Chapter 7 (Strings) starts by reviewing Sec 25 (Ch4)

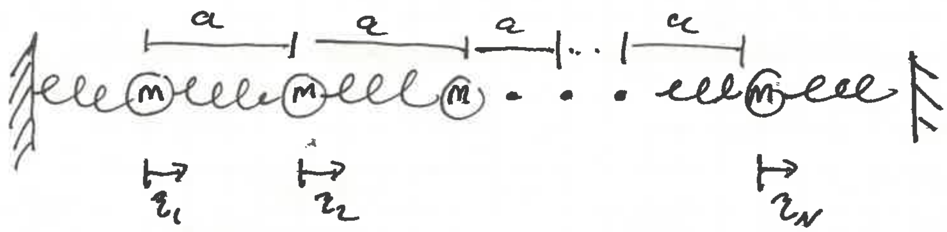
I decided to start strings w/ Ch4 Sec 24 & build up to strings from there.

24: Many degrees of freedom (N-body problems)

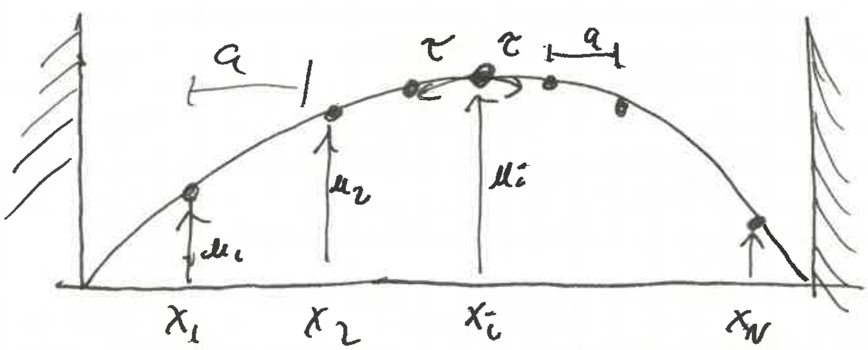
$\det|A_{N\times N}|$ = Hard but not impossible

Exploit symmetries to find complete solutions in two cases

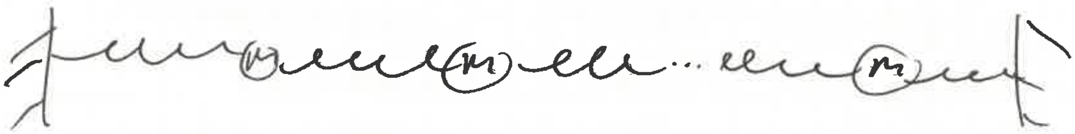
1) longitudinal oscillations of particles connected by massless springs



2) transverse planar oscillations of particles on stretched massless string



1)



masses are equal
springs are identical & massless

Model for 1D crystal lattice w/ nearest neighbor interactions

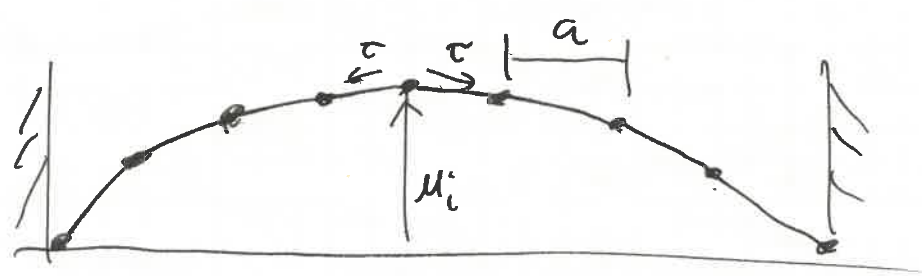
$$L = \tfrac{1}{2} m \sum_{i=1}^{N} \dot{z}_i^2 - \sum_{i=0}^{N} \tfrac{1}{2} K (z_{i+1} - z_i)^2 \qquad \text{w/ } z_0 = z_{N+1} = 0$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{z}_i}\right] = \frac{\partial L}{\partial z_i} \Rightarrow m\ddot{z}_i = \underbrace{+\tfrac{1}{2}K(z_{i+1}-z_i)}_{\text{right neighbor}} - \underbrace{\tfrac{1}{2}K(z_i - z_{i-1})}_{\text{left neighbor}} \qquad \underline{M}\ddot{\vec{z}} = -\underline{k}\vec{z}$$

