Physics 451/551
Theoretical Mechanics

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Lecture 19

Problem 9.4

- Sound in wind! Linearize about uniformly moving fluid

$$\rho(\vec{x},t)=\rho_0+\rho' \quad \vec{v}(\vec{x},t)=\vec{u}+\vec{v}' \quad p(\vec{x},t)=p_0+p'$$

- Continuity equation

$$\frac{\partial\rho'}{\partial t}+\nabla\cdot(\rho_0\vec{v}'+\rho'\vec{u})=0\rightarrow\frac{1}{\rho_0}\left[\frac{\partial}{\partial t}+\vec{u}\cdot\frac{\partial}{\partial\vec{x}}\right]\rho'+\nabla\cdot\vec{v}'=0$$

- Velocity equation

$$\left[\frac{\partial}{\partial t}+\vec{u}\cdot\frac{\partial}{\partial\vec{x}}\right]\vec{v}'=-\frac{1}{\rho_0}\nabla p'$$

- Isentropic equation of state the same (go into the moving frame)

$$p_0+p'=p(s,\rho_0+\rho')\approx p_0+\left.\frac{\partial p}{\partial\rho}\right|_s\rho'\rightarrow p'=c^2\rho'$$

Flow Still Irrotational

- Take curl of velocity equation. Curl commutes with uniform velocity. Conclude flow still irrotational

$$\vec{v}' = -\nabla\Phi \rightarrow \left[\frac{\partial}{\partial t} + \vec{u}\cdot\frac{\partial}{\partial \vec{x}}\right]\vec{v}' = -\nabla\left[\frac{\partial}{\partial t} + \vec{u}\cdot\frac{\partial}{\partial \vec{x}}\right]\Phi$$

$$\rightarrow \left[\frac{\partial}{\partial t} + \vec{u}\cdot\frac{\partial}{\partial \vec{x}}\right]\Phi = \frac{p'}{\rho_0} = c^2\frac{\rho'}{\rho_0} \rightarrow \left[\frac{\partial}{\partial t} + \vec{u}\cdot\frac{\partial}{\partial \vec{x}}\right]^2\Phi = c^2\frac{1}{\rho_0}\left[\frac{\partial}{\partial t} + \vec{u}\cdot\frac{\partial}{\partial \vec{x}}\right]\rho'$$

- Scalar wave equation now

$$\frac{1}{c^2}\frac{D^2\Phi}{Dt^2} \equiv \frac{1}{c^2}\left[\frac{\partial}{\partial t} + \vec{u}\cdot\frac{\partial}{\partial \vec{x}}\right]^2\Phi = \nabla\cdot\left(\nabla\Phi\right)$$

Wave moves before only in the rest frame of medium

- Boundary conditions same as before

Problem 9.13

- Fourier Transform the solution for 9.4

$$\frac{1}{c^2}\left[\frac{\partial}{\partial t}+\vec{u}\cdot\frac{\partial}{\partial\vec{x}}\right]^2\Phi-\nabla^2\Phi=S\left(\vec{r},t\right)$$

$$-\frac{1}{c^2}\left[\omega+i\eta-\vec{u}\cdot\vec{k}\right]^2\tilde{\Phi}\left(\vec{k},\omega\right)+k^2\tilde{\Phi}\left(\vec{k},\omega\right)=\tilde{S}\left(\vec{k},\omega\right)$$

$$G\left(\vec{r}-\vec{r}',t-t'\right)=\frac{1}{(2\pi)^4}\iiint\int\frac{e^{i\vec{k}\cdot\left(\vec{r}-\vec{r}'\right)}e^{-i\omega\left(t-t'\right)}}{k^2-\left[\omega+i\eta-\vec{u}\cdot\vec{k}\right]^2/c^2}d^3k\,d\omega$$

- For point source of given frequency at origin

$$G_{\omega_0}\left(\vec{r}\right)=\frac{1}{(2\pi)^3}\iiint\frac{e^{i\vec{k}\cdot\vec{r}}}{k^2-\left[\omega_0+i\eta-\vec{u}\cdot\vec{k}\right]^2/c^2}d^3k\,e^{-i\omega_0 t}$$

- Next show

$$k^2 - \left[\omega_0 + i\eta - \vec{u}\cdot\vec{k}\right]^2 / c^2$$

$$= k_x^2 + k_y^2 + \left(1 - \frac{u^2}{c^2}\right)\left[k_z + \frac{(\omega_0 + i\eta)u}{c^2 - u^2}\right]^2 - \frac{(\omega_0 + i\eta)^2}{c^2 - u^2}$$

