Accelerator Physics
Instabilties and Landau Damping

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Lecture 11

Collisionless Damping

- For Lorentzian distribution

$$\frac{F_0}{n_0} = \frac{\Delta}{\pi\left[p_z^2 + \Delta^2\right]}$$

$$1 = \omega_p^2 \int_{-\infty}^{\infty} dp_z \frac{1}{\left(\omega - p_z\left(2\pi l / Lm\right)\right)^2} \frac{\Delta}{\pi\left[p_z^2 + \Delta^2\right]}$$

$$= \frac{\omega_p^2}{\left(\omega + i\left(2\pi l / Lm\right)\Delta\right)^2}$$

- Landau damping rate

$$\omega = \pm\omega_p - i\frac{2\pi l}{L}\Delta$$

Linear Beam-Beam Tune Shift

$$\therefore \Delta\gamma\beta_{1x}mc = q_1 \frac{(1+\beta_{1z}\beta_{2z})}{\beta_{1z}+\beta_{2z}} \frac{q_2 x}{2\pi\varepsilon_0 c} \frac{1}{\sigma_x(\sigma_x+\sigma_y)}$$

$$1/f = \frac{2N_2}{\gamma_1}\frac{(1+\beta_{1z}\beta_{2z})}{\beta_{1z}+\beta_{2z}}\frac{r_1}{\sigma_x(\sigma_x+\sigma_y)} \qquad r_1 = \frac{e^2}{4\pi\varepsilon_0 m_1 c^2}$$

$$1/f \doteq \frac{2N_2 r_1}{\gamma_1\sigma_x(\sigma_x+\sigma_y)}$$

Both beams relativistic

From linear Bassetti-Erskine model, and replacing the beam size

$$\xi_x^1 = \frac{N_2 r_1}{2\pi\gamma_1}\frac{1}{\varepsilon_x^1(1+\sigma_y/\sigma_x)} \qquad \xi_y^1 = \frac{N_2 r_1}{2\pi\gamma_1}\frac{1}{\varepsilon_y^i(1+\sigma_y/\sigma_x)(\sigma_x/\sigma_y)}$$

Argument entirely symmetric wrt choice of bunch 1 and 2

$$\xi_x^i = \frac{N_{\bar{i}} r_i}{2\pi\gamma_i}\frac{1}{\varepsilon_x^i(1+\sigma_y/\sigma_x)} \qquad \xi_y^i = \frac{N_{\bar{i}} r_i}{2\pi\gamma_i}\frac{1}{\varepsilon_y^i(1+\sigma_y/\sigma_x)(\sigma_x/\sigma_y)}$$

Luminosity Beam-Beam tune-shift relationship

- Express Luminosity in terms of the (larger!) vertical tune shift (i either 1 or 2)

$$\mathcal{L} = \frac{f N_i \xi_y^i \gamma_i}{2 r_i \beta_{iy}^*}\left(1+\sigma_y/\sigma_x\right) = \frac{I_i}{e}\frac{\xi_y^i \gamma_i}{2 r_i \beta_{iy}^*}\left(1+\sigma_y/\sigma_x\right)$$

- Necessary, but not sufficient, for self-consistent design
- Expressed in this way, and given a known limit to the beam-beam tune shift, the only variables to manipulate to increase luminosity are the stored current, the aspect ratio, and the β^* (beta function value at the interaction point)
- Applies to ERL-ring colliders, stored beam (ions) only

Luminosity-Deflection Theorem

- Luminosity-tune shift formula is linearized version of a much more general formula discovered by Krafft and generalized by V. Ziemann.
- Relates easy calculation (luminosity) to a hard calculation (beam-beam force), and contains all the standard results in beam-beam interaction theory.
- Based on the fact that the relativistic beam-beam force is almost entirely transverse, i. e., 2-D electrostatics applies.