$$m\ddot{z}_i + 2Kz_i - K(z_{i+1} + z_{i-1}) = 0 \qquad i = 1, \dots, N$$ 24.4

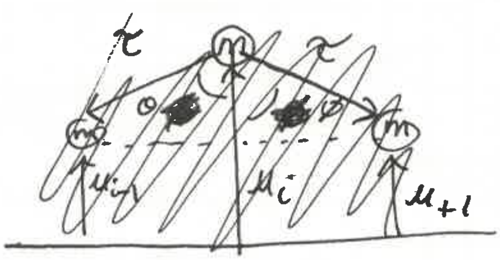
$$\underline{M} = \begin{bmatrix} m & & & \\ & m & & 0 \\ & & \ddots & \\ 0 & & & m \end{bmatrix} = m\,\mathbb{1}_{N\times N} \quad , \quad \underline{K} = \begin{bmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & \ddots & 1 \\ 0 & & & 1 & -2 \end{bmatrix} K$$

2)



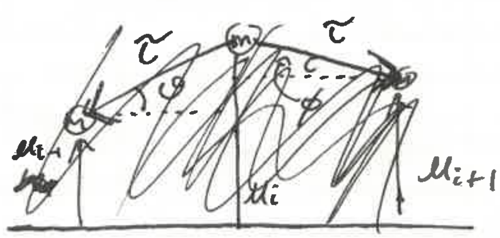
at equilibrium, const. uniform tension τ, particles separated by a

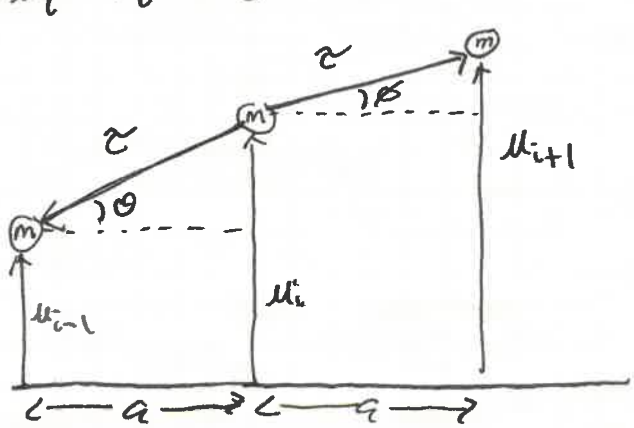
consider only transverse motion w/ displacements μ_i



~~Newton's 2nd Law for transverse motion~~

$\cancel{m\ddot{\mu}_i = F_i^{net} = \tau}$





Newton's 2nd law : $m\ddot{u}_i = \tau \sin\phi - \tau \sin\theta$
(transverse component)

$$m\ddot{u}_i = \tau\left(\frac{u_{i+1}-u_i}{\sqrt{a^2+(u_{i+1}-u_i)^2}} - \frac{u_i - u_{i-1}}{\sqrt{a^2+(u_i-u_{i-1})}}\right)$$

$$= \frac{\tau}{a}\left(\frac{u_{i+1}-u_i}{\sqrt{1+\left(\frac{u_{i+1}-u_i}{a}\right)^2}} - \frac{u_i-u_{i-1}}{\sqrt{1+\left(\frac{u_i-u_{i-1}}{a}\right)^2}}\right) \qquad a \gg u_{i+1}-u_i$$

$$\approx \frac{\tau}{a}\Big((u_{i+1}-u_i) - (u_i-u_{i-1})\Big)$$

$$\boxed{m\ddot{u}_i \approx \frac{\tau}{a}\left(-2u_i + (u_{i+1}+u_{i-1})\right)} \qquad (\text{require } u_0 = u_{N+1} = 0) \qquad (24.8)$$

Compare 24.4 & 24.8 , $\eta_i \leftrightarrow u_i$ & $k \leftrightarrow \frac{\tau}{a}$

These two problems are identical!

$$* \quad L = \tfrac{1}{2} m \sum_{i=1}^{N} \dot{u}_i^{\,2} - \frac{\tau}{2a}\sum_{i=0}^{N}(u_{i+1}-u_i)^2 \qquad u_0 = u_{N+1} = 0$$

