

Accelerator Physics

Landau Damping and Beam-Beam Effect

G. A. Krafft

Old Dominion University

Jefferson Lab

Lecture 10

Debye Length Picture*

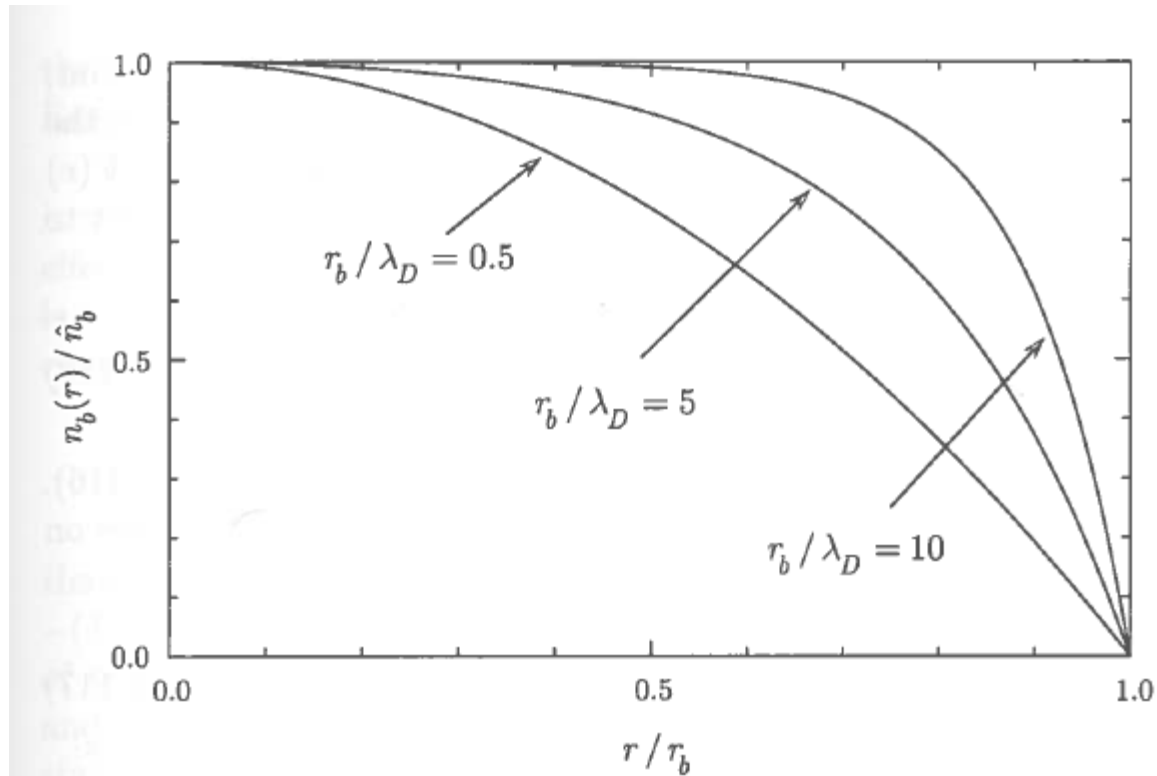


Figure 5.4. Plot of the normalized density profile $n_b(r)/\hat{n}_b$ versus r/r_b obtained from Eq. (5.115) for the choice of equilibrium distribution in Eq. (5.109). Here, the three cases correspond to the choices $r_b/\lambda_D = 0.5$, $r_b/\lambda_D = 5$ and $r_b/\lambda_D = 10$ [see Eq. (5.114)].

