Physics 451/551
Theoretical Mechanics

G. A. Krafft
Old Dominion University
Jefferson Lab

Lecture 18

Sound Waves

- Properties of Sound
  - Requires medium for propagation
  - Mainly longitudinal (displacement along propagation direction)
  - Wavelength much longer than interatomic spacing so can treat medium as continuous
- Fundamental functions
  - Mass density $\rho(x,y,z,t)$
  - Velocity field $\vec{v}(x,y,z,t)$
- Two fundamental equations
  - Continuity equation (Conservation of mass)
  - Velocity equation (Conservation of momentum)
    - Newton's Law in disguise

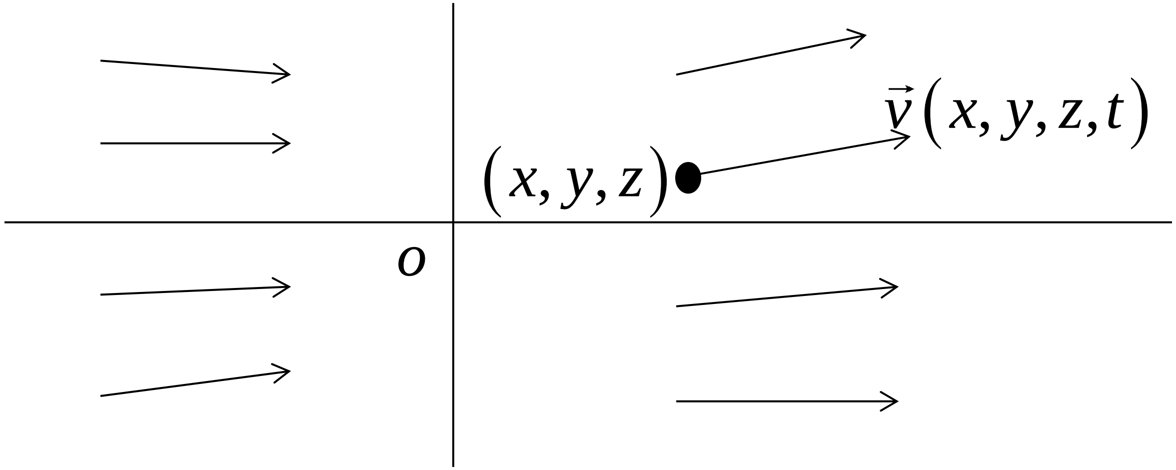
Fundamental Functions

- Density $\rho(x,y,z)$, mass per unit volume

$$\rho\left(x,y,z,t\right)=\lim_{\Delta V\to 0}\frac{\Delta M}{\Delta V}$$

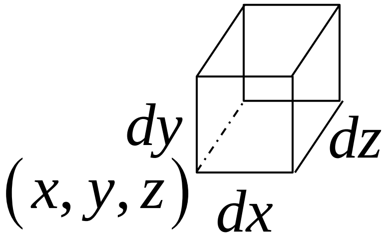
$$dM=\rho\left(x,y,z,t\right)dxdydz$$

- Velocity field $\vec{v}\left(x,y,z,t\right)$



Continuity Equation

- Consider mass entering differential volume element



- Mass entering box in a short time Δt

$$
\begin{aligned}
&\left[\rho v_x(x,y,z,t) - \rho v_x(x+dx,y,z,t)\right] dydz\Delta t \\
&+\left[\rho v_y(x,y,z,t) - \rho v_y(x,y+dy,z,t)\right] dzdx\Delta t \\
&+\left[\rho v_z(x,y,z,t) - \rho v_z(x,y,z+dz,t)\right] dxdy\Delta t \\
&=\left[\rho(x,y,z,t+\Delta t) - \rho(x,y,z,t)\right] dxdydz = -\Delta t \int_{\partial dV} \omega^2_{\rho\vec{v}}
\end{aligned}
$$

- Take limit $\Delta t \rightarrow 0$

$$\frac{\partial \rho}{\partial t} dV = \int_{dV} \frac{\partial \rho}{\partial t} dxdydz = -\int_{dV} \nabla \cdot \left(\rho \vec{v} \right) dxdydz$$

- By Stoke's Theorem. Because true for all *dV*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{v} \right) = 0$$

- Mass current density (flux) (kg/(sec m^2))

$$J_m = \rho \vec{v}$$

- Sometimes rendered in terms of the total time derivative (moving along with the flow)

$$\frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho + \rho \nabla \cdot \vec{v} = 0 = \frac{d\rho}{dt} + \rho \nabla \cdot \vec{v}$$

