field $\vec{H}$:

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \oint_C \vec{H} \cdot d\vec{\ell}, \tag{4}$$

where the closed loop C bounds the surface S.

Applied to Maxwell's equation $\nabla \times \vec{H} = \vec{J}$, Stokes' theorem tells us that the magnetic field $\vec{H}$ integrated around a closed loop equals the total current passing through that loop:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \vec{J} \cdot d\vec{S} = I. \tag{5}$$



Linearity and superposition

Maxwell's equations are *linear*:

$$\nabla \cdot \left(\vec{B}_1 + \vec{B}_2\right) = \nabla \cdot \vec{B}_1 + \nabla \cdot \vec{B}_2, \tag{6}$$

and:

$$\nabla \times \left(\vec{H}_1 + \vec{H}_2\right) = \nabla \times \vec{H}_1 + \nabla \times \vec{H}_2. \tag{7}$$

This means that if two fields $\vec{B}_1$ and $\vec{B}_2$ satisfy Maxwell's equations, so does their sum $\vec{B}_1 + \vec{B}_2$.

As a result, we can apply the *principle of superposition* to construct complicated magnetic fields just by adding together a set of simpler fields.

Multipole fields

Let us first consider fields that satisfy Maxwell's equations in free space, e.g. the interior of an accelerator vacuum chamber. Here, we have $\vec{J} = 0$, and $\vec{B} = \mu_0 \vec{H}$; hence, Maxwell's equations (1) and (2) become:

$$\nabla \cdot \vec{B} = 0, \quad \text{and} \quad \nabla \times \vec{B} = 0. \tag{8}$$

Consider the field given by $B_z =$ constant, and:

$$B_y + iB_x = C_n \left(x + iy\right)^{n-1}, \tag{9}$$

where n is a positive integer, and C_n is a complex number.

Note that the field components B_x, B_y and B_z are all *real*; we are only using complex numbers for convenience.

Multipole fields

Now consider the differential operator:

$$\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}. \tag{10}$$

Applying this operator to the left hand side of (9) gives:

$$\begin{aligned}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(B_y + iB_x) &= \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) + i\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right), \\ &= \left[\nabla \times \vec{B}\right]_z + i\nabla \cdot \vec{B}. \end{aligned} \tag{11}$$

In the final step, we have used the fact that B_z is constant. Also using this fact, and the fact that B_x and B_y are independent of z, we see that the x and y components of $\nabla \times \vec{B}$ vanish.

Applying the operator (10) to the right hand side of (9) gives:

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x+iy)^{n-1} = (n-1)(x+iy)^{n-2} + i^2(n-1)(x+iy)^{n-2} = 0. \tag{12}$$

Multipole fields

Hence, applying the operator (10) to both sides of equation (9), we find that:

$$\nabla \cdot \vec{B} = 0, \qquad \nabla \times \vec{B} = 0. \tag{13}$$

Therefore, the field (9) satisfies Maxwell's equations for a magnetostatic system in free space.

Of course, this analysis simply tells us that the field (9):

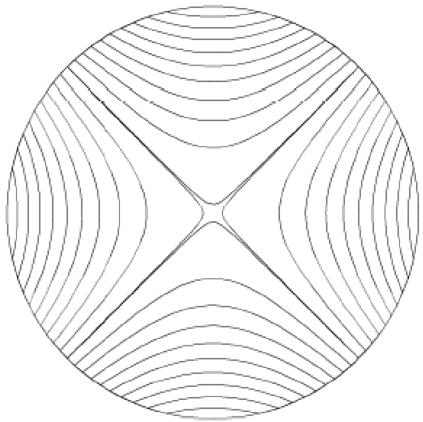
$$B_y + iB_x = C_n \left(x + iy\right)^{n-1}$$

is a possible solution to Maxwell's equations in the situation we have described: it does not tell us how to generate such a field.