Two methods to solve this

1) Normal modes of endpoints, then manipulate $\det(K-\omega^2 M)=0$ to get explicit sol. for N eigenvalues
   - method independent of order of N
   - pg 110 – 114

2) Find normal mode in form of propagating plane waves

$$u_j \equiv u(x_j)$$

$$x_j \equiv ja \qquad j = \text{integer}$$

$$\boxed{\text{Ansatz: } u(x_j,t) = A\exp[i(kx_j - \omega t)]} \qquad 24.41$$

claim u_j (actual displacement) is

$$u_j = \mathrm{Re}[u(x_j,t)]$$

sub into 24.8

* recall τ is tension in string

$$-m\omega^2 A e^{i(kx_i-\omega t)} = \frac{\tau}{a}\left[-2Ae^{i(kx_i-\omega t)} + Ae^{-i\omega t}\left(e^{ikx_{i+1}} + e^{ikx_{i-1}}\right)\right]$$

$$x_{i\pm1} = a(i\pm1)$$

$$-m\omega^2 = -2\tau/a + \tau/a\left(e^{ika} + e^{-ika}\right)$$

$$\omega^2 = \frac{2\tau}{ma}(1-\cos ka) = \frac{4\tau}{ma}\sin^2\left(\tfrac{1}{2}ka\right) \qquad \text{(dispersion relation)}$$ 24.43

• $\pm k$ are degenerate in ω, so superimpose

$$u(x_j,t) = A\left[e^{i(kx_j-\omega t)} - e^{i(-kx_j-\omega t)}\right]$$ 24.49

w/ B.C.s $\quad u(x_0) = u(0) = 0$

$\quad u(x_{N+1}) = u([N+1]a) = 0$

Apply $u(x_{N+1})$ to 24.49

$$e^{ik(N+1)a} - e^{-ik(N+1)a} = 0 \qquad \sin\phi = \frac{e^{i\phi}-e^{-i\phi}}{2i}$$

$$\sin[k(N+1)a] = 0$$

$$k(N+1)a = 2\pi n$$

$$\boxed{k = \frac{\pi}{a}\,\frac{n}{N+1}, \quad n = 1, 2, \dots, N}$$ 24.55

sub into dispersion relation

$$\omega_n^2 = \frac{4\tau}{ma}\sin^2\left(\frac{n\pi}{2(N+1)}\right) \qquad n = 1, 2, \dots, N$$ 24.38

we have found our N eigenvalues!

sub ω_n^2 into 24.41

$$u(x_j,t) = u_j(t) = 2A_n \sin\left(\frac{\pi n j}{N+1}\right)\sin\omega_n t$$

Hw prob 4.14 to determine 4.14 $\qquad$ used 22.34 $\quad \vec{\rho}_i^{\,T}\underline{M}\,\vec{\rho}_j = \delta_{ij}$

let $\quad l \equiv (N+1)a$, $\quad c \equiv \left(\frac{\tau}{m/a}\right)^{1/2}$ (characteristic velocity)

then $\quad \frac{\omega_n}{c} = \frac{2}{a}\sin\left(\frac{n\pi}{2}\frac{a}{l}\right) \qquad n = 1, 2, \dots, N$ 24.58

~~for down to~~
for small n, the normal mode frequencies become

$$\frac{\omega_n}{c} \xrightarrow[n \to 0]{} \frac{n\pi}{l}$$

we can also measure the wavelength

eq 24.59 ~~k~~ $K = \dfrac{n\pi}{a(N+1)}$

k = wave # $= \dfrac{2\pi}{\lambda}$

$$\frac{2\pi}{\lambda} = \frac{n\pi}{a(N+1)} \implies \lambda_n = \frac{2l}{n} \quad n = 1, 2, \dots, N$$ 24.63

* Our system of discrete, N, particles have wave properties

25: Discrete → Continuous systems

$N \to \infty$

$a \to 0$ (inter particle spacing)

$l \equiv (N+1)a$ = const (length of string)