- Change of variable

$$k_z' = \frac{\sqrt{c^2 - u^2}}{c}\left(k_z + \frac{(\omega_0 + i\eta)u}{c^2 - u^2}\right)$$

$$G_{\omega_0}(\vec{r}) = \frac{c}{\sqrt{c^2 - u^2}}\frac{1}{(2\pi)^3}\iiint \frac{e^{i\left(k_x x + k_y y + k_z' z\left(c/\sqrt{c^2-u^2}\right)\right)}}{k_x^2 + k_y^2 + k_z'^2 - [\omega_0 + i\eta]^2/\left(c^2 - u^2\right)} d^2k\, dk_z' e^{-i\omega_0 uz/\left(c^2-u^2\right)} e^{-i\omega_0 t}$$

$$= \frac{c}{\sqrt{c^2 - u^2}}\frac{e^{i\omega_0 R/\sqrt{c^2-u^2}}}{4\pi R} e^{-i\omega_0 uz/\left(c^2-u^2\right)} e^{-i\omega_0 t} \qquad R^2 = x^2 + y^2 + \frac{z^2 c^2}{c^2 - u^2}$$

- Somwhat clearer physical picture if change of variable on frequency

$$G(\vec{r}-\vec{r}',t-t')=\frac{1}{(2\pi)^4}\iiiint\frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}e^{-i\omega(t-t')}}{k^2-\left[\omega+i\eta-\vec{u}\cdot\vec{k}\right]^2/c^2}d^3k\,d\omega$$

$$=\frac{1}{(2\pi)^4}\iiiint\frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}e^{-i\vec{u}\cdot\vec{k}(t-t')}e^{-i\omega'(t-t')}}{k^2-[\omega'+i\eta]^2/c^2}d^3k\,d\omega'$$

- For $u > c$ pole structure is somewhat different

$$\frac{u^2-c^2}{c^2}\left(k_z-\frac{(\omega_0+i\eta)u}{u^2-c^2}\right)^2=k_x^2+k_y^2+\frac{(\omega_0+i\eta)^2}{u^2-c^2}$$

$$k_z=\frac{(\omega_0+i\eta)u}{u^2-c^2}\pm\frac{c}{\sqrt{u^2-c^2}}\sqrt{k_x^2+k_y^2+\frac{(\omega_0+i\eta)^2}{u^2-c^2}}$$

- For $z < 0$ close in LHP, no poles so response 0
- for $z > 0$ close in UHP, have two poles

$$G_{\omega_0}(\vec{r}) = -\frac{c^2}{(u^2-c^2)(2\pi)^3}\iiint \frac{e^{i(k_x x + k_y y + k_z z)}}{\left(k_z - \frac{(\omega_0+i\eta)u}{u^2-c^2}\right)^2 - \frac{c^2}{u^2-c^2}\left(k_x^2+k_y^2+\frac{(\omega_0+i\eta)^2}{u^2-c^2}\right)} dk_x dk_y dk_z e^{-i\omega_0 t}$$

$$= -\frac{c^2 i}{(u^2-c^2)(2\pi)^2}\iint \frac{e^{i(k_x x+k_y y)} e^{i\frac{c^2}{u^2-c^2}\left(k_x^2+k_y^2+\frac{(\omega_0+i\eta)^2}{u^2-c^2}\right)} - e^{i(k_x x+k_y y)} e^{-i\frac{c^2}{u^2-c^2}\left(k_x^2+k_y^2+\frac{(\omega_0+i\eta)^2}{u^2-c^2}\right)}}{2\frac{c}{\sqrt{u^2-c^2}}\left(k_x^2+k_y^2+\frac{(\omega_0+i\eta)^2}{u^2-c^2}\right)^{1/2}} dk_x dk_y e^{i\frac{(\omega_0+i\eta)u}{u^2-c^2}z} e^{-i\omega_0 t}$$

$$= \frac{c}{\sqrt{u^2-c^2}}\frac{1}{2\pi}\int \frac{J_0(k_\perp \rho)\sin\left(\frac{c}{\sqrt{u^2-c^2}}\left(k_\perp^2+\frac{(\omega_0+i\eta)^2}{u^2-c^2}\right)^{1/2} z\right)}{\left(k_\perp^2+\frac{(\omega_0+i\eta)^2}{u^2-c^2}\right)^{1/2}} k_\perp dk_\perp e^{i\frac{(\omega_0+i\eta)u}{u^2-c^2}z} e^{-i\omega_0 t} \qquad J_0(k_\perp\rho) = \frac{1}{2\pi}\int_0^{2\pi} e^{ik_\perp\rho\cos\phi} d\phi$$

$$\rho = \sqrt{x^2+y^2}$$

$$G_{\omega_0}(\vec{r}) = \frac{c}{\sqrt{u^2-c^2}}\frac{1}{2\pi}\int_0^\infty \frac{J_0(k_\perp\rho)\sin\left(\frac{c}{\sqrt{u^2-c^2}}\left(k_\perp^2+\frac{(\omega_0+i\eta)^2}{u^2-c^2}\right)^{1/2} z\right)}{\left(k_\perp^2+\frac{(\omega_0+i\eta)^2}{u^2-c^2}\right)^{1/2}} k_\perp dk_\perp e^{i\frac{(\omega_0+i\eta)u}{u^2-c^2}z} e^{-i\omega_0 t}$$