*Davidson and Qin

Collisionless (Landau) Damping



- Other important effect of thermal spreads in accelerator physics
- Longitudinal Plasma Oscillations (1 D)

$$\frac{\partial n}{\partial t} + \nabla \cdot v_z n = 0$$

$$\frac{dv_z}{dt} = \frac{-e}{m} E_z$$

$$\frac{\partial E_z}{\partial z} = \frac{-en}{\epsilon_0}$$

Linearized



$$\frac{\partial \delta n}{\partial t} + n_0 \frac{\partial \delta v_z}{\partial z} = 0$$

$$\frac{\partial \delta v_z}{\partial t} = \frac{-e}{m} \delta E_z$$

$$\frac{\partial \delta E_z}{\partial z} = \frac{-e \delta n}{\epsilon_0}$$

$$\frac{\partial^2 \delta n}{\partial t^2} = \frac{e n_0}{m} \frac{\partial \delta E_z}{\partial z} = -\frac{e^2 n_0}{\epsilon_0 m} \delta n$$

$$\delta n \propto e^{\pm i \omega_p t} \quad \omega_p = \sqrt{\frac{e^2 n_0}{\epsilon_0 m}}$$

In fluid limit *plasma* oscillations are undamped

Vlasov Analysis of Problem

$$F_e(z, p_z, t) = \iint \Psi(z, \vec{p}) dp_x dp_y$$

$$\left\{ \frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} + e \frac{\partial \phi}{\partial z} \frac{\partial}{\partial p_z} \right\} F_e = 0$$

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{e}{\epsilon_0} \left(\int F_e dp_z - n_i \right)$$

0th order solution

$$F_e = F_0(p_z), \quad \phi_0 = 0$$

linearized

$$\left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} \right) \delta F_e = -e \frac{\partial F_0}{\partial p_z} \frac{\partial \delta \phi}{\partial z}$$

$$\frac{\partial^2 \delta \phi}{\partial z^2} = \frac{e}{\epsilon_0} \int \delta F_e dp_z$$

Initial Value Problem

- Laplace in t and Fourier in z

$$\hat{F}(\omega) = \int_0^{\infty} dt e^{i\omega t} F(t) \quad \text{Im}\omega \text{ large enough to converge}$$

$$F(t) = \frac{1}{2\pi} \int_C d\omega e^{-i\omega t} \hat{F}(\omega)$$

$$\int_0^{\infty} dt e^{i\omega t} \frac{d}{dt} F(t) = -i\omega \hat{F}(\omega) - F(t=0)$$

$$\delta\phi(z, t) = \sum_{l=-\infty}^{\infty} \delta\hat{\phi}(l, t) e^{2\pi i l z / L}$$

$$\delta F_e(l, p_z, \omega) = \frac{i\delta F_e(l, p_z, t=0)}{\omega - v_z(2\pi l / L)} + \frac{e 2\pi l \partial F_0 / \partial p_z}{L(\omega - v_z(2\pi l / L))} \delta\hat{\phi}(l, \omega)$$

$$\delta\hat{\phi}(l, \omega) = - \left(\frac{L}{2\pi l} \right)^2 \left[\frac{e^2}{\epsilon_0} \int_{-\infty}^{\infty} dp_z \frac{2\pi l \partial F_0(p_z) / \partial p_z}{L(\omega - v_z(2\pi l / L))} \delta\hat{\phi}(l, \omega) + \frac{ei}{\epsilon_0} \int_{-\infty}^{\infty} dp_z \frac{\delta F_e(l, p_z, t=0)}{\omega - v_z(2\pi l / L)} \right]$$

Dielectric function



- Landau (self-consistent) dielectric function

$$D(l, \omega) \delta \hat{\phi}(l, \omega) = N(l, \omega)$$

$$D(l, \omega) = 1 + \frac{e^2}{\epsilon_0} \left(\frac{L}{2\pi l} \right) \int_{-\infty}^{\infty} dp_z \frac{\partial F_0(p_z) / \partial p_z}{\omega - v_z (2\pi l / L)}$$

- Solution for normal modes are

$$D(l, \omega) = 0$$

$$\begin{aligned} D(l, \omega) &= 1 - \frac{e^2}{\epsilon_0 m} \int_{-\infty}^{\infty} dp_z \frac{F_0(p_z)}{(\omega - v_z (2\pi l / L))^2} \\ &= 1 - \omega_p^2 \int_{-\infty}^{\infty} dp_z \frac{F_0(p_z) / n_0}{(\omega - v_z (2\pi l / L))^2} \end{aligned}$$