$\dfrac{m}{a} \equiv \sigma$ = const (string mass density)

$$\omega_n^2 = \frac{4\tau}{ma}\sin^2\left(\frac{n\pi}{2(N+1)}\right) \qquad N \gg n$$

$$\omega_n^2 \Rightarrow \frac{4\tau}{ma}\left(\frac{n\pi}{2(N+1)}\right)^2$$

$$\omega_n^2 = \frac{\tau}{\sigma}\left(\frac{n\pi}{l}\right)^2$$

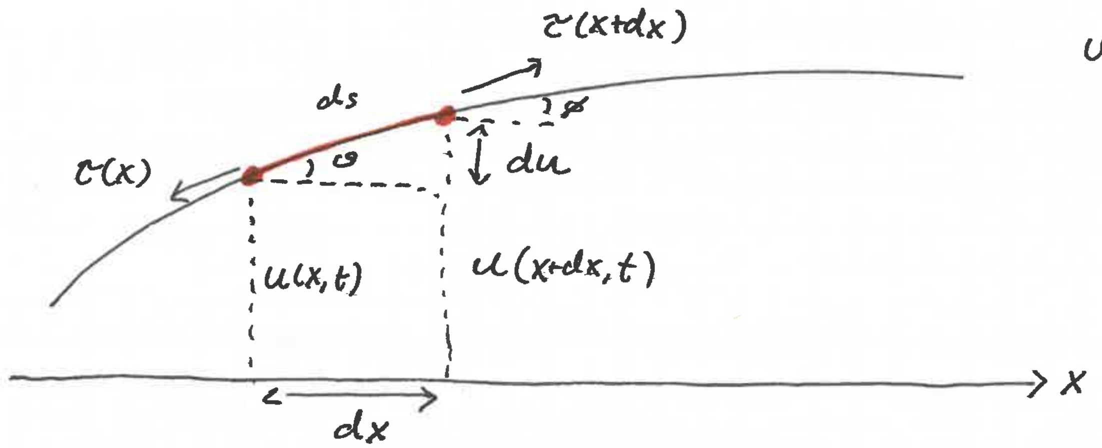
$$\omega_n = \left(\frac{n\pi}{l}\right)c \quad , \quad c \equiv \left(\frac{\tau}{\sigma}\right)^{1/2}$$

$$K_n = \frac{2\pi}{l}n \quad , \quad n = 0, \pm 1, \pm 2, \dots, \pm \infty$$

Plane wave sol: $u(x,t) = A e^{i(kx - \omega t)}$

* We could ~~keep~~ continue applying the limit to previous sol
- tedious
- more insightful to recast problem into macroscopic variables

consider ~~σdx~~ $dm = \sigma dx$



u - vertical displacement

$$\tan\theta = \frac{(du)}{(dx)}$$ (Note this expression is a ratio of infinitesimals) ~~not~~ NOT ~~d~~ $\frac{d}{dx}(u)$

$$\sin\theta \approx \frac{du}{dx}$$ (~~the~~ vertical displacement depends on transverse displacement)

$$\theta \approx \frac{\partial}{\partial x} u(x,t)$$

~~N~~ 2nd Law on dm

$$dm \frac{\partial^2}{\partial t^2}[u(x,t)] = \tau(x+dx)\sin\phi - \tau(x)\sin\theta$$

~~$[\sigma(x) dx$~~

$$[\sigma dx] \frac{\partial^2 u(x,t)}{\partial^2 t} \approx \tau(x+dx)\frac{\partial u(x+dx,t)}{\partial x} - \tau(x)\frac{\partial u(x,t)}{\partial x}$$ 25.11

$$RHS = \left(\tau(x)\frac{\partial u(x,t)}{\partial x} + dx\frac{\partial}{\partial x}\left[\tau(x)\frac{\partial u(x,t)}{\partial x}\right] + \dots\right) - \tau(x)\frac{\partial u(x,t)}{\partial x}$$

general 1D string Eq

$$\boxed{\sigma \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left[\tau \frac{\partial u(x,t)}{\partial x}\right] \qquad \sigma = \sigma(x) \quad \tau = \tau(x)}$$

our problem $\sigma = \text{const}$, $\tau = \text{const}$

$$\boxed{\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}}$$ 1D Wave Eqn 25.16

