Accelerator Physics
Stabilization by Landau Damping

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Lecture 12

Oscillation Frequency

$$\Delta\Omega^2 = \left(\Omega - n\omega_0\right)^2 = -i\,\frac{nq\eta_c\omega_0^2 I_0}{2\pi\beta^2 E_0} Z_{||}$$

$\operatorname{Re} Z_{||} \neq 0 \rightarrow 1$ mode has positive imaginary part

$\rightarrow$ instability

Resistive impedance has positive real part

"Resistive wall instability"

If $\operatorname{Re} Z_{||} = 0$ (e.g. space charge impedance at long wavelengths)

stability/instability depends on sign of RHS

$\operatorname{Im} Z_{||} < 0$ (inductive, stable if $\eta_c < 0$, unstable if $\eta_c > 0$)

$\operatorname{Im} Z_{||} > 0$ (capacitive, space charge is this way,

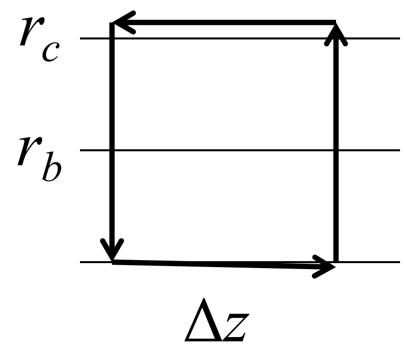
stable if $\eta_c > 0$, unstable if $\eta_c < 0$)

Later case is negative mass instability

NMI Growth time

Impedance?

$$E_r = \begin{cases} \dfrac{e\lambda r}{2\pi\varepsilon_0 r_b^2} & r < r_b \\ \dfrac{e\lambda}{2\pi\varepsilon_0 r} & r > r_b \end{cases} \qquad B_\theta = \begin{cases} \beta c \dfrac{\mu_0 e\lambda r}{2\pi r_b^2} & r < r_b \\ \beta c \dfrac{\mu_0 e\lambda}{2\pi r} & r > r_b \end{cases}$$



$$E_z = -\frac{e}{4\pi\varepsilon_0}\left(1-\beta^2\right)\frac{\partial\lambda}{\partial z}\left(1+2\ln\left(r_c/r_b\right)\right)$$

$$\lambda \propto \lambda_n e^{i(n\theta-\Omega t)}$$

$$V_{SC} = \frac{-in}{2\varepsilon_0\gamma^2}\lambda_n\left(1+2\ln\left(r_c/r_b\right)\right) = \frac{-in}{2\varepsilon_0\gamma^2\beta c} I_n\left(1+2\ln\left(r_c/r_b\right)\right)$$

$$\left(\Omega - n\omega_0\right)^2 = -i\frac{nq\eta_c\omega_0^2 I_0}{2\pi\beta^2 E_0} Z_{SC} = -\frac{n^2\omega_0^2}{\beta^2\gamma^2}\left(\frac{q\eta_c I_0}{4\pi\varepsilon_0 c\beta E_0}\left(1+2\ln\left(r_c/r_b\right)\right)\right)$$

Stabilization by Beam Temperature?

Canonical variables $\theta, \delta \equiv \Delta p / p_0$

$$\left[\frac{\partial}{\partial t} + \dot{\theta}\frac{\partial}{\partial \theta} + \dot{\delta}\frac{\partial}{\partial \delta}\right]\psi = 0$$

$$\psi = \psi_0 + \psi_n e^{i(n\theta - \omega_n t)}$$

$$i(\omega_n - n\omega)\psi_n = \frac{\dot{\delta}}{e^{i(n\theta - \omega_n t)}}\frac{\partial \psi_0}{\partial \delta}$$

$$\frac{\partial \psi_0}{\partial \delta} = \frac{\partial \psi_0}{\partial \omega}\frac{\partial \omega}{\partial \delta} = \eta_c \omega_0 \frac{\partial \psi_0}{\partial \omega}$$

current perturbation is

$$I_n = q\omega_0 \int_{-\infty}^{\infty} \psi_n d\delta$$

Dispersion Relation

$$\psi_0(\delta) = \eta_c \omega_0 \Phi_0(\omega)$$

$$\dot{\delta} = \frac{1}{\eta_c \omega_0} \dot{\omega} = \left(\frac{dE}{dt}\right) / \left(\beta^2 E_0\right)$$

$$1 = i \frac{q^2 \omega_0^3 \eta_c Z_{||}}{2\pi\beta^2 E_0} \int \frac{\partial \Phi_0 / \partial \omega}{\omega_n - n\omega} d\omega$$

recover before

$$\Phi_0 = N_b \delta(\omega - \omega_0) / 2\pi$$

$$\int \frac{\partial \Phi_0 / \partial \omega}{\omega_n - n\omega} d\omega = -\frac{N_b n}{2\pi\left(\omega_n - n\omega_0\right)^2}$$