Collisionless Damping



- For Lorentzian distribution

$$\begin{aligned}\frac{F_0}{n_0} &= \frac{\Delta}{\pi [p_z^2 + \Delta^2]} \\ 1 &= \omega_p^2 \int_{-\infty}^{\infty} dp_z \frac{1}{(\omega - p_z (2\pi l / Lm))^2} \frac{\Delta}{\pi [p_z^2 + \Delta^2]} \\ &= \frac{\omega_p^2}{(\omega + i(2\pi l / Lm) \Delta)^2}\end{aligned}$$

- Landau damping rate

$$\omega = \pm \omega_p - i \frac{2\pi l}{L} \Delta$$

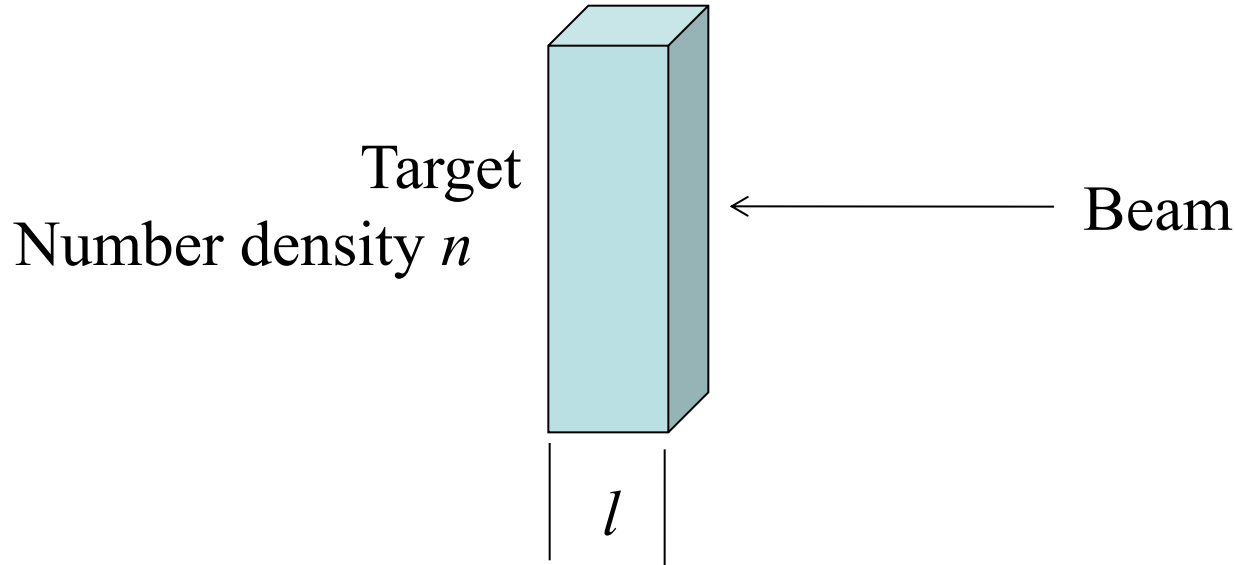
Luminosity and Beam-Beam Effect



- Luminosity Defined
- Beam-Beam Tune Shift
- Luminosity Tune-shift Relationship (Krafft-Ziemann Thm.)
- Beam-Beam Effect

Events per Beam Crossing

- In a nuclear physics experiment with a beam crossing through a thin fixed target