2-D Electrostatics Theorem

$$\vec{E}(\vec{x}) = \frac{2Q'}{4\pi\varepsilon_0}\frac{\vec{x}-\vec{x}'}{\left|\vec{x}-\vec{x}'\right|^2}$$

$$\vec{F}'_{21} = -\vec{F}'_{12} = \frac{1}{2\pi\varepsilon_0}\iint \rho_2(\vec{x}_2)\frac{\vec{x}_2-\vec{x}_1}{\left|\vec{x}_2-\vec{x}_1\right|^2}\rho_1(\vec{x}_1)\,d^2\vec{x}_1 d^2\vec{x}_2 \quad \text{1 on 2}$$

$$n_1(\vec{x}_1) = \rho_1(\vec{x}_1)/Q_1' \quad n_2(\vec{x}_2) = \rho_1\left(\vec{x}_2+\vec{b}\right)/Q_1' \text{ zero centerred}$$

$$Q_i' = \iint \rho_i(\vec{x})\,d^2\vec{x} \qquad \vec{b} = \iint \vec{x}\rho_2(\vec{x})\,d^2\vec{x}/Q_2'$$

$$\vec{F}'_{21} = -\vec{F}'_{12} = \frac{Q_1'Q_2'}{2\pi\varepsilon_0}\iint n_2(\vec{x}_2)\frac{\vec{x}_1+\vec{b}-\vec{x}_2}{\left|\vec{x}_1+\vec{b}-\vec{x}_2\right|^2}n_1(\vec{x}_1)\,d^2\vec{x}_1 d^2\vec{x}_2$$

$$\vec{\nabla}_{\vec{b}} \cdot \frac{\vec{x}_1 + \vec{b} - \vec{x}_2}{\left|\vec{x}_1 + \vec{b} - \vec{x}_2\right|^2} = 2\pi\delta\left(x_2 + b_x + x_1\right)\delta\left(y_2 + b_y + y_1\right)$$

$$\vec{\nabla}_{\vec{b}} \cdot \vec{F}'_{21} = \frac{1}{\varepsilon_0}\iint \rho_2\left(\vec{x} + \vec{b}\right)\rho_1\left(\vec{x}\right) d^2\vec{x}$$

Generalizes $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ $\left(\text{take } \rho_2\left(\vec{x}\right) \propto \delta^2\left(\vec{x} + \vec{b}\right)\right)$

Transverse interaction in the beam-beam problem

$$\Delta p_1 = \frac{q_1 q_2}{2\pi\varepsilon_0 c} \frac{\vec{x}_1 - \vec{x}_2}{\left|\vec{x}_1 - \vec{x}_2\right|^2}$$

$$\vec{D}\left(\vec{b}\right)=\Delta\gamma_1\vec{\beta}_1=-\Delta m_2\gamma_2\vec{\beta}_2/m_1$$

$$=\frac{q_1q_2}{m_1c^2}\iint n_2\left(\vec{x}_2\right)\frac{\vec{x}_1-\vec{x}_2-\vec{b}}{\left|\vec{x}_1-\vec{x}_2-\vec{b}\right|^2}n_1\left(\vec{x}_1\right)d^2\vec{x}_1d^2\vec{x}_2$$

$$\vec{\nabla}_{\vec{b}}\cdot\vec{D}\left(\vec{b}\right)=4\pi N_2r_e\iint n_2\left(\vec{x}-\vec{b}\right)n_1\left(\vec{x}\right)d^2\vec{x} \qquad r_e=\frac{e^2}{4\pi\varepsilon_0mc^2}$$

$$L\left(\vec{b}\right)=N_1N_2\iint n_2\left(\vec{x}-\vec{b}\right)n_1\left(\vec{x}\right)d^2\vec{x}$$

$$L\left(\vec{b}\right)=\frac{N_1}{4\pi r_e}\vec{\nabla}_{\vec{b}}\cdot\vec{D}\left(\vec{b}\right)$$

$$L\left(\vec{b}\right)=-\frac{N_2}{4\pi r_e}\vec{\nabla}_{\vec{b}}\cdot\left(\Delta\gamma_2\vec{\beta}_2\right)$$

$$\begin{pmatrix} D_x \\ D_y \end{pmatrix} = \frac{\gamma_1}{2f} \begin{pmatrix} \sigma_y / \sigma_x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_x \\ b_y \end{pmatrix}$$

$$L = \frac{N_1 \gamma \xi}{2 r_e \beta^*} \left(1 + \sigma_y / \sigma_x \right) \quad \text{as before}$$

Maximum when

$$\frac{\partial}{\partial b_x} \left[\frac{\partial D_x}{\partial b_x} \right] = 0, \qquad \frac{\partial}{\partial b_y} \left[\frac{\partial D}{\partial b_y} \right] = 0$$

Luminosity-Deflection Pairs

ODU

- Round Beam Fast Model

$$\vec{D}\left(\vec{b}\right)=\frac{2N_2 r_e \vec{b}}{\sigma^2+b^2} \qquad L\left(\vec{b}\right)=\frac{N_1 N_2 \sigma^2}{\pi\left(\sigma^2+b^2\right)^2}$$