- Incompressible flow $\nabla \cdot \vec{v} = 0$ and ρ constant

Pressure Scalar

- Displace material from a small volume dV with sides given by dA. The pressure p is defined to the force acting on the area element

$$p = \frac{dF}{dA}$$

  - Pressure is normal to the area element
  - Doesn't depend on orientation of volume
- External forces (e.g., gravitational force) must be balanced by a pressure gradient to get a stationary fluid in equilibrium
- Pressure force (per unit volume)

$$\vec{F}_{pr} = -\frac{\partial p}{\partial x}$$

Hydrostatic Equilibrium

- Fluid at rest

$$\left(\rho\vec{f}_{app}-\nabla p\right)dV=0$$

$$\vec{f}_{app}=\frac{\nabla p}{\rho}$$

- Fluid in motion

$$F_{net}=\left(-\nabla p+\rho\vec{f}_{app}\right)dV=m\frac{d\vec{v}}{dt}=\rho dV\frac{d\vec{v}}{dt}$$

- As with density use total derivative (sometimes called material derivative or convective derivative)

$$\frac{d\vec{v}}{dt}=\frac{\partial\vec{v}}{\partial t}+\left(\vec{v}\cdot\nabla\right)\vec{v}$$

Fluid Dynamic Equations

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = \vec{f}_{app} - \frac{\nabla p}{\rho}$$

- Manipulate with vector identity

$$(\vec{v}\cdot\nabla)\vec{v} = \nabla\left(\frac{\vec{v}\cdot\vec{v}}{2}\right) - \left[\vec{v}\times(\nabla\times\vec{v})\right]$$

- Final velocity equation

$$\frac{\partial \vec{v}}{\partial t} + \nabla\left(\frac{\vec{v}\cdot\vec{v}}{2}\right) - \left[\vec{v}\times(\nabla\times\vec{v})\right] = \vec{f}_{app} - \frac{\nabla p}{\rho}$$

- One more thing: equation of state relating p and ρ

Energy Conservation

- For energy in a fixed volume

$$E_{tot} = \int_V d^3x \left(\frac{\rho v^2}{2} + \rho\varepsilon \right)$$

ε internal energy per unit mass

- Work done (first law in co-moving frame)

$$M d\varepsilon = -p dV = \frac{Mp}{\rho^2} d\rho$$

$$\varepsilon(s,\rho) = \int^{\rho} \frac{p}{\rho^2} d\rho$$

- Isentropic process (s constant, no heat transfer in)

$$\frac{\partial \varepsilon}{\partial t} = \frac{p}{\rho^2} \frac{\partial \rho}{\partial t}$$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho v^2\right] = -\nabla\cdot\left(\frac{1}{2}\rho v^2\vec{v}\right) - \vec{v}\cdot\nabla p + \rho\vec{v}\cdot\vec{f}_{app}$$

$$\frac{\partial}{\partial t}\rho\varepsilon = -\left(\varepsilon + \frac{p}{\rho}\right)\nabla\cdot\left(\rho\vec{v}\right) = -\nabla\cdot\left[\left(\rho\varepsilon + p\right)\vec{v}\right] + \rho v\cdot\nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\nabla\left(\varepsilon + \frac{p}{\rho}\right) = \frac{\nabla p}{\rho}$$

$$\frac{\partial}{\partial t}\rho\varepsilon = -\nabla\cdot\left[\left(\rho\varepsilon + p\right)\vec{v}\right] + \vec{v}\cdot\nabla p$$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho v^2 + \rho\varepsilon\right] + \nabla\cdot\left[\left(\frac{1}{2}\rho v^2 + \rho\varepsilon + p\right)\vec{v}\right] = \rho\vec{f}_{app}\cdot\vec{v}$$

$$\vec{j}_E = \frac{1}{2}\rho v^2 + \rho\varepsilon + p$$

Bernoulli's Theorem

- Exact first integral of velocity equation when
  - Irrotational motion $\nabla \times \vec{v} = 0 \rightarrow v = -\nabla \Phi$
  - External force conservative $\vec{f}_{app} = -\nabla U$
  - Flow incompressible with fixed ρ
- Bernouli's Theorem

$$\frac{p}{\rho} + U + \frac{(\nabla \Phi)^2}{2} - \frac{\partial \Phi}{\partial t} = 0$$

- If flow compressible but isentropic

$$\varepsilon + \frac{p}{\rho} + U + \frac{(\nabla \Phi)^2}{2} - \frac{\partial \Phi}{\partial t} = 0$$

Kelvin's Theorem on Circulation

- Already discussed this in the Arnold material

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \frac{\partial \vec{v}}{\partial t} = -\nabla\left(\varepsilon + \frac{p}{\rho} + U\right)$$

$$\Gamma(t) = \int_{C(t)} \omega_{\vec{v}}^{1} = \oint_{C} d\vec{s} \cdot \vec{v}$$