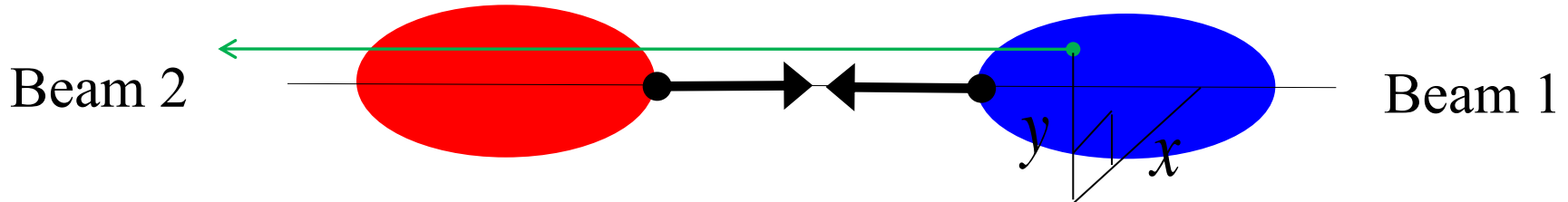


- Probability of single event, per beam particle passage is

$$P = n\sigma l$$

- σ is the “cross section” for the process (area units)

Collision Geometry



- Probability an event is generated by a single particle of Beam 1 crossing Beam 2 bunch with Gaussian density*

$$P = \sigma \frac{N_2 \exp\left(-x^2 / 2\sigma_{2x}^2\right) \exp\left(-y^2 / 2\sigma_{2y}^2\right)}{(2\pi)^{3/2} \sigma_{2x} \sigma_{2y} \sigma_{2z}} \int_{-\infty}^{\infty} \exp\left(-z^2 / 2\sigma_{2z}^2\right) dz$$
$$= \frac{N_2 \exp\left(-x^2 / 2\sigma_{2x}^2\right) \exp\left(-y^2 / 2\sigma_{2y}^2\right)}{2\pi\sigma_{2x}\sigma_{2y}} \sigma$$

* This expression still correct when relativity done properly

Collider Luminosity



- Probability an event is generated by a Beam 1 bunch with Gaussian density crossing a Beam 2 bunch with Gaussian density

$$P = \frac{N_1 N_2}{2\pi \sqrt{\sigma_{1x}^2 + \sigma_{2x}^2} \sqrt{\sigma_{1y}^2 + \sigma_{2y}^2}} \sigma$$

- Event rate with equal transverse beam sizes

$$\frac{dN}{dt} = \frac{f N_1 N_2}{4\pi \sigma_x \sigma_y} \sigma = \mathcal{L} \sigma$$

- Luminosity

$$\mathcal{L} = \frac{f N_1 N_2}{4\pi \sigma_x \sigma_y} \sim 10^{33} \text{ sec}^{-1} \text{ cm}^{-2},$$

for $f = 100$ MHz, $N_1 = N_2 = 10^{10}$, $\sigma_x = \sigma_y = 10$ microns

Beam-Beam Tune Shift



- As we've seen previously, in a ring accelerator the number of transverse oscillations a particle makes in one circuit is called the “betatron tune” Q .
- Any deviation from the design values of the tune (in either the horizontal or vertical directions), is called a “tune shift”. For long term stability of the beam in a ring accelerator, the tune must be highly controlled.

$$\begin{aligned} M_{tot} &= \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} \cos \mu & \beta^* \sin \mu \\ -\sin \mu / \beta^* & \cos \mu \end{pmatrix} \\ &= \begin{pmatrix} \cos \mu & \beta^* \sin \mu \\ -\cos \mu / f - \sin \mu / \beta^* & \cos \mu - (\beta^* / f) \sin \mu \end{pmatrix} \end{aligned}$$

$$\cos(\mu + \Delta\mu) = \frac{\text{Tr}(M_{tot})}{2} = \cos \mu - \frac{\beta^*}{2f} \sin \mu$$

$$\xi = \Delta Q = \frac{\Delta\mu}{2\pi} = \frac{\beta^*}{4\pi f} \quad \beta^* \ll f$$

Bessetti-Erskine Solution



- 2-D potential of Bi-Gaussian transverse distribution

$$\rho(x, y) = \frac{Q'}{2\pi\sigma_x\sigma_y} \exp\left(-\frac{x^2}{2\sigma_x^2}\right) \exp\left(-\frac{y^2}{2\sigma_y^2}\right)$$