- Gaussian Macroparticles

$$\vec{D}\left(\vec{b}\right)=\vec{D}_{Bassetti\_Erskine}\left(\vec{b};\sqrt{\sigma_{1x}^2+\sigma_{2x}^2};\sqrt{\sigma_{1y}^2+\sigma_{2y}^2}\right)$$

$$L\left(\vec{b}\right)=\frac{N_1 N_2}{2\pi\sqrt{\sigma_{1x}^2+\sigma_{2x}^2}\sqrt{\sigma_{1y}^2+\sigma_{2y}^2}}\exp\left(-\frac{b_x^2}{\sqrt{\sigma_{1x}^2+\sigma_{2x}^2}}\right)\exp\left(-\frac{b_y^2}{\sqrt{\sigma_{1y}^2+\sigma_{2y}^2}}\right)$$

- Smith-Laslett Model

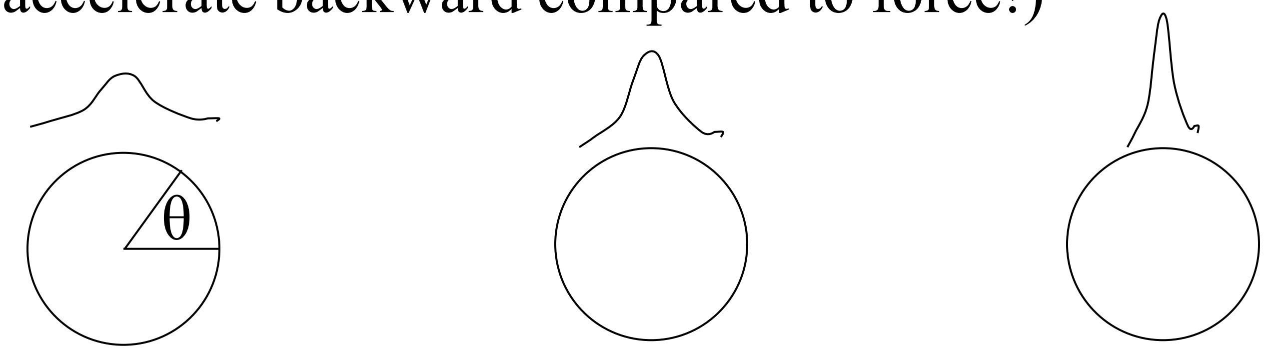
$$\vec{D}\left(\vec{b}\right)=\frac{2N_2 r_e \vec{b}}{\hat{b}^2 AB}\left\{\frac{\left(4\hat{b}^2+2\hat{b}^4\right)}{\left(4\hat{b}^2+\hat{b}^4\right)}-\frac{4\hat{b}^2}{\left(4\hat{b}^2+\hat{b}^4\right)^{3/2}}\left\{\sinh^{-1}\left[\frac{\hat{b}^3}{2}+\frac{3\hat{b}}{2}\right]+\sinh^{-1}\left[\frac{\hat{b}}{2}\right]\right\}\right\}$$

$$L\left(\vec{b}\right)=\frac{N_1 N_2}{\pi AB}\left\{\frac{\left(2\hat{b}^2-4\right)\hat{b}^2}{\left(4\hat{b}^2+\hat{b}^4\right)^2}-\frac{4\hat{b}^2\left(1+\hat{b}^2\right)}{\left(4\hat{b}^2+\hat{b}^4\right)^{5/2}}\left\{\sinh^{-1}\left[\frac{\hat{b}^3}{2}+\frac{3\hat{b}}{2}\right]+\sinh^{-1}\left[\frac{\hat{b}}{2}\right]\right\}\right\}$$

$$\hat{b}^2=\left(\frac{b_x}{A}\right)^2+\left(\frac{b_y}{B}\right)^2$$

Negative Mass Instability

- Simplified argument: assume longitudinal clump on otherwise uniform beam
- Particles pushed away from clump centroid
- If above transition, come back LATER if ahead of clump center and EARLIER if behind it
- The clump is therefore enhanced!
- INSTABILITY; particles act as if they have negative mass (they accelerate backward compared to force!)