Landau Damping

Use our favorite analytic distribution

$$\psi_0(\delta) \propto \frac{1}{\pi} \frac{\delta_0}{\delta_0^2 + \delta^2} \qquad \Phi_0(\omega) \propto \frac{1}{\pi} \frac{\hat{\omega}}{\hat{\omega}^2 + (\omega - \omega_0)^2}$$

$$\hat{\omega} = \delta_0 \eta_c \omega_0$$

$$1 = -i \frac{q\omega_0^2 \eta_c Z_{||} I_0 n}{2\pi\beta^2 E_0} \int \frac{\hat{\omega}}{(\omega_n - n\omega)^2 \pi\left(\hat{\omega}^2 + (\omega - \omega_0)^2\right)} d\omega$$

$$1 = -i \frac{q\eta_c Z_{||} I_0 n}{2\pi\beta^2 E_0} \frac{\omega_0^2}{(\omega_n - n\omega_0 + ni\hat{\omega})^2}$$

$$\omega_n = n\omega_0 - ni\hat{\omega} + \sqrt{V + iU}$$

LD from another view

Single Oscillator

$$\ddot{u} + \Omega^2 u = Fe^{-i\omega t}$$

$$u(t) = \frac{Fe^{-i\omega t}}{2\omega}\left(\frac{1}{\Omega - \omega} - \frac{1}{\Omega + \omega}\right)$$

Many oscillators distributed in frequency

$$\psi(\Omega) = \frac{1}{N}\frac{dN}{d\Omega}$$

$$U = \frac{\sum_{i=1}^{N} u_i}{N}$$

$$U = \frac{Fe^{-i\omega t}}{2\omega}\int\left[\frac{1}{\Omega - \omega} - \frac{1}{\Omega + \omega}\right]d\Omega$$

for $\psi(\Omega) = \psi(-\Omega)$

$$U = \frac{Fe^{-i\omega t}}{\omega}\int\frac{\psi(\Omega)}{\Omega - \omega}d\Omega$$

Resonance Effect

$$U = \frac{Fe^{-i\omega t}}{\omega}\left[+i\pi\psi\left(\omega\right) + P.V.\int \frac{\psi\left(\Omega\right)}{\Omega - \omega} d\Omega \right]$$

$$\dot{U} = Fe^{-i\omega t}\left[\pi\psi\left(\omega\right) - iP.V.\int \frac{\psi\left(\Omega\right)}{\Omega - \omega} d\Omega \right]$$

For our analytic Lorentzian

$$\psi\left(\Omega\right) = \frac{\Delta}{\pi\left(\Delta^2 + \Omega^2\right)}$$

$$\dot{U} = \frac{Fe^{-i\omega t}}{-i\Delta - \omega} = \frac{Fe^{-i\omega t}}{\Delta^2 + \omega^2}\left(\Delta - i\omega\right)$$

Energy goes in!

Where does it go?

Inhomogeneous Solution

$$u(t) = a \sin \Omega t + \frac{F}{\Omega^2 - \omega^2} \sin \omega t$$

Solution with zero initial excitation

$$a = -\frac{\omega}{\Omega} \frac{F}{\Omega^2 - \omega^2}$$

$$\therefore u_{\Omega \neq \omega} = \frac{F}{\Omega^2 - \omega^2} \left(\sin \omega t - \frac{\omega}{\Omega} \sin \Omega t \right)$$

No energy flow

$$\therefore u_{\Omega = \omega} = \frac{F}{\Omega^2 - \omega^2} \left(t \cos \omega t - \frac{\sin \omega t}{\omega} \right)$$

Resonant particles capture energy and oscillation generated out of phase

Oscillators Similtaneously Excited

$$u_i(t) = 1$$

$$\ddot{u} + \Omega^2 u = Fe^{-i\omega t}$$

$$u(t) = \frac{Fe^{-i\omega t}}{2\omega}\left(\frac{1}{\Omega-\omega} - \frac{1}{\Omega+\omega}\right)$$

Many oscillators distributed in frequency

$$\psi(\Omega) = \frac{1}{N}\frac{dN}{d\Omega}$$

$$U = \frac{\sum_{i=1}^{N} u_i}{N}$$

$$U = \frac{Fe^{-i\omega t}}{2\omega}\int\left[\frac{1}{\Omega-\omega} - \frac{1}{\Omega+\omega}\right]d\Omega$$

for $\psi(\Omega) = \psi(-\Omega)$

$$U = \frac{Fe^{-i\omega t}}{\omega}\int\frac{\psi(\Omega)}{\Omega-\omega}d\Omega$$