- To linear order

$$\vec{C}(s, t+\Delta t) = \vec{C}(s,t) + \Delta t\ \vec{v}\left(\vec{C}(s,t), t\right)$$

$$\Gamma(t+\Delta t) = \int \vec{v}\left(\vec{C}(s, t+\Delta t), t+\Delta t\right) \cdot \frac{\partial \vec{C}(s, t+\Delta t)}{\partial s} ds$$

$$\Gamma(t) = \int \vec{v}\left(\vec{C}(s,t), t\right) \cdot \frac{\partial \vec{C}(s,t)}{\partial s} ds$$

$$\frac{d\Gamma}{dt} = \int \frac{d\vec{v}}{dt}\left(\vec{C}\left(s,t\right),t\right)\cdot\frac{\partial\vec{C}}{\partial s}ds + \int \vec{v}\left(\vec{C}\left(s,t\right),t\right)\cdot\frac{\partial}{\partial s}\vec{v}\left(\vec{C}\left(s,t\right),t\right)ds$$

$$= -\int \nabla\left(\varepsilon + \frac{p}{\rho} + U\right)\cdot\frac{\partial\vec{C}}{\partial s}ds + \int \nabla\left(\frac{\left|\vec{v}\right|^2}{2}\right)\cdot\frac{\partial\vec{C}}{\partial s}ds$$

$$= 0 \qquad \text{(the integrand is exact!)}$$

- The circulation is constant about any closed curve that moves with the fluid. If a fluid is stationary and acted on by a conservative force, the flow in a simply connected region necessarily remains irrotational.

Lagrangian for Isentropic Flow

- Two independent field variables: ρ and Φ

$$\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho \nabla \Phi) = 0$$

$$\varepsilon + \frac{p}{\rho} + U + \frac{(\nabla \Phi)^2}{2} - \frac{\partial \Phi}{\partial t} = 0$$

- Lagrangian density

$$\mathscr{L} = \rho \frac{\partial \Phi}{\partial t} - \frac{\rho (\nabla \Phi)^2}{2} - \rho U - \rho \varepsilon(\rho)$$

- Canonical momenta

$$\mathscr{P}_\Phi = \frac{\partial \mathscr{L}}{\partial (\partial \Phi / \partial t)} = \rho$$

$$\mathscr{P}_\rho = \frac{\partial \mathscr{L}}{\partial (\partial \rho / \partial t)} = 0$$

- Euler Lagrange Equations

$$\frac{\partial}{\partial t}\mathcal{P}_{\Phi} + \nabla \cdot \frac{\partial \mathcal{L}}{\partial (\nabla \Phi)} = \frac{\partial \mathcal{L}}{\partial \Phi} = 0 \rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \nabla \Phi) = 0$$

$$\frac{\partial}{\partial t}\mathcal{P}_{\rho} + \nabla \cdot \frac{\partial \mathcal{L}}{\partial (\nabla \rho)} = \frac{\partial \mathcal{L}}{\partial \rho} = \varepsilon + \frac{p}{\rho} + U + \frac{(\nabla \Phi)^2}{2} - \frac{\partial \Phi}{\partial t} = 0$$

$$\frac{\partial (\rho \varepsilon)}{\partial \rho} = \varepsilon + \frac{p}{\rho}$$

- Hamiltonian Density

$$\mathcal{H} = \mathcal{P}_{\Phi}\frac{\partial \Phi}{\partial t} + \mathcal{P}_{\rho}\frac{\partial \rho}{\partial t} - \mathcal{L} = \rho \varepsilon (\rho) + \rho U + \rho \frac{(\nabla \Phi)^2}{2}$$

internal energy plus potential energy plus kinetic energy

Sound Waves

- Linearize about a uniform stationary state

$$\rho(\vec{x},t) = \rho_0 + \rho' \qquad \vec{v}(\vec{x},t) = 0 + \vec{v}' \qquad p(\vec{x},t) = p_0 + p'$$

- Continuity equation

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_0 \vec{v}') + 0 = 0 \rightarrow \frac{1}{\rho_0}\frac{\partial \rho'}{\partial t} + \nabla \cdot \vec{v}' = 0$$

- Velocity equation

$$\frac{\partial \vec{v}'}{\partial t} = -\frac{1}{\rho_0}\nabla p'$$

- Isentropic equation of state

$$p_0 + p' = p(s, \rho_0 + \rho') \approx p_0 + \left.\frac{\partial p}{\partial \rho}\right|_s \rho' \rightarrow p' = c^2 \rho'$$

Flow Irrotational

- Take curl of velocity equation. Conclude flow irrotational

$$\vec{v}' = -\nabla\Phi \rightarrow \frac{\partial \vec{v}'}{\partial t} = -\nabla \frac{\partial \Phi}{\partial t} \rightarrow \frac{\partial \Phi}{\partial t} = \frac{p'}{\rho_0}$$

$$\frac{\partial^2 \Phi}{\partial t^2} = \frac{1}{\rho_0}\frac{\partial p'}{\partial t} = \frac{c^2}{\rho_0}\frac{\partial \rho'}{\partial t}$$

- Scalar wave equation

$$\frac{\partial^2 \Phi}{\partial t^2} = c^2 \nabla \cdot \left(\nabla\Phi\right) \rightarrow \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = \nabla^2 \Phi$$