Longitudinal Impedance

$W_{||}$ longitudinal wake function

ξ distance between exciting charge q and test charge

$$W_{||}(\xi) \equiv \frac{1}{q} \int_{ring} E_z\left(z, t_{q\ arrival} + \xi / \beta c\right) dz \qquad \text{units V/C}$$

trailing particle (singly charged) picks up voltage per turn of

$$\Delta V(\bar{z}) = -e \int_{\bar{z}}^{"\infty"} \lambda(z) W_{||}(z - \bar{z})\, dz$$

total energy loss

$$\Delta U = -\int_{-\infty}^{\infty} e\lambda(\bar{z})\, d\bar{z} \int_{\bar{z}}^{"\infty"} e\lambda(z) W_{||}(z - \bar{z})\, dz$$

Frequency Domain

$I(\bar{z},t) = \beta c \lambda(\bar{z},t)$ note the coordinate $\bar{z}$ moves with beam

$$\Delta V(\bar{z},t) = -\frac{1}{\beta c} \int_{\bar{z}}^{"\infty"} I\left(z, t + \frac{z-\bar{z}}{\beta c}\right) W_{\|}(z-\bar{z})\, dz$$

Fourier Transform

$$\Delta V(\omega) = -I(\omega)\frac{1}{\beta c} \int_{\bar{z}}^{"\infty"} e^{-i\omega\xi/\beta c} W_{\|}(\xi)\, dz \equiv -Z_{\|}(\omega) I(\omega)$$

$$Z_{\|}(\omega) = \frac{1}{\beta c} \int_{-\infty}^{\infty} e^{-i\omega\xi/\beta c} W_{\|}(\xi)\, d\xi$$

$$W_{\|}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega z/\beta c} Z_{\|}(\omega)\, d\omega$$

Loss factor

$$k = \frac{\Delta U}{q^2} = \frac{2}{q^2} \int_0^{\infty} \mathrm{Re}\left[Z(\omega)\right] |I|^2(\omega)\, d\omega$$

NMI Simple Analysis

ω revolution frequency of particle

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + \frac{\partial \omega}{\partial \theta}\frac{\partial \theta}{\partial t}$$

$$\frac{d\omega}{dt} = \frac{d\omega}{dE}\frac{dE}{dt} = \frac{\eta_c \omega_0}{\beta^2 E_0}\frac{dE}{dt}$$

$$\frac{dE}{dt} = qV_{zn}\frac{\omega_0}{2\pi} = -qZ_{\|}I_n e^{i(n\theta - \Omega t)}\frac{\omega_0}{2\pi}$$

$$\omega = \omega_0 + \omega_n e^{i(n\theta - \Omega t)}$$ Ω oscillation frequency of disturbance

$$\omega_n\left(\Omega - n\omega_0\right) = -i\frac{q\eta_c \omega_0^2}{2\pi\beta^2}\frac{Z_{\|}I_n}{E_0}$$

Linearized Continuity Equation

$$I = v_z \rho \pi r_b^2 = v_z \lambda$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(v_z \rho) = 0$$

$$\frac{\partial \lambda}{\partial t} + \frac{1}{R}\frac{\partial}{\partial \theta}(v_z \lambda) =$$

$$\frac{\partial \delta\lambda}{\partial t} + \omega_0 \frac{\partial \delta\lambda}{\partial \theta} + \lambda_0 \frac{\partial \delta\omega}{\partial \theta} = 0$$

$$(\Omega - n\omega_0) I_n = \omega_n n I_0$$

Oscillation Frequency

$$\Delta\Omega^2 = (\Omega - n\omega_0)^2 = -i\,\frac{nq\eta_c\omega_0^2 I_0}{2\pi\beta^2 E_0} Z_{||}$$

$\mathrm{Re}\, Z_{||} \neq 0 \rightarrow 1$ mode has positive imaginary part

$\rightarrow$ instability

Resistive impedance has positive real part

"Resistive wall instability"

If $\mathrm{Re}\, Z_{||} = 0$ (e.g. space charge impedance at long wavelengths)

stability/instability depends on sign of RHS

$\mathrm{Im}\, Z_{||} < 0$ (inductive, stable if $\eta_c < 0$, unstable if $\eta_c > 0$)

$\mathrm{Im}\, Z_{||} > 0$ (capacitive, space charge is this way,

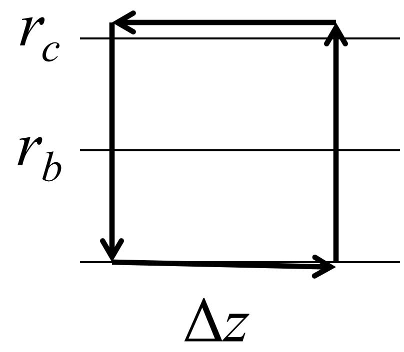
stable if $\eta_c > 0$, unstable if $\eta_c < 0$)