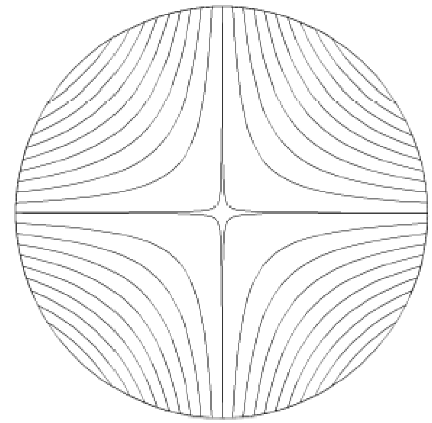
Fields given by (9) are called *multipole fields*. Note that, since Maxwell's equations are linear, we can superpose any number of multipole fields, and obtain a valid solution to Maxwell's equations.

Multipole fields

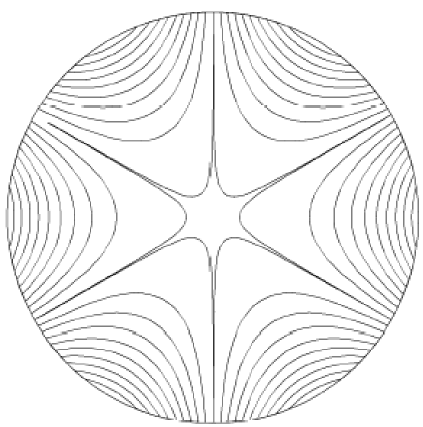
C_2 =real, normal quadrupole



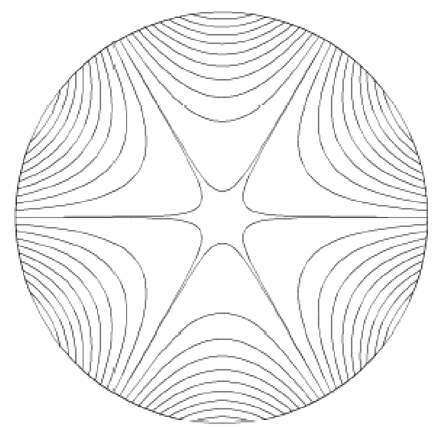
C_2 =imaginary, skew quadrupole



C_3 =real, normal sextupole



C_3 =imaginary, skew sextupole



Multipole fields

For $C_n = 0$ for all n, we have:

$$B_x = B_y = 0, \qquad B_z = \text{constant}. \tag{14}$$

This is a solenoid field, and is not generally regarded as a multipole field.

In the conventional notation (see Chao and Tigner), we rewrite the field (9) as:

$$B_y + iB_x = B_{\text{ref}} \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x + iy}{R_{\text{ref}}} \right)^{n-1}. \tag{15}$$

The b_n are the "normal multipole coefficients", and the a_n are the "skew multipole coefficients". B_{ref} and R_{ref} are a reference field strength and a reference radius, whose values may be chosen arbitrarily; however their values will affect the values of the multipole coefficients for a given field.

Multipole fields

The interpretation of the multipole coefficients is probably best understood by considering the field behaviour in the plane $y = 0$:

$$B_y = B_{\text{ref}} \sum_{n=1}^{\infty} b_n \left(\frac{x}{R_{\text{ref}}} \right)^{n-1}, \quad \text{and} \quad B_x = B_{\text{ref}} \sum_{n=1}^{\infty} a_n \left(\frac{x}{R_{\text{ref}}} \right)^{n-1}. \tag{16}$$

A single multipole component with $n = 1$ is a dipole field: $B_y = b_1 B_{\text{ref}}$ is constant, and $B_x = a_1 B_{\text{ref}}$ is also constant.

A single multipole component with $n = 2$ is a quadrupole field:

$$B_y = b_2 B_{\text{ref}} \frac{x}{R_{\text{ref}}}, \quad \text{and} \quad B_x = a_2 B_{\text{ref}} \frac{x}{R_{\text{ref}}}. \tag{17}$$

Both B_y and B_x vary linearly with x.

For $n = 3$ (sextupole), the field components vary as x^2, etc.