- Boundary conditions

$$-\hat{n}\cdot\nabla\Phi = \hat{n}\cdot\vec{V} \qquad = 0 \ \text{ for a fixed boundary}$$

$$\frac{\partial \phi}{\partial t} = 0 \qquad \text{free surface}$$

3-D Plane Wave Solutions

- Ansatz

$$\rho', \Phi \propto \mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}\right)$$

$$-\frac{\omega^2}{c^2}\Phi_0 = -\left|\vec{k}\right|^2 \Phi_0$$

$$\vec{v}' = -\left(i\vec{k}\Phi_0\right)$$

$$-i\omega\vec{v}' = -i\vec{k}\frac{c^2\rho'}{\rho_0} \to \rho' = -\rho_0\frac{i\omega}{c^2}\Phi_0$$

- Energy flux

$$j_E = -\rho\nabla\Phi\frac{\partial\Phi}{\partial t} = \frac{1}{2}\mathrm{Re}\left(\rho_0\left(-i\vec{k}\Phi_0\right)\left(i\omega\Phi_0^*\right)\right) = \frac{1}{2}\rho_0 k^2 c\left|\Phi_0\right|^2\hat{k}$$

Helmholz Equation and Organ Pipes

- Velocity potential solves Helmholtz equation

$$\nabla^2\Phi(\vec{r})+k^2\Phi(\vec{r})=0$$

- BCs

$$\frac{\partial\Phi}{\partial r}=v_r=0 \quad r=a \qquad \frac{\partial\Phi}{\partial z}=v_z=0 \quad z=0,L$$

- Cylindrical Solutions

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}+k^2\Phi=0$$

$$\Phi(r,\phi,z)=R(r)F(\phi)Z(z)$$

$$F(\phi)=e^{im\phi} \qquad Z(z)=\cos(\alpha z) \qquad \alpha=\frac{p\pi}{L}$$ zero possible

Bessel Function Solutions

- Bessel Functions solve

$$\frac{1}{r}\frac{d}{dr}r\frac{d\left(\kappa r\right)}{dr}-\frac{m^2}{r^2}J_m\left(\kappa r\right)+\kappa^2 J_m\left(\kappa r\right)=0 \qquad \kappa^2=k^2-\alpha^2$$

- Eigenfunctions

$$\Phi_{mnp}\left(\vec{r},t\right)=\mathrm{Re}\,\Phi_0 J_m\left(\frac{\alpha_{mn}}{a}r\right)\cos\left(\frac{p\pi}{L}z\right)\exp\left(im\phi-i\omega t\right)$$

$$\omega_{mnp}=c^2k_{mnp}^2=c^2\left[\left(\frac{\alpha_{mn}}{a}\right)^2+\left(\frac{p\pi}{L}\right)^2\right]$$

- Fundamental

$$\omega_{011}=ck_{011}=c\frac{\pi}{L}$$

- Open ended

$$\omega_{010}=ck_{010}=\frac{c}{2}\frac{\pi}{L}$$

Green Function for Wave Equation

- Green Function in 3-D

$$\nabla^2 u(\vec{r}) - \lambda^2 u(\vec{r}) = -f(\vec{r})$$

- Apply Fourier Transforms

$$\tilde{f}(\vec{p}) = \iiint d^3 r e^{-i\vec{p}\cdot\vec{r}} f(\vec{r})$$

$$f(r) = \frac{1}{(2\pi)^3} \iiint d^3 p e^{i\vec{p}\cdot\vec{r}} \tilde{f}(\vec{p})$$

- Fourier transform equation to solve and integrate by parts twice

$$-p^2 \tilde{u}(\vec{p}) - \lambda^2 \tilde{u}(\vec{p}) = -\tilde{f}(\vec{p})$$

Green Function Solution

- The Fourier transform of the solution is

$$\tilde{u}(\vec{p}) = \frac{\tilde{f}(\vec{p})}{p^2 + \lambda^2}$$

- The solution is

$$u(\vec{r}) = \frac{1}{(2\pi)^3} \iiint \frac{\tilde{f}(\vec{p})}{p^2 + \lambda^2} e^{i\vec{p}\cdot\vec{x}} d^3 p$$

- The Green function is

$$u(\vec{r}) = \frac{1}{(2\pi)^3} \iiint \frac{1}{p^2 + \lambda^2} e^{i\vec{p}\cdot\vec{r}} e^{-i\vec{p}\cdot\vec{r}'} d^3 p f(\vec{r}') d^3 r'$$

$$\rightarrow G(\vec{r} - \vec{r}') = \frac{1}{(2\pi)^3} \iiint \frac{1}{p^2 + \lambda^2} e^{i\vec{p}\cdot\vec{r}} e^{-i\vec{p}\cdot\vec{r}'} d^3 p$$

- Alternate equation for Green function

$$\left[\nabla^2 - \lambda^2\right] G\left(\vec{r} - \vec{r}'\right) = -\frac{1}{(2\pi)^3} \iiint e^{i\vec{p}\cdot\vec{r}} e^{-i\vec{p}\cdot\vec{r}'} d^3 p = -\delta\left(\vec{r} - \vec{r}'\right)$$