Later case is negative mass instability

NMI Growth time

Impedance?

$$E_r = \begin{cases} \dfrac{e\lambda r}{2\pi\varepsilon_0 r_b^2} & r < r_b \\ \dfrac{e\lambda}{2\pi\varepsilon_0 r} & r > r_b \end{cases} \qquad B_\theta = \begin{cases} \beta c \dfrac{\mu_0 e\lambda r}{2\pi r_b^2} & r < r_b \\ \beta c \dfrac{\mu_0 e\lambda}{2\pi r} & r > r_b \end{cases}$$



$$E_z = -\frac{e}{4\pi\varepsilon_0}\left(1-\beta^2\right)\frac{\partial\lambda}{\partial z}\left(1+2\ln\left(r_c/r_b\right)\right)$$

$$\lambda \propto \lambda_n e^{i(n\theta-\Omega t)}$$

$$V_{SC} = \frac{-in}{2\varepsilon_0\gamma^2}\lambda_n\left(1+2\ln\left(r_c/r_b\right)\right) = \frac{-in}{2\varepsilon_0\gamma^2\beta c}I_n\left(1+2\ln\left(r_c/r_b\right)\right)$$

$$\left(\Omega-n\omega_0\right)^2 = -i\frac{nq\eta_c\omega_0^2 I_0}{2\pi\beta^2 E_0}Z_{SC} = -\frac{n^2\omega_0^2}{\beta^2\gamma^2}\left(\frac{q\eta_c I_0}{4\pi\varepsilon_0 c\beta E_0}\left(1+2\ln\left(r_c/r_b\right)\right)\right)$$

Stabilization by Beam Temperature?

Canonical variables $\theta, \delta \equiv \Delta p / p_0$

$$\left[\frac{\partial}{\partial t} + \dot{\theta}\frac{\partial}{\partial \theta} + \dot{\delta}\frac{\partial}{\partial \delta}\right]\psi = 0$$

$$\psi = \psi_0 + \psi_n e^{i(n\theta - \omega_n t)}$$

$$i(\omega_n - n\omega)\psi_n = \frac{\dot{\delta}}{e^{i(n\theta - \omega_n t)}}\frac{\partial \psi_0}{\partial \delta}$$

$$\frac{\partial \psi_0}{\partial \delta} = \frac{\partial \psi_0}{\partial \omega}\frac{\partial \omega}{\partial \delta} = \eta_c \omega_0 \frac{\partial \psi_0}{\partial \omega}$$

current perturbation is

$$I_n = q\omega_0 \int_{-\infty}^{\infty} \psi_n d\delta$$

Dispersion Relation

$$\psi_0(\delta) = \eta_c \omega_0 \Phi_0(\omega)$$

$$\dot{\delta} = \frac{1}{\eta_c \omega_0} \dot{\omega} = \left(\frac{dE}{dt}\right) / \left(\beta^2 E_0\right)$$

$$1 = i \frac{q^2 \omega_0^3 \eta_c Z_{||}}{2\pi \beta^2 E_0} \int \frac{\partial \Phi_0 / \partial \omega}{\omega_n - n\omega} d\omega$$

recover before

$$\Phi_0 = N_b \delta(\omega - \omega_0) / 2\pi$$

$$\int \frac{\partial \Phi_0 / \partial \omega}{\omega_n - n\omega} d\omega = -\frac{N_b n}{2\pi (\omega_n - n\omega_0)^2}$$

Landau Damping

Use our favorite analytic distribution

$$\psi_0(\delta) \propto \frac{1}{\pi} \frac{\delta_0}{\delta_0^2 + \delta^2} \qquad \Phi_0(\omega) \propto \frac{1}{\pi} \frac{\hat{\omega}}{\hat{\omega}^2 + (\omega - \omega_0)^2}$$

$$\hat{\omega} = \delta_0 \eta_c \omega_0$$

$$1 = -i \frac{q\omega_0^2 \eta_c Z_{||} I_0 n}{2\pi\beta^2 E_0} \int \frac{\hat{\omega}}{(\omega_n - n\omega)^2 \pi \left(\hat{\omega}^2 + (\omega - \omega_0)^2\right)} d\omega$$

$$1 = -i \frac{q\eta_c Z_{||} I_0 n}{2\pi\beta^2 E_0} \frac{\omega_0^2}{(\omega_n - n\omega_0 + ni\hat{\omega})^2}$$