Generating multipole fields from a current distribution

To see how to generate a multipole field, we start with the magnetic field around a thin wire carrying a current I_0. Generally, the magnetic field in the presence of a current density $\vec{J}$ is given by Maxwell's equation (2):

$$\nabla \times \vec{H} = \vec{J}.$$

Consider a thin straight wire of infinite length, oriented along the z axis. Let us integrate Maxwell's equation (2) over a circular disc of radius r centered on the wire, and normal to the wire:

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \int_S \vec{J} \cdot d\vec{S} = I_0, \tag{18}$$

where we have used the fact that the integral of the current density over the cross section of the wire equals the total current flowing in the wire.

Multipole fields from a current distribution

Now we apply Stokes' theorem, which tells us that for *any* smooth vector field F:

$$\int_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{\ell}, \tag{19}$$

where C is the closed curve bounding the surface S.

Applied to equation (18), Stokes' theorem gives us:

$$\oint_C \vec{H} \cdot d\vec{\ell} = I_0, \tag{20}$$

By symmetry, the magnetic field must be the same magnitude at equal distances from the wire. We also know, from Gauss' theorem applied to $\nabla \cdot \vec{B} = 0$, that there can be no radial component to the magnetic field.

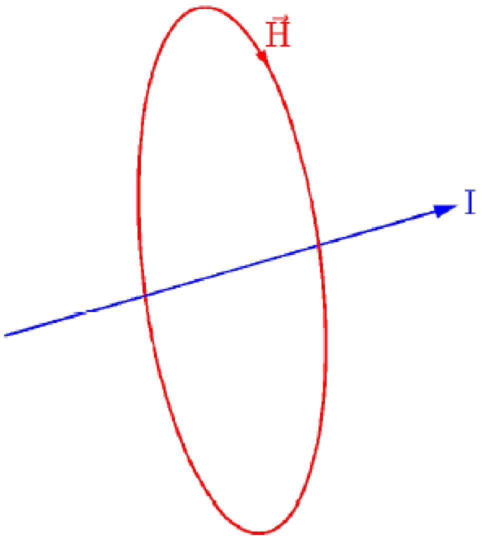
Multipole fields from a current distribution

Hence, the magnetic field at any point is tangential to a circle centered on the wire and passing through that point. We also find, by performing the integral in (20), that the magnitude of the magnetic field at distance r from the wire is given by:

$$\vec{H} = \frac{I_0}{2\pi r}. \tag{21}$$

If there are no magnetic materials present, $\mu = \mu_0$, so:

$$\vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I_0}{2\pi r}. \tag{22}$$



Multipole fields from a current distribution

Now, let us work out the field at a point $\vec{r} = (x, y, z)$ from a current parallel to the z axis, but displaced from it. The line of current is defined by $x = x_0$, $y = y_0$.

The magnitude of the field is given, from (22) by:

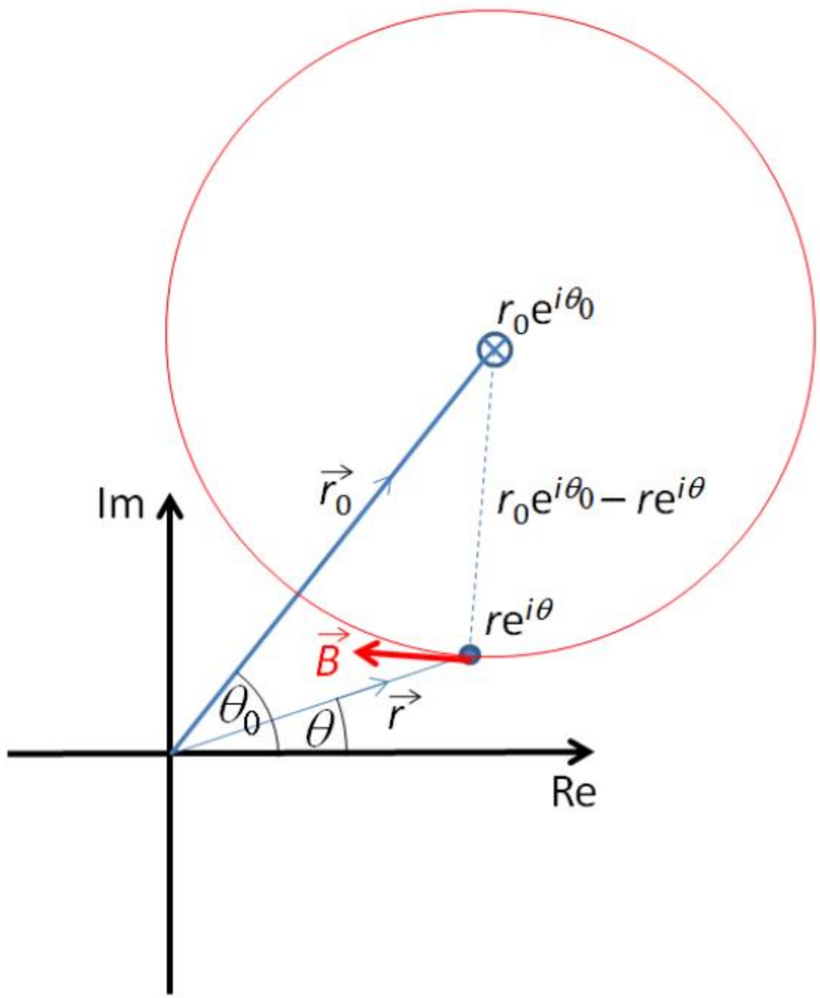
$$B = \frac{\mu_0 I_0}{2\pi \left|\vec{r} - \vec{r}_0\right|}, \tag{23}$$

where the vector $\vec{r}_0$ has components $\vec{r}_0 = (x_0, y_0, z)$.

Since the field at $\vec{r}$ is perpendicular to $\vec{r} - \vec{r}_0$, the field vector is given by:

$$\vec{B} = \frac{\mu_0 I_0}{2\pi} \frac{(y - y_0, -x + x_0, 0)}{\left|\vec{r} - \vec{r}_0\right|^2}. \tag{24}$$

Multipole fields from a current distribution



Multipole fields from a current distribution

It is convenient to express the field (24) in complex notation. Writing:

$$x + iy = re^{i\theta}, \quad \text{and} \quad x_0 + iy_0 = r_0 e^{i\theta_0}, \tag{25}$$

we find that:

$$B_y + iB_x = \frac{\mu_0 I_0}{2\pi} \frac{\left(r_0 e^{-i\theta_0} - re^{-i\theta} \right)}{\left| r_0 e^{i\theta_0} - re^{i\theta} \right|^2}. \tag{26}$$

Using the fact that for any complex number ζ, we have $|\zeta|^2 = \zeta\zeta^*$:

$$\begin{aligned} B_y + iB_x &= \frac{\mu_0 I_0}{2\pi} \frac{1}{r_0 e^{i\theta_0} - re^{i\theta}} \\ &= \frac{\mu_0 I_0}{2\pi r_0} \frac{e^{-i\theta_0}}{\left(1 - \frac{r}{r_0} e^{i(\theta - \theta_0)} \right)}. \end{aligned} \tag{27}$$

Multipole fields from a current distribution

Using the Taylor series expansion:

$$(1-\zeta)^{-1} = \sum_{n=0}^{\infty} \zeta^n, \qquad (28)$$

(valid for $|\zeta| < 1$) we can express the magnetic field (27) as:

$$B_y + iB_x = \frac{\mu_0 I_0}{2\pi r_0} e^{-i\theta_0} \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} e^{i(n-1)(\theta-\theta_0)}, \qquad (29)$$

which is valid for $r < r_0$.