- Simplify

$$G\left(\vec{R}\right) = \frac{1}{(2\pi)^3} \iiint \frac{e^{ipR\cos\theta}}{p^2 + \lambda^2} d^3 p = \frac{1}{(2\pi)^2} \int_0^\infty p^2 dp \int_0^\pi \sin\theta \frac{e^{ipR\cos\theta}}{p^2 + \lambda^2} d\theta$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty \frac{2p\sin pR}{R\left(p^2 + \lambda^2\right)} dp = \frac{1}{(2\pi)^2} \int_{-\infty}^\infty \frac{p\sin pR}{R\left(p^2 + \lambda^2\right)} dp = \frac{e^{-\lambda R}}{4\pi R}$$

- Yukawa potential (Green function)

$$G\left(\vec{r} - \vec{r}'\right) = \frac{e^{-\lambda\left|\vec{r} - \vec{r}'\right|}}{4\pi\left|\vec{r} - \vec{r}'\right|}$$

Helmholtz Equation

- Driven (Inhomogeneous) Wave Equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial}{\partial t^2}\right]\Phi\left(\vec{r},t\right) = -f\left(\vec{r},t\right)$$

- Time Fourier Transform

$$\Phi\left(\vec{r},t\right) = \frac{1}{2\pi}\int d\omega e^{-i\omega t}\tilde{\Phi}\left(\vec{r},\omega\right)$$

$$f\left(\vec{r},t\right) = \frac{1}{2\pi}\int d\omega e^{-i\omega t}\tilde{f}\left(\vec{r},\omega\right)$$

- Wave Equation Fourier Transformed

$$\left[\nabla^2 + \frac{\omega^2}{c^2}\right]\tilde{\Phi}\left(\vec{r},\omega\right) = -\tilde{f}\left(\vec{r},\omega\right)$$

Green Function

- Green function satisfies

$$\Phi\left(\vec{r},t\right)=\int d^3r'\int dt' G\left(\vec{r}-\vec{r}',t-t'\right) f\left(\vec{r}',t'\right)$$

$$\left(\frac{\omega^2}{c^2}-k^2\right)\tilde{\Phi}\left(\vec{k},\omega\right)=-\tilde{f}\left(\vec{k},\omega\right)$$

$$\Phi\left(\vec{r},t\right)=\frac{1}{\left(2\pi\right)^4}\int d^3k\int d\omega \frac{-f\left(\vec{k},\omega\right)}{\left(\frac{\omega^2}{c^2}-k^2\right)} e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}$$

$$\Phi\left(\vec{r},t\right)=\frac{1}{\left(2\pi\right)^4}\int d^3r'\int dt'\int d^3k\int d\omega \frac{-1}{\left(\frac{\omega^2}{c^2}-k^2\right)} e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)} e^{-i\left(\vec{k}\cdot\vec{r}'-\omega t'\right)} f\left(\vec{r}',t'\right)$$

- Green function is

$$G\left(\vec{r}-\vec{r}',t-t'\right)=\frac{1}{\left(2\pi\right)^4}\int d^3k\int d\omega\frac{-e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}e^{-i\left(\vec{k}\cdot\vec{r}'-\omega t'\right)}}{\left(\frac{\omega^2}{c^2}-k^2\right)}$$

- Satisfies

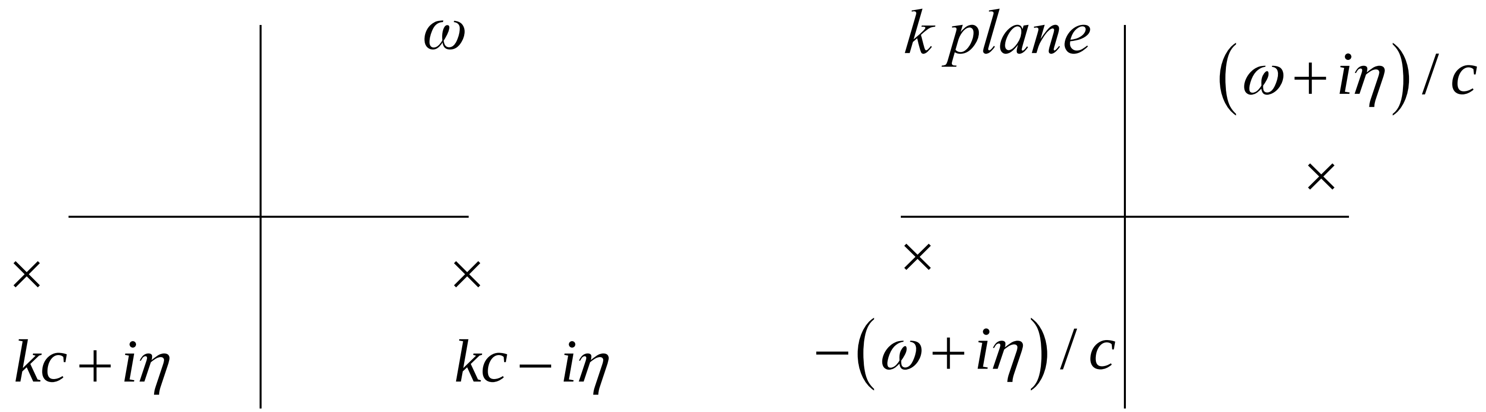
$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial}{\partial t^2}\right]G\left(\vec{r}-\vec{r}',t-t'\right)=-\delta\left(\vec{r}-\vec{r}'\right)\delta\left(t-t'\right)$$