$$\omega_n = n\omega_0 - ni\hat{\omega} + \sqrt{V + iU}$$

$$\ddot{u} + \Omega_2^2 u = F e^{i\omega t}$$

$$u = F \frac{e^{i\omega t}}{2\omega} \left(\frac{1}{\Omega - \omega} - \frac{1}{\Omega + \omega} \right)$$

$$\psi(\omega) = \frac{1}{N_b} \frac{dN_b}{d\Omega}$$

$$\ddot{u} = F \frac{e^{i\omega t}}{2\omega} \int_{-\infty}^{\infty} \left[\frac{\psi(\Omega)}{\Omega - \omega} - \frac{\psi(\Omega)}{\Omega + \omega} \right] d\Omega$$

$$\ddot{u} = F \frac{e^{i\omega t}}{\omega} \int_{-\infty}^{\infty} \frac{\psi(\Omega)}{\Omega - \omega} d\Omega$$

LD from another view

Single Oscillator

$$\ddot{u}+\Omega^2 u = Fe^{-i\omega t}$$

$$u(t)=\frac{Fe^{-i\omega t}}{2\omega}\left(\frac{1}{\Omega-\omega}-\frac{1}{\Omega+\omega}\right)$$

Many oscillators distributed in frequency

$$\psi(\Omega)=\frac{1}{N}\frac{dN}{d\Omega}$$

$$U=\frac{\sum_{i=1}^{N} u_i}{N}$$

$$U=\frac{Fe^{-i\omega t}}{2\omega}\int\left[\frac{1}{\Omega-\omega}-\frac{1}{\Omega+\omega}\right]d\Omega$$

for $\psi(\Omega)=\psi(-\Omega)$

$$U=\frac{Fe^{-i\omega t}}{\omega}\int\frac{\psi(\Omega)}{\Omega-\omega}d\Omega$$

Resonance Effect

$$U=\frac{Fe^{-i\omega t}}{\omega}\left[+i\pi\psi(\omega)+P.V.\int\frac{\psi(\Omega)}{\Omega-\omega}d\Omega\right]$$

$$\dot{U}=Fe^{-i\omega t}\left[\pi\psi(\omega)-iP.V.\int\frac{\psi(\Omega)}{\Omega-\omega}d\Omega\right]$$

For our analytic Lorentzian

$$\psi(\Omega)=\frac{\Delta}{\pi\left(\Delta^2+\Omega^2\right)}$$

$$\dot{U}=\frac{Fe^{-i\omega t}}{-i\Delta-\omega}=\frac{Fe^{-i\omega t}}{\Delta^2+\omega^2}(\Delta-i\omega)$$

Energy goes in!

Where does it go?

Inhomogeneous Solution

$$u(t) = a\sin\Omega t + \frac{F}{\Omega^2 - \omega^2}\sin\omega t$$

Solution with zero initial excitation

$$a = -\frac{\omega}{\Omega}\frac{F}{\Omega^2 - \omega^2}$$

$$\therefore u_{\Omega \neq \omega} = \frac{F}{\Omega^2 - \omega^2}\left(\sin\omega t - \frac{\omega}{\Omega}\sin\Omega t\right)$$

No energy flow

$$\therefore u_{\Omega = \omega} = \frac{F}{\Omega^2 - \omega^2}\left(t\cos\omega t - \frac{\sin\omega t}{\omega}\right)$$

Resonant particles capture energy and oscillation generated out of phase

Oscillators Similtaneously Excited

$$u_i(t) = 1$$

$$\ddot{u} + \Omega^2 u = F e^{-i\omega t}$$

$$u(t) = \frac{F e^{-i\omega t}}{2\omega}\left(\frac{1}{\Omega - \omega} - \frac{1}{\Omega + \omega}\right)$$

Many oscillators distributed in frequency

$$\psi(\Omega) = \frac{1}{N}\frac{dN}{d\Omega}$$

$$U = \frac{\sum_{i=1}^{N} u_i}{N}$$

$$U = \frac{F e^{-i\omega t}}{2\omega}\int\left[\frac{1}{\Omega - \omega} - \frac{1}{\Omega + \omega}\right] d\Omega$$

for $\psi(\Omega) = \psi(-\Omega)$

$$U = \frac{F e^{-i\omega t}}{\omega}\int \frac{\psi(\Omega)}{\Omega - \omega} d\Omega$$