Multipole fields from a current distribution

The advantage of writing the field in the form (29) is that by comparing with equation (15) we immediately see that the field is a sum over an infinite number of multipoles, with coefficients given by:

$$\frac{B_{\mathrm{ref}}}{R_{\mathrm{ref}}^{n-1}}(b_n + ia_n) = \frac{\mu_0 I_0}{2\pi r_0}\frac{e^{-in\theta_0}}{r_0^{n-1}}. \tag{30}$$

If we choose:

$$B_{\mathrm{ref}} = \frac{\mu_0 I_0}{2\pi r_0}, \quad \text{and} \quad R_{\mathrm{ref}} = r_0, \tag{31}$$

we see that:

$$b_n + ia_n = e^{-in\theta_0}. \tag{32}$$

Multipole fields from a current distribution

Now, let us consider the total field generated by a set of wires distributed around a cylinder of radius r_0, such that the current flowing in a region at angle θ_0 and subtending angle $d\theta_0$ at the origin is:

$$I_0 = I_m \cos m(\theta_0 - \theta_m)\, d\theta_0, \tag{33}$$

where m is an integer.

The total field is found by integrating over all θ_0. From (29):

$$\begin{aligned} B_y + iB_x &= \frac{\mu_0 I_m}{2\pi r_0} \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} e^{i(n-1)\theta} \int_0^{2\pi} e^{-in\theta_0} \cos m(\theta_0 - \theta_m)\, d\theta_0 \\ &= \frac{\mu_0 I_m}{2\pi r_0} \left(\frac{r}{r_0}\right)^{m-1} e^{i(m-1)\theta}\, \pi e^{-im\theta_m}. \end{aligned} \tag{34}$$

We see that the cosine current distribution (33) generates a pure $2m$-pole field within the cylinder on which the current flows.

Multipole fields from a current distribution

Choosing the reference field and radius (31) as we did above:

$$B_{\text{ref}} = \frac{\mu_0 I_m}{2\pi r_0}, \quad \text{and} \quad R_{\text{ref}} = r_0,$$

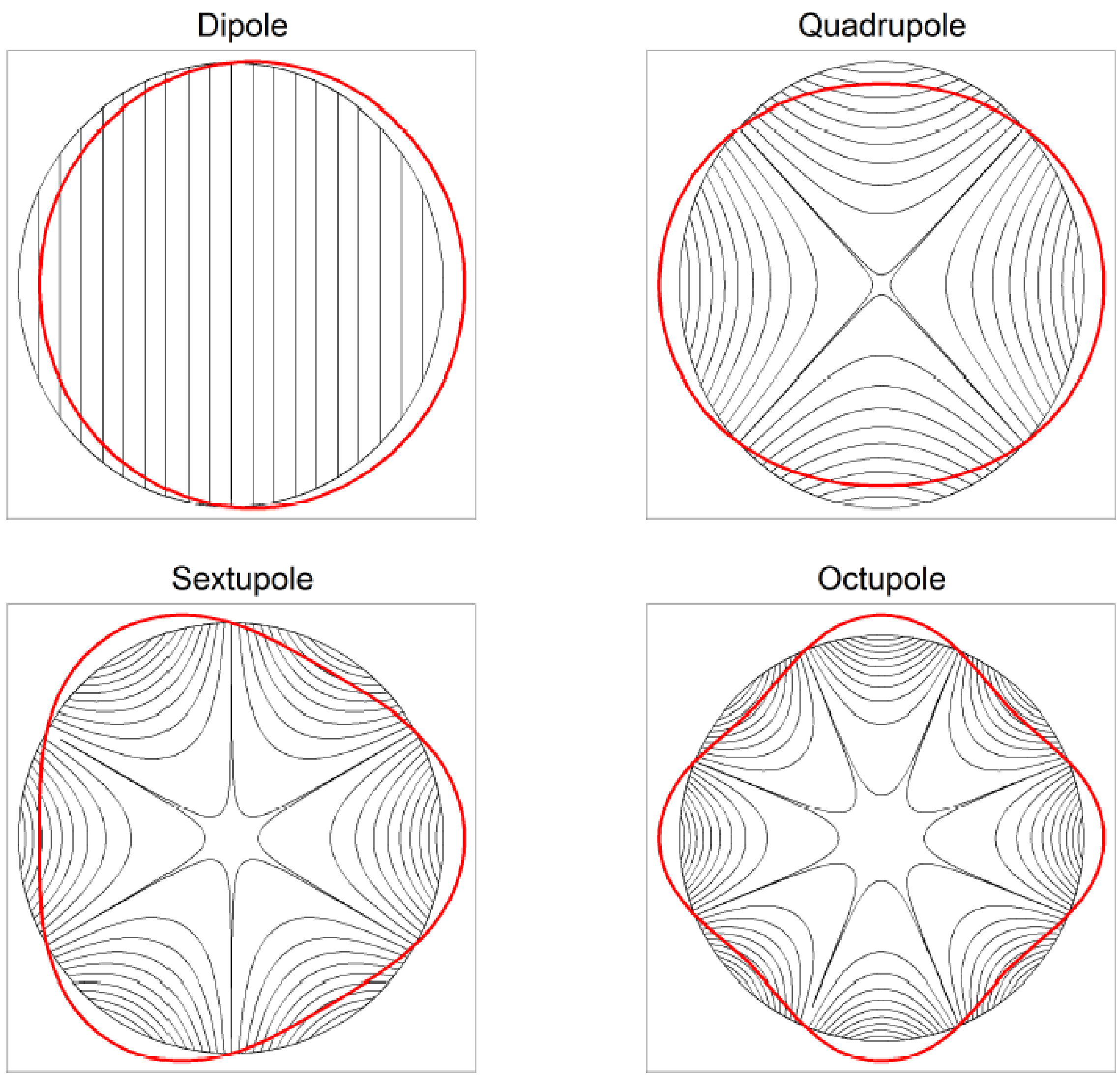
we find that the multipole coefficients for the field generated by the cosine current distribution (33) are:

$$b_m + i a_m = \pi e^{-i m \theta_m}. \tag{35}$$

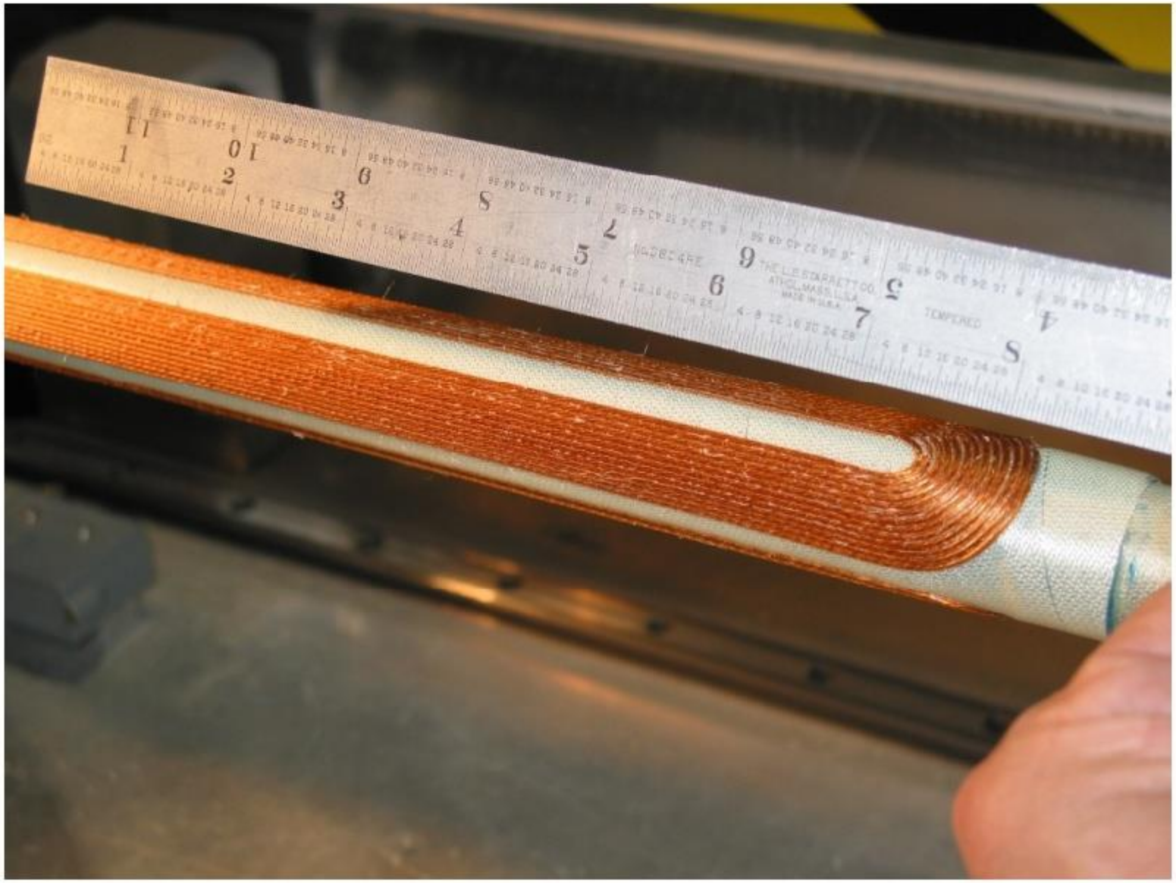
For $\theta_m = 0$ or $\theta_m = \pi$, we have a normal $2m$-pole field.