- Also, with causal boundary conditions is

$$\tilde{G}\left(\vec{r}-\vec{r}',\omega\right)=\frac{e^{i\omega\left|\vec{r}-\vec{r}'\right|/c}}{4\pi\left|\vec{r}-\vec{r}'\right|}$$

Causal Boundary Conditions

- Can get causal B. C. by correct pole choice



- Gives so-called retarded Green function
- Green function evaluated

$$\tilde{G}\left(\vec{R},\omega\right)=\int\frac{d^3k}{\left(2\pi\right)^3}\frac{e^{i\vec{k}\cdot\vec{R}}}{k^2-\left(\omega+i\eta\right)^2/c^2}$$

$$=\frac{1}{8\pi^2 iR}\int kdk\,\frac{e^{ikR}-e^{-ikR}}{k^2-\left(\omega+i\eta\right)^2/c^2}=\frac{e^{i\omega R/c}}{4\pi R}$$

Method of Images

- Suppose have homogeneous boundary conditions on the *x*-*y* half plane. The can solve the problem by making an image source and making a combined Green function. The rigid boundary solution has

$$\tilde{G}(\vec{r}-\vec{r}') = \frac{e^{i\omega|\vec{r}-\vec{r}'|/c}}{4\pi|\vec{r}-\vec{r}'|} + \frac{e^{i\omega|\vec{r}-\vec{\bar{r}}|/c}}{4\pi|\vec{r}-\vec{\bar{r}}|} \qquad \vec{\bar{r}} = (r_x', r_y', -r_z')$$

- To satisfy the boundary condition so that the solution vanishes on the boundary

$$\tilde{G}(\vec{r}-\vec{r}') = \frac{e^{i\omega|\vec{r}-\vec{r}'|/c}}{4\pi|\vec{r}-\vec{r}'|} - \frac{e^{i\omega|\vec{r}-\vec{\bar{r}}|/c}}{4\pi|\vec{r}-\vec{\bar{r}}|} \qquad \vec{\bar{r}} = (r_x', r_y', -r_z')$$

Kirchhoff's Approximation

- We all know sound waves diffract (easily pass around corners). Standard approximation "schema"

$$\nabla^2\Phi(\vec{r})+k^2\Phi(\vec{r})=-\delta(\vec{r}-\vec{r}_0)$$

- Zeroth solution the Image GF

$$\tilde{G}(\vec{r}-\vec{r}')=\frac{e^{i\omega|\vec{r}-\vec{r}'|/c}}{4\pi|\vec{r}-\vec{r}'|}+\frac{e^{i\omega|\vec{r}-\vec{\bar{r}}|/c}}{4\pi|\vec{r}-\vec{\bar{r}}|} \qquad \vec{\bar{r}}=(r_x',r_y',-r_z')$$

- Boundary condition not correct at hole

$$\int_V d^3r\left(\Phi\nabla^2\Psi-\Psi\nabla^2\Phi\right)=\int_A dA\cdot\left(\Phi\nabla\Psi-\Psi\nabla\Phi\right)$$

$$\int d^3r\left(\Phi\nabla^2G-G\nabla^2\Phi\right)=\int d^3r\left(-\Phi k^2G-\Phi\delta(\vec{r}-\vec{r}')+Gk^2\Phi\right)=-\Phi(\vec{r}')$$

$$=\int_{\sigma+R} dA\cdot\left(\Phi\nabla G-G\nabla\Phi\right)\ \lim_{R\to\infty}\to-\int_H dA\cdot G\nabla\Phi$$

In RHP

- Exact relation

$$\Phi(\vec{r}') = \int_H dA \cdot G \nabla \Phi = -\int_H dA \cdot \frac{e^{ik|\vec{r}-\vec{r}'|}}{2\pi|\vec{r}-\vec{r}'|} \frac{\partial \Phi}{\partial z}$$

- For short wavelengths, evaluate RHS as if screen not there!

$$\Phi(\vec{r}') = \int_H dA \cdot G \nabla \Phi = \frac{ik}{8\pi^2} \int_H dA \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} \frac{z_0}{|\vec{r}-\vec{r}_0|}$$