- Potential Theory gives solution to Poisson Equation

$$\nabla^2 \phi = \frac{\rho(x, y)}{\epsilon_0}$$

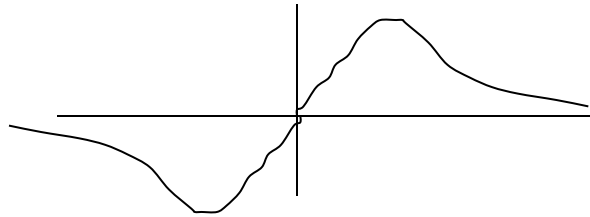
$$\phi(x, y) = \frac{Q'}{4\pi\epsilon_0} \int_0^\infty \frac{\exp\left(-\frac{x^2}{2\sigma_x^2 + q}\right) \exp\left(-\frac{y^2}{2\sigma_y^2 + q}\right)}{\sqrt{2\sigma_x^2 + q} \sqrt{2\sigma_y^2 + q}} dq$$

- Bassetti and Erskine manipulate this to

$$E_x = \frac{Q'}{2\epsilon_0 \sqrt{2\pi(\sigma_x^2 - \sigma_y^2)}} \operatorname{Im} \left[w \left(\frac{x+iy}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \right) - \exp \left(-\frac{x^2}{2\sigma_x^2} - \frac{x^2}{2\sigma_y^2} \right) w \left(\frac{x \left(\frac{\sigma_y}{\sigma_x} \right) + iy \left(\frac{\sigma_x}{\sigma_y} \right)}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \right) \right]$$

$$E_y = \frac{Q'}{2\epsilon_0 \sqrt{2\pi(\sigma_x^2 - \sigma_y^2)}} \operatorname{Re} \left[w \left(\frac{x+iy}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \right) - \exp \left(-\frac{x^2}{2\sigma_x^2} - \frac{x^2}{2\sigma_y^2} \right) w \left(\frac{x \left(\frac{\sigma_y}{\sigma_x} \right) + iy \left(\frac{\sigma_x}{\sigma_y} \right)}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \right) \right]$$

$w(z)$ Complex error function



- We need 2-D linear field for small displacements

$$E_x(x, 0) = -\frac{\partial \phi}{\partial x} \doteq \frac{Q'x}{2\pi\epsilon_0} \int_0^\infty \frac{1}{\left(\sqrt{2\sigma_x^2 + q}\right)^{3/2} \sqrt{2\sigma_y^2 + q}} dq$$

- Can do the integral analytically

$$\begin{aligned}
 \int_0^{\infty} \frac{1}{\left(\sqrt{2\sigma_x^2 + q}\right)^3 \sqrt{2\sigma_y^2 + q}} dq &= \int_{\sigma_x^2 + \sigma_y^2}^{\infty} \frac{\sigma_y^2 - \sigma_x^2 + q'}{\left(\sqrt{\sigma_x^2 - \sigma_y^2 + q'}\right)^3 \left(\sqrt{\sigma_y^2 - \sigma_x^2 + q'}\right)^3} dq' \\
 &= \int_{\sigma_x^2 + \sigma_y^2}^{\infty} \frac{\sigma_y^2 - \sigma_x^2 + q'}{\left(q'^2 - (\sigma_y^2 - \sigma_x^2)^2\right)^{3/2}} dq' = \left[-\frac{(\sigma_y^2 - \sigma_x^2) q'}{\left(\sigma_y^2 - \sigma_x^2\right)^2 \left(q'^2 - (\sigma_y^2 - \sigma_x^2)^2\right)^{1/2}} - \frac{1}{\left(q'^2 - (\sigma_y^2 - \sigma_x^2)^2\right)^{1/2}} \right] \Bigg|_{\sigma_x^2 + \sigma_y^2}^{\infty} \\
 &= -\frac{1}{\sigma_y^2 - \sigma_x^2} + \frac{\sigma_y^2 + \sigma_x^2}{\sigma_y^2 - \sigma_x^2} \frac{1}{2\sigma_x \sigma_y} + \frac{1}{2\sigma_x \sigma_y} = \frac{-2\sigma_x \sigma_y + \sigma_y^2 + \sigma_x^2 + \sigma_y^2 - \sigma_x^2}{\left(\sigma_y^2 - \sigma_x^2\right) 2\sigma_x \sigma_y} = \frac{1}{\left(\sigma_x + \sigma_y\right) \sigma_x}
 \end{aligned}$$