For $\theta_m = \pm\pi/2$, we have a skew $2m$-pole field.

Multipole fields from a current distribution



Superconducting quadrupole - collider final focus

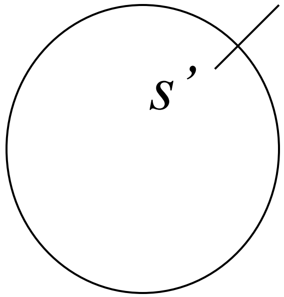


Second layer of a six-layer superconducting quadrupole devel-oped by Brookhaven National Laboratory for a linear collider. The design goal is a gradient of 140 T/m.

Dipole Error

Kick at every turn. Solve a toy model:

$$\Delta\theta = \frac{\rho\Delta B}{(B\rho)} \qquad \left[\frac{d^2}{ds^2} + \frac{k_B}{Q}\frac{d}{ds} + k_B^2\right] x(s) = \Delta\theta \sum_{i=0}^{\infty} \delta(s - s' + iL)$$



$$x_{ih}(s) = \sum_{i=0}^{\infty} \exp\left(-k_B (s - s' + iL)/2Q\right) \sin\left(k_B' (s - s' + iL)\right)$$

$$k_B' = k_B \sqrt{1 - 1/4Q^2} \qquad q = \exp\left(-k_B (s - s')/2Q\right)$$

Geometric series summed

$$x_{ih}(s) \propto \exp\left(-k_B (s - s')/2Q\right) \frac{\sin\left(k_B' (s - s')\right)\left(1 - q\cos k_B' L\right) + \cos\left(k_B' (s - s')\right) q \sin k_B' L}{1 - 2q\cos k_B' L + q^2}$$

$$x_{ih}(s) \doteq \frac{\Delta\theta}{k_B} \frac{\cos\left(k_B (s - s') + k_B L/2\right)}{2\sin\left(k_B L/2\right)}$$

integer resonance

blows up when

$$k_B' L = n\pi$$

Closed Orbit Distortion

Perform summation over all kick sources

$$x_{co}(s) \doteq \sum_i \frac{\Delta\theta_i}{k_B} \frac{\cos\left(k_B(s-s_i)+k_B L/2\right)}{2\sin\left(k_B L/2\right)}$$

(bound) oscillation generated by error

Source (dipole powering, quad displacement, etc.)