Huygens' Principle

Babinet's Principle

$$\Phi(\vec{r}') = -\int_H dA \cdot \tilde{G}(\vec{r},\vec{r}') \frac{\partial}{\partial z} \frac{e^{ik|\vec{r}-\vec{r}_0|}}{4\pi|\vec{r}-\vec{r}_0|}$$

$$\bar{\Phi}(\vec{r}') = -\int_{P-H} dA \cdot \tilde{G}(\vec{r},\vec{r}') \frac{\partial}{\partial z} \frac{e^{ik|\vec{r}-\vec{r}_0|}}{4\pi|\vec{r}-\vec{r}_0|}$$

$$\therefore \Phi(\vec{r}') + \bar{\Phi}(\vec{r}') = -\int_P dA \cdot \tilde{G}(\vec{r},\vec{r}') \frac{\partial}{\partial z} \frac{e^{ik|\vec{r}-\vec{r}_0|}}{4\pi|\vec{r}-\vec{r}_0|}$$

- Apply Green's identity

$$\Phi(\vec{r}) = \Phi_{inc}(\vec{r}) = \frac{e^{ik|\vec{r}-\vec{r}_0|}}{4\pi|\vec{r}-\vec{r}_0|} \qquad \Psi(\vec{r}) = \tilde{G}(\vec{r},\vec{r}')$$

$$\Phi(\vec{r}') + \bar{\Phi}(\vec{r}') = \Phi_{inc}(\vec{r}') = \frac{e^{ik|\vec{r}'-\vec{r}_0|}}{4\pi|\vec{r}'-\vec{r}_0|}$$

$$\therefore \Phi_{diff} = -\bar{\Phi}_{diff}$$

Diffracted Amplitude

$$\left|\vec{r}-\vec{r}_0\right| = r_0 - \hat{r}_0\cdot\vec{r} + \frac{\left[r^2-\left(\hat{r}_0\cdot\vec{r}\right)^2\right]}{2r_0}$$

$$\Phi\left(\vec{r}'\right) = -\frac{ik}{8\pi^2}\frac{e^{ik\left(r_0+r'\right)\left|\vec{r}-\vec{r}'\right|}}{r_0 r'}\int_H dA e^{ik\vec{r}\cdot\left(\hat{k}-\hat{k}'\right)}\cos\theta_0 \exp\left(\begin{array}{l}\frac{ik}{2r_0}\left[r^2-\left(\hat{k}\cdot\vec{r}\right)^2\right] \\ +\frac{ik}{2r_0}\left[r^2-\left(\hat{k}'\cdot\vec{r}\right)^2\right]\end{array}\right)$$

- Fresnel diffraction: phase shifts across the aperture important. Full integral must be completed
- Fraunhofer diffraction $ka^2/r \ll 1 \quad ka^2/r' \ll 1$

Pattern is the transverse Fourier Transform!

$$\Phi\left(\vec{r}'\right) = -\frac{ik}{8\pi^2}\frac{e^{ik\left(r_0+r'\right)}}{r_0 r'}\int_H dA e^{ik\vec{r}\cdot\left(\hat{k}-\hat{k}'\right)} = -\frac{ik}{8\pi^2}\frac{e^{ik\left(r_0+r'\right)}}{r_0 r'}\int_H dA e^{-i\vec{r}\cdot\vec{q}}$$

Two Cases

- Rectangular aperture

$$I(\vec{r}') = I_0 \left(\frac{\sin q_x a}{q_x a} \right)^2 \left(\frac{\sin q_y b}{q_y b} \right)^2$$

- Destructive interference at $q_x a = \pi$

- Circular aperture

$$I(\vec{r}') = I_0 \left(\frac{2 J_1 \sin q_\perp a}{q_\perp a} \right)^2$$

- Airy disk (angle of first zero)

$$\sin\theta' \approx 0.61 \frac{\lambda}{a}$$

Equation for Heat Conduction

- Field variable: temperature scalar
- Additional inputs: heat capacity (at constant pressure) c_p, thermal conductivity k_{th}

$$dT = c_p dE$$

$$j_H = -k_{th} \nabla T$$

- Thermal diffusivity

$$\kappa = \frac{k_{th}}{\rho c_p}$$

- Heat Equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T + \frac{\dot{q}}{c_p}$$

Boundary Conditions

- Closed boundary surface held at constant T_{ex}
- Insulating surface

$$n \cdot \nabla T = 0$$

- Separate variables

$$T(\vec{r},t) = T(\vec{r}) e^{-\lambda t}$$

- Helmholtz again

$$\nabla^2 T(\vec{r}) + k^2 T(\vec{r}) = -\frac{\dot{q}}{\kappa c_p}$$

Long Rectangular Rod

- Long ends held at temperature T_0
- Eigensolutions

$$\delta T(\vec{r}) = X(x)Y(y)Z(z)$$

$$X(x) = \sin\frac{m\pi x}{a} \qquad m = 1, 2, 3\cdots$$

$$Y(y) = \cos\frac{n\pi y}{b} \qquad n = 0, 1, 2, 3\cdots$$

$$Z(z) = \cos\frac{p\pi z}{c} \qquad p = 0, 1, 2, 3\cdots$$

$$k_{mnp}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2$$

General Solution

$$\delta T\left(\vec{r},t\right)=\sum_{mnp}C_{mnp}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cos\frac{p\pi z}{c}e^{-\lambda_{mnp}t}$$

$$\delta T\left(\vec{r},t=0\right)=\sum_{mnp}C_{mnp}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cos\frac{p\pi z}{c}$$