- Similarly for the y -direction

$$E_y(0, y) = -\frac{\partial \phi}{\partial y} \doteq \frac{Q'y}{2\pi\epsilon_0} \frac{1}{\sigma_y (\sigma_x + \sigma_y)}$$

Linear Beam-Beam Kick



- Linear kick received after interaction with bunch

$$\Delta(\gamma_1 \beta_{1x} mc) = q_1 \int_{-\infty}^{\infty} \left(\vec{E}_{2x} + (\vec{v} \times \vec{B})_{2x} \right) (\vec{x}_1(t), t) dt$$

by relativity, for oppositely moving beams

$$\Delta \gamma \beta_{1x} mc = q_1 (1 + \beta_{1z} \beta_{2z}) \int_{-\infty}^{\infty} \left(\vec{E}_{2x} \right) (\vec{x}_1(t), t) dt$$

Following linear Bassetti-Erskine model

$$E_{2x}(x, 0, z, t) = \frac{q_2 x}{2\pi\epsilon_0} \frac{1}{\sigma_x (\sigma_x + \sigma_y)} \frac{1}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{(z - \beta_{2z} ct)^2}{2\sigma_z^2}\right)$$

q_1 moves with $\vec{x}(t) = (x, 0, -\beta_{1z} ct)$

Linear Beam-Beam Tune Shift



$$\therefore \Delta\gamma\beta_{1x}mc = q_1 \frac{(1 + \beta_{1z}\beta_{2z})}{\beta_{1z} + \beta_{2z}} \frac{q_2 x}{2\pi\epsilon_0 c} \frac{1}{\sigma_x (\sigma_x + \sigma_y)}$$

$$1/f = \frac{2N_2}{\gamma_1} \frac{(1 + \beta_{1z}\beta_{2z})}{\beta_{1z} + \beta_{2z}} \frac{r_1}{\sigma_x (\sigma_x + \sigma_y)} \quad r_1 = \frac{e^2}{4\pi\epsilon_0 m_1 c^2}$$

$$1/f \doteq \frac{2N_2 r_1}{\gamma_1 \sigma_x (\sigma_x + \sigma_y)} \quad \text{Both beams relativistic}$$

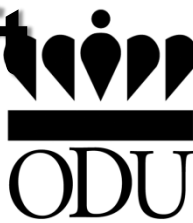
From linear Bassetti-Erskine model, and replacing the beam size

$$\xi_x^1 = \frac{N_2 r_1}{2\pi\gamma_1} \frac{1}{\epsilon_x^1 (1 + \sigma_y / \sigma_x)} \quad \xi_y^1 = \frac{N_2 r_1}{2\pi\gamma_1} \frac{1}{\epsilon_y^i (1 + \sigma_y / \sigma_x) (\sigma_x / \sigma_y)}$$

Argument entirely symmetric wrt choice of bunch 1 and 2

$$\xi_x^i = \frac{N_{\bar{i}} r_i}{2\pi\gamma_i} \frac{1}{\epsilon_x^i (1 + \sigma_y / \sigma_x)} \quad \xi_y^i = \frac{N_{\bar{i}} r_i}{2\pi\gamma_i} \frac{1}{\epsilon_y^i (1 + \sigma_y / \sigma_x) (\sigma_x / \sigma_y)}$$