Gradshteyn and Ryzhik: Formula 6.737.5

$$\int_0^\infty \frac{J_0(cx)\sin\left(a\left(x^2+b^2\right)^{1/2}\right)}{\left(x^2+b^2\right)^{1/2}} x dx = \begin{cases} 0 & 0<a<c \\ \cos\left(b\left(a^2-c^2\right)^{1/2}\right)/\left(a^2-c^2\right)^{1/2} & 0<c<a \end{cases}$$

$$G_{\omega_0}(\vec{r}) = \frac{c}{\sqrt{u^2-c^2}}\frac{e^{i\frac{(\omega_0+i\eta)u}{u^2-c^2}z} e^{-i\omega_0 t}}{2\pi} \times \begin{cases} 0 & \frac{cz}{\sqrt{u^2-c^2}} < \rho \\ \cos\left(\frac{(\omega_0+i\eta)}{\sqrt{u^2-c^2}} R\right)/R & \frac{cz}{\sqrt{u^2-c^2}} > \rho \end{cases}$$

$$R^2 = \frac{c^2 z^2}{u^2-c^2} - x^2 - y^2$$

Accoustic point source

- Source function is

$$S(\vec{r},t) = S_0 \delta(\vec{r}) e^{-i\omega t}$$

$$G(\vec{r}-\vec{r}',t-t') = \frac{1}{(2\pi)^4} \iiint\int \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} e^{-i\omega(t-t')}}{k^2 - [\omega + i\eta]^2 / c^2} d\omega$$

$$\Phi(\vec{r},t) = \frac{1}{(2\pi)^3} \iiint \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}{k^2 - [\omega_0 + i\eta]^2 / c^2} d^3k e^{-i\omega_0 t} = \frac{e^{i\omega_0 r/c}}{4\pi r} e^{-i\omega_0 t}$$

- Amplitude is uniform in angle (spherical wave). The energy flux is

$$j_e = \frac{1}{2} c k^2 \rho_0 \frac{S_0^2}{16\pi^2 r^2} \hat{r} = \frac{dP_0}{d\Omega}$$

$$P_0 = 4\pi r^2 j_e = \frac{1}{8\pi} c k^2 \rho_0 S_0^2$$

Two point sources

$$\Phi(\vec{r},t) = S_0\left[\frac{e^{i\omega|\vec{r}-d\hat{z}|/c}e^{i\alpha}}{4\pi|\vec{r}-d\hat{z}|} + \frac{e^{i\omega|\vec{r}+d\hat{z}|/c}e^{-i\alpha}}{4\pi|\vec{r}+d\hat{z}|}\right]e^{-i\omega t}$$

$$|\Phi|^2(\vec{r},t) \doteq \frac{S_0^2}{16\pi^2}\left[2+2\cos\left(\frac{\omega}{c}\left[|\vec{r}+d\hat{z}|-|\vec{r}-d\hat{z}|\right]-2\alpha\right)\right]$$

$$= \frac{S_0^2}{16\pi^2}\left[2+2\cos\left(2kd\cos\theta-2\alpha\right)\right]$$

$$\therefore \frac{dP_0}{d\Omega} = \frac{P_0}{4\pi}\left[2+2\cos\left(2kd\cos\theta-2\alpha\right)\right]$$

$$P_0 = 2P_0\left[1+\frac{1}{4\pi}\int d\cos\theta\, d\phi \cos\left(2kd\cos\theta - 2\alpha\right)\right]$$

$$= 2P_0\left[1+\frac{1}{4kd}\int_{-1}^{1} d\left(2kd\cos\theta - 2\alpha\right)\cos\left(2kd\cos\theta - 2\alpha\right)\right]$$

$$= 2P_0\left[1+\frac{1}{4kd}\left[\sin\left(2kd - 2\alpha\right) - \sin\left(-2kd - 2\alpha\right)\right]\right]$$

$$= 2P_0\left[1+\frac{1}{2kd}\left[\sin\left(2kd\right)\cos\left(2\alpha\right)\right]\right]$$