- Find expansion coefficients with the orthogonality relations

- Long term solution dominated by slowest decaying mode

$$T\left(\vec{r},t\right)=T_0+C_{100}\sin\frac{\pi x}{a}e^{-\lambda_{100}t}$$

Thermal Waves

- Put periodic boundary condition on plane $z = 0$

$$T(z=0,t) = T_0 \cos \omega t$$

- 1-D problem

$$\frac{\partial^2 T}{\partial z^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t}$$

$$T(z,t) = \operatorname{Re} T(z) e^{-i\omega t}$$

$$\frac{d^2 T}{dz^2} = \frac{-i\omega}{\kappa} T$$

$$T(z) \propto e^{\alpha z}$$

$$\alpha^2 = \frac{i\omega}{\kappa} \to \alpha = \pm \frac{1-i}{\sqrt{2}}\sqrt{\frac{\omega}{\kappa}}$$

Penetration Depth

- Exponential falloff length (for amplitude)

$$\delta = \left(\frac{2\kappa}{\omega} \right)^{1/2} = \left(\frac{\kappa T}{\pi} \right)^{1/2}$$

- Solution for thermal wave

$$T(z,t) = T_0 e^{-z/\delta} \cos\left(\frac{z}{\delta} - \omega t \right)$$

- On earth, 3.2 m with a one year period!

Green Function for Heat Equation

- Fourier Transform spatial dependence

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T$$

$$\frac{\partial \tilde{T}\left(\vec{k},t\right)}{\partial t} = -\kappa k^2 \tilde{T}\left(\vec{k},t\right)$$

$$\tilde{T}\left(\vec{k},t\right) = A\left(\vec{k}\right) e^{-\kappa k^2 t}$$

- Solve using initial condition

$$\int T\left(\vec{r},t=0\right) e^{-i\vec{k}\cdot\vec{r}} d^3 r = \tilde{T}\left(\vec{k},t=0\right) = A\left(k\right)$$

$$\therefore T\left(\vec{r},t\right) = \frac{1}{\left(2\pi\right)^3} \int \left(\int T\left(\vec{r}',t=0\right) e^{-i\vec{k}\cdot\vec{r}'} d^3 r' \right) e^{-\kappa k^2 t} e^{i\vec{k}\cdot\vec{r}} d^3 k$$

$$G\left(\vec{r}-\vec{r}',t\right) = \frac{1}{\left(2\pi\right)^3} \int e^{-\kappa k^2 t} e^{i\vec{k}\cdot\left(\vec{r}-\vec{r}'\right)} d^3 k$$

- Complete the square

$$G(\vec{r}-\vec{r}',t)=\frac{1}{(2\pi)^3}\int e^{-\kappa k^2 t}e^{ik|\vec{r}-\vec{r}'|\cos\theta}k^2dkd\cos\theta d\phi$$

$$=\frac{1}{(2\pi)^2}\int e^{-\kappa k^2 t}e^{ik|\vec{r}-\vec{r}'|\cos\theta}k^2dkd\cos\theta=\frac{|\vec{r}-\vec{r}'|^{-1}}{(2\pi)^2 i}\int_0^\infty e^{-\kappa k^2 t}\left[e^{ik|\vec{r}-\vec{r}'|}-e^{-ik|\vec{r}-\vec{r}'|}\right]kdk$$

$$=\frac{|\vec{r}-\vec{r}'|^{-1}}{(2\pi)^2 i}\int_{-\infty}^{\infty} e^{-\kappa k^2 t}e^{ik|\vec{r}-\vec{r}'|}kdk=\frac{|\vec{r}-\vec{r}'|^{-1}}{(2\pi)^2 i}\left[\int_{-\infty}^{\infty} e^{-\kappa t\left(k^2-ik|\vec{r}-\vec{r}'|/2\kappa t-|\vec{r}-\vec{r}'|^2/4\kappa^2 t^2\right)}kdk\right]e^{-|\vec{r}-\vec{r}'|^2/4\kappa t}$$

$$l=\sqrt{\kappa t}\left(k-i|\vec{r}-\vec{r}'|/2\kappa t\right)$$

$$G=\frac{|\vec{r}-\vec{r}'|^{-1}}{(2\pi)^2 i}e^{-|\vec{r}-\vec{r}'|^2/4\kappa t}\left[\int_{-\infty}^{\infty} e^{-l^2}\left(l+i|\vec{r}-\vec{r}'|/2\kappa t\right)\frac{dl}{\sqrt{\kappa t}}\right]=\frac{1}{(4\pi\kappa t)^{3/2}}e^{-|\vec{r}-\vec{r}'|^2/4\kappa t}$$