Luminosity Beam-Beam tune-shift relationship



- Express Luminosity in terms of the (larger!) vertical tune shift (i either 1 or 2)

$$\mathcal{L} = \frac{fN_i \xi_y^i \gamma_i}{2r_i \beta_{iy}^*} \left(1 + \sigma_y / \sigma_x\right) = \frac{I_i}{e} \frac{\xi_y^i \gamma_i}{2r_i \beta_{iy}^*} \left(1 + \sigma_y / \sigma_x\right)$$

- Necessary, **but not sufficient**, for self-consistent design
- Expressed in this way, and given a known limit to the beam-beam tune shift, the only variables to manipulate to increase luminosity are the stored current, the aspect ratio, and the β^* (beta function value at the interaction point)
- Applies to ERL-ring colliders, stored beam (ions) only

Luminosity-Deflection Theorem



- Luminosity-tune shift formula is linearized version of a much more general formula discovered by Krafft and generalized by V. Ziemann.
- Relates easy calculation (luminosity) to a hard calculation (beam-beam force), and contains all the standard results in beam-beam interaction theory.
- Based on the fact that the relativistic beam-beam force is almost entirely transverse, i. e., 2-D electrostatics applies.

2-D Electrostatics Theorem



$$\vec{E}(\vec{x}) = \frac{2Q'}{4\pi\epsilon_0} \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^2}$$

$$\vec{F}'_{21} = -\vec{F}'_{12} = \frac{1}{2\pi\epsilon_0} \iint \rho_2(\vec{x}_2) \frac{\vec{x}_2 - \vec{x}_1}{|\vec{x}_2 - \vec{x}_1|} \rho_1(\vec{x}_1) d^2\vec{x}_1 d^2\vec{x}_2 \quad 1 \text{ on } 2$$

$$n_1(\vec{x}_1) = \rho_1(\vec{x}_1) / Q'_1 \quad n_2(\vec{x}_2) = \rho_1(\vec{x}_2 + \vec{b}) / Q'_1 \quad \text{zero centered}$$

$$Q'_i = \iint \rho_i(\vec{x}) d^2\vec{x} \quad \vec{b} = \iint \vec{x} \rho_2(\vec{x}) d^2\vec{x} / Q'_2$$

$$\vec{F}'_{21} = -\vec{F}'_{12} = \frac{Q'_1 Q'_2}{2\pi\epsilon_0} \iint n_2(\vec{x}_2) \frac{\vec{x}_1 + \vec{b} - \vec{x}_2}{|\vec{x}_1 + \vec{b} - \vec{x}_2|^2} n_1(\vec{x}_1) d^2\vec{x}_1 d^2\vec{x}_2$$

$$\vec{\nabla}_{\vec{b}} \cdot \frac{\vec{x}_1 + \vec{b} - \vec{x}_2}{|\vec{x}_1 + \vec{b} - \vec{x}_2|^2} = 2\pi\delta(x_2 + b_x + x_1)\delta(y_2 + b_y + y_1)$$

$$\vec{\nabla}_{\vec{b}} \cdot \vec{F}'_{21} = \frac{1}{\epsilon_0} \iint \rho_2(\vec{x} + \vec{b})\rho_1(\vec{x})d^2\vec{x}$$

Generalizes $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ (take $\rho_2(\vec{x}) \propto \delta^2(\vec{x} + \vec{b})$)

Transverse interaction in the beam-beam problem

$$\Delta p_1 = \frac{q_1 q_2}{2\pi\epsilon_0 c} \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^2}$$

$$\vec{D}(\vec{b}) = \Delta\gamma_1 \vec{\beta}_1 = -\Delta m_2 \gamma_2 \vec{\beta}_2 / m_1$$

$$= \frac{q_1 q_2}{m_1 c^2} \iint n_2(\vec{x}_2) \frac{\vec{x}_1 - \vec{x}_2 - \vec{b}}{|\vec{x}_1 - \vec{x}_2 - \vec{b}|^2} n_1(\vec{x}_1) d^2 \vec{x}_1 d^2 \vec{x}_2$$