Oscillation can be observed and corrected

Using the real betatron motion

$$\left(M_{s,s'}\right)_{12} = \sqrt{\beta(s)\beta(s')}\sin\left(\varphi(s)-\varphi(s')\right)$$

the proper result is

$$x_{co}(s) \doteq \sum_i \sqrt{\beta(z)\beta_i}\,\Delta\theta_i \frac{\cos\left(\varphi(s)-\varphi(s_i')+\nu\pi\right)}{2\sin\left(\nu\pi\right)}$$

Beta Measurement

If BPM close to steerer (there is little phase advance between them), and the tune has been measured, induce a closed orbit distortion to measure the β

$$x_{co}\left(s_{bpm}\right)=\frac{\beta_i \Delta\theta_i}{2}\cot \nu\pi$$

Dipole Error Distribution

$$u_{co}^2(s) = \beta(s)\sum_i \sum_j \sqrt{\beta(s_i)}\Delta\theta_i(s_i)\sqrt{\beta(s_j)}\Delta\theta_i(s_j)$$

$$\times \frac{\cos(\varphi(s)-\varphi(s_i)+\nu\pi)}{2\sin(\nu\pi)}\frac{\cos(\varphi(s)-\varphi(s_j)+\nu\pi)}{2\sin(\nu\pi)}ds_1 ds_2$$

Angular stuff averages assuming independence of error distributions

$$\langle \Delta\theta^2 \rangle / 2$$

$$\langle u_{co}^2(s)\rangle = \frac{\beta(s)}{8\sin^2(\nu\pi)}\sum_i \beta_i \langle \Delta\theta\rangle_i^2$$

$$\sigma_u^2 = \frac{\beta(s)N}{8\sin^2(\nu\pi)}\langle\beta\rangle\sigma_{\Delta\theta}^2$$

For quad displacements replace

$$\sigma_{\Delta\theta} = \sigma_x / f$$

Closed Orbit Correction

Suppose orbit does not go through center of all BPMs. What do you do? (At CEBAF just steer to BPM centers!)

Trim magnets added whose purpose is to bring CO as close to zero as possible.

$$u_{co}(s) = \sum_i \sqrt{\beta(s)\beta_i}\Delta\theta_i \frac{\cos(\varphi(s)-\varphi(s_i)+\nu\pi)}{2\sin(\nu\pi)}$$

At BPM j closed orbit reads

$$u_j = u_{co}(s_j) = \sum_i \sqrt{\beta(s_j)\beta_i}\Delta\theta_i \frac{\cos(\varphi(s_j)-\varphi(s_i)+\nu\pi)}{2\sin(\nu\pi)}$$

Measure response matrix as trim magnets (index k) varied

$$\Delta u_j = \sum_i \sqrt{\beta(s_j)\beta_k}\Delta\theta_k \frac{\cos(\varphi(s_j)-\varphi(s_i)-\nu\pi)}{2\sin(\nu\pi)} \equiv R_{jk}\Delta\theta_k$$

Correction Algorithm

Desire $\Delta\vec{u}_j = -u_j$. If have enough trims simply update

$$\Delta\vec{\theta}_k = -R_{kj}^{-1}\Delta\vec{u}_j$$

More sophisticated when less trims than BPMs, minimize

$$\sum_{i=1}^{N_{BPM}} x_{BPM}^2\left(\Delta\theta_1,\cdots,\Delta\theta_{N_{trims}}\right) = \left(\vec{u} - \left(R\cdot\vec{\theta}\right)\right)^2$$

analogous to "least squares fitting" and generally uses the same types of computer algorithms, including Singular Value Decomposition (SVD).

How many BPMs/trims?

Fourier Analyzing closed orbit equation

$$u_{co}(s) = \sqrt{\beta(s)}\sum_{l=-\infty}^{\infty}\frac{\nu^2 F_l e^{il\varphi(s)}}{\nu^2 - l^2} \qquad F_l = \int \beta^{3/2}(s)\Delta\theta(s)e^{-il\varphi(s)}ds$$

Need enough to resolve the betatron orbit and distribute uniformly in betatron phase