Gen sol to wave eqn w/ ICs

Determine normal mode sols to wave eq (25.16) of form

$$u(x,t) = C\,\rho(x)\cos(\omega t+\phi)$$

$$-\frac{1}{c^2}\omega^2 C\rho(x)\cos(\omega t+\phi) = C\frac{d^2\rho}{dx^2}\cos(\omega t+\phi)$$

$$\frac{d^2\rho}{dx^2}+k^2\rho(x)=0 \quad , \quad k\equiv\frac{\omega}{c}$$ 25.20

$\ni$ eigenvalue problem we've solved before in discrete case

B.C. $\rho(0)=\rho(l)=0$

$$\rho(x)=\left(\frac{2}{l\sigma}\right)^{1/2}\sin kx$$ (normalization we'll discuss later)

$$\rho(l)=\left(\frac{2}{l\sigma}\right)^{1/2}\sin(kl)=0$$

$$kl=n\pi$$

$$k_n=n\frac{\pi}{l} \qquad n=1,2,\dots,\infty$$

$$\omega_n=\frac{n\pi}{l}c$$

$$\lambda_n=\frac{2l}{n}$$

Take superposition to get gen sol

$$u(x,t)=\sum_{n=1}^{\infty}\left(C_n\,\rho^{(n)}(x)\cos(\omega_n t+\phi_n)\right)$$ 25.29

recall discrete sol 22.40 is same expression but a finite sum of terms

- we'll need to develop new techniques for ∞ sum sols Ch 27

let $a_n\equiv C_n\cos\phi_n \quad , \quad b_n\equiv -C_n\sin\phi_n$

$$u(x,t)=\sum_{n=1}^{\infty}\left(\frac{2}{l\sigma}\right)^{1/2}\sin k_n x\,(a_n\cos\omega_n t+b_n\sin\omega_n t)$$ 25.30

ICs have to specify position & speed of every point of string

~~$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \rho^{(n)}(x)$~~

$$f(x) \equiv u(x,0) = \sum_{n=1}^{\infty} \left(\frac{2}{\ell\sigma}\right)^{1/2} a_n \sin k_n x$$

$$g(x) \equiv \dot{u}(x,0) = \sum_{n=1}^{\infty} \left(\frac{2}{\ell\sigma}\right)^{1/2} \omega_n b_n \sin k_n x$$

} Fourier series 25.33

- we will prove later that $\rho^{(n)}(x)$ form a complete set of functions on $0 \le x \le \ell$

How to set a_n & b_n? Recall

$$\vec{\rho}^{(i)T} \underline{M} \, \vec{\rho}^{(j)} = \delta_{ij}$$

$\Downarrow$ similarly

$$\int_0^\ell \rho^{(i)} \sigma \rho^{(j)} \, dx$$

$$\frac{2}{\ell} \int_0^\ell \sin(k_i x) \sin(k_j x) \, dx$$

$$\frac{2}{\ell} \int_0^\ell \sin\left(i \frac{\pi x}{\ell}\right) \sin\left(j \frac{\pi x}{\ell}\right) dx = \delta_{ij}$$

The $\rho^{(i)}(x)$ are an orthogonal set w/ the normalization constant $\left(\frac{2}{\ell\sigma}\right)^{1/2}$

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{2}{\ell\sigma}\right)^{1/2} a_n \sin k_n x$$

~~$\int_0^L f(x) \sigma \sin(k_m x) dx = \int_0^L \left(\frac{2}{\ell\sigma}\right)^{1/2} \sum_{n=1}^{\infty} a_n \sin k_n x \, \sigma \sin k_m x \, dx$~~

$$\int_0^\ell f(x) \sigma \left(\frac{2}{L\sigma}\right)^{1/2} \sin(k_m x) dx = \underbrace{\int_0^\ell \sum_{n=1}^{\infty} \left(\frac{2}{\ell\sigma}\right)^{1/2} a_n \sin k_n x \, \sigma \left(\frac{2}{\ell\sigma}\right)^{1/2} \sin k_m x \, dx}_{\text{only } m=n \text{ term survives}}$$

$$a_n = \left(\frac{2}{\ell\sigma}\right)^{1/2} \int_0^\ell f(x) \sigma \sin(k_n x) \, dx$$

similarly (25.35)

$$\omega_n b_n = \left(\frac{2}{\ell\sigma}\right)^{1/2} \int_0^\ell g(x) \sigma \sin(k_n x) \, dx$$