$$\vec{\nabla}_{\vec{b}} \cdot \vec{D}(\vec{b}) = 4\pi N_2 r_e \iint n_2(\vec{x} - \vec{b}) n_1(\vec{x}) d^2 \vec{x} \quad r_e = \frac{e^2}{4\pi\epsilon_0 m c^2}$$

$$L(\vec{b}) = N_1 N_2 \iint n_2(\vec{x} - \vec{b}) n_1(\vec{x}) d^2 \vec{x}$$

$$L(\vec{b}) = \frac{N_1}{4\pi r_e} \vec{\nabla}_{\vec{b}} \cdot \vec{D}(\vec{b})$$

$$L(\vec{b}) = -\frac{N_2}{4\pi r_e} \vec{\nabla}_{\vec{b}} \cdot (\Delta\gamma_2 \vec{\beta}_2)$$

$$\begin{pmatrix} D_x \\ D_y \end{pmatrix} = \frac{\gamma_1}{2f} \begin{pmatrix} \sigma_y / \sigma_x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_x \\ b_y \end{pmatrix}$$

$$L = \frac{N_1 \gamma \xi}{2r_e \beta^*} (1 + \sigma_y / \sigma_x) \quad \text{as before}$$

Maximum when

$$\frac{\partial}{\partial b_x} \left[\frac{\partial D_x}{\partial b_x} \right] = 0, \quad \frac{\partial}{\partial b_y} \left[\frac{\partial D}{\partial b_y} \right] = 0$$

Luminosity-Deflection Pairs



- Round Beam Fast Model

$$\vec{D}(\vec{b}) = \frac{2N_2 r_e \vec{b}}{\sigma^2 + b^2} \quad L(\vec{b}) = \frac{N_1 N_2 \sigma^2}{\pi (\sigma^2 + b^2)^2}$$

- Gaussian Macroparticles

$$\vec{D}(\vec{b}) = \vec{D}_{Bassetti_Erskine}(\vec{b}; \sqrt{\sigma_{1x}^2 + \sigma_{2x}^2}; \sqrt{\sigma_{1y}^2 + \sigma_{2y}^2})$$

$$L(\vec{b}) = \frac{N_1 N_2}{2\pi \sqrt{\sigma_{1x}^2 + \sigma_{2x}^2} \sqrt{\sigma_{1y}^2 + \sigma_{2y}^2}} \exp\left(-\frac{b_x^2}{\sigma_{1x}^2 + \sigma_{2x}^2}\right) \exp\left(-\frac{b_y^2}{\sigma_{1y}^2 + \sigma_{2y}^2}\right)$$

- Smith-Laslett Model

$$\vec{D}(\vec{b}) = \frac{2N_2 r_e \vec{b}}{\hat{b}^2 AB} \left\{ \frac{(4\hat{b}^2 + 2\hat{b}^4)}{(4\hat{b}^2 + \hat{b}^4)} - \frac{4\hat{b}^2}{(4\hat{b}^2 + \hat{b}^4)^{3/2}} \left\{ \sinh^{-1} \left[\frac{\hat{b}^3}{2} + \frac{3\hat{b}}{2} \right] + \sinh^{-1} \left[\frac{\hat{b}}{2} \right] \right\} \right\}$$

$$L(\vec{b}) = \frac{N_1 N_2}{\pi AB} \left\{ \frac{(2\hat{b}^2 - 4)\hat{b}^2}{(4\hat{b}^2 + \hat{b}^4)^2} - \frac{4\hat{b}^2 (1 + \hat{b}^2)}{(4\hat{b}^2 + \hat{b}^4)^{5/2}} \left\{ \sinh^{-1} \left[\frac{\hat{b}^3}{2} + \frac{3\hat{b}}{2} \right] + \sinh^{-1} \left[\frac{\hat{b}}{2} \right] \right\} \right\}$$

$$\hat{b}^2 = \left(\frac{b_x}{A}\right)^2 + \left(\frac{b_y}{B}\right)^2$$