Physics 451/551
Theoretical Mechanics

G. A. Krafft

Old Dominion University

Jefferson Lab

Lecture 11

Action-Angle Variables

- Suppose a mechanical system has a periodic motion
  - Libration (return in phase space)
  - Rotation (oscillating momentum)
- In some situations knowledge of the full motion is not so interesting as *knowing the frequencies of motion in the system*
- Frequencies determined by the following procedure

1. Define the action

$$J_i = \int_{H=C} p_i dq^i \qquad \text{no summation}$$

2. Determine the Hamiltonian as a function of action

$$H = \left(J_i, \cdots, J_n\right)$$

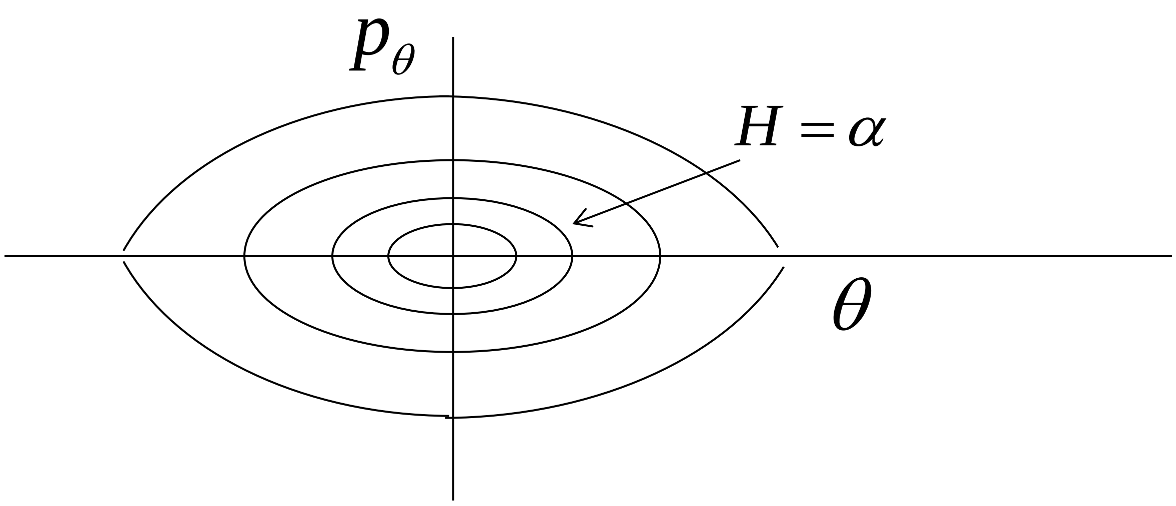
3. Frequencies are the derivatives

Real Pendulum Phase Space

- Hamiltonian

$$H = \frac{P_\theta^2}{2mL^2} - mgL\cos\theta$$

- Action is the phase space area



Relationship to period

- Action is

$$J = 2\int_{-\theta_0}^{\theta_0} \sqrt{2mL}\sqrt{\alpha + mgL\cos\theta}\,d\theta \qquad \theta_0 = \cos^{-1}\left(-\frac{\alpha}{mgL}\right)$$

- Derivative with respect to energy is

$$\frac{dJ}{dE} = 2\int_{-\theta_0}^{\theta_0} \frac{\sqrt{2mL}}{2\sqrt{\alpha + mgL\cos\theta}}d\theta = 2\int_{-\theta_0}^{\theta_0} \frac{\sqrt{2mL}}{2}\frac{d\theta}{\sqrt{m/2L}\dot{\theta}} = T$$

The RHS is simply the oscillation period

More General Argument

- Classical Action for Motion

$$S = W(q,\alpha) - \alpha t \qquad p = \partial W / \partial q$$

- Oscillation action

$$J = 2\left[W(\theta_0,\alpha) - W(-\theta_0,\alpha)\right] = f(\alpha)$$

Depends only on α, not q

- Invert to get

$$\alpha = H(J) = f^{-1}(J)$$

$$S = W\left(q, H(J)\right) - H(J)t$$

Angle Variable

- Define "angle" variable

$$w = \frac{\partial W}{\partial J}$$

- Constant of motion

$$\beta = \frac{\partial S}{\partial J} = w - \left[\frac{\partial \alpha}{\partial J} \right] t$$

Angle variable increases linearly with time. Again

$$\partial H / \partial J$$

gives the frequency

- Fetter and Walecka have generalization for many "separable" degrees of freedom

Symplectic Matrices

- Assume the even-dimensional manifold (and vector space) $\mathbf{R}^{2n}$. A matrix acting on vectors in $\mathbf{R}^{2n}$ is called *symplectic* if it preserves the canonical symplectic structure

$$\omega^2\left(S\vec{\eta},S\vec{\xi}\right)=\sum_{i=1}^{n} dp_i \wedge dq^i\left(S\vec{\eta},S\vec{\xi}\right)=\omega^2\left(\vec{\eta},\vec{\xi}\right)$$

$$\left[S\vec{\eta},S\vec{\xi}\right]=\left[\vec{\eta},\vec{\xi}\right]$$

- Such matrices form a matrix Lie group (like rotations!)

$$\left[S_1S_2\vec{\eta},S_1S_2\vec{\xi}\right]=\left[S_2\vec{\eta},S_2\vec{\xi}\right]=\left[\vec{\eta},\vec{\xi}\right]$$

$$\left[S^{-1}\vec{\eta},S^{-1}\vec{\xi}\right]=\left[SS^{-1}\vec{\eta},SS^{-1}\vec{\xi}\right]=\left[\vec{\eta},\vec{\xi}\right]$$

Symplectic Condition

- Note (co-ordinate convention $(q^1,\ldots,q^n,p_1,\ldots,p_n)$)

$$\omega^2\left(\vec{\xi},\vec{\eta}\right)=\left[\vec{\xi},\vec{\eta}\right]=\vec{\xi}^{\,t}\begin{bmatrix}0 & -I\\ I & 0\end{bmatrix}\vec{\eta}$$

$$J\equiv\begin{bmatrix}0 & -I\\ I & 0\end{bmatrix}\qquad J^2=-I$$

- Symplectic means

$$\left[S\vec{\xi},S\vec{\eta}\right]=\vec{\xi}^{\,t}S^t\begin{bmatrix}0 & -I\\ I & 0\end{bmatrix}S\vec{\eta}$$

$$S^tJS=J$$

- Determinate is always +1
- Another definition of canonical transformation: $\dfrac{\partial\left(\vec{Q},\vec{P}\right)}{\partial\left(\vec{q},\vec{p}\right)}$ a symplectic matrix

Note on J

- There is wide conformance that the canonical symplectic structure should be

$$\omega^2 = \sum_{i=1}^{n} dp_i \wedge dq^i$$

- Not so uniform convention on J

$$\left(q^1,\ldots,q^n,p_1,\ldots p_n\right) \qquad J \equiv \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

$$\left(q^1,p_1,\ldots,q^n,p_n\right) \qquad J \equiv \begin{bmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{bmatrix}$$

$$\left(p_1,\ldots p_n,q^1,\ldots,q^n\right) \qquad J \equiv \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

Definitions Equivalent

- Invariance of fundamental form

$$dQ^i = \sum_{j=1}^{n} \frac{\partial Q^i}{\partial q^j} dq^j + \sum_{j=1}^{n} \frac{\partial Q^i}{\partial p_k} dp_k \quad dP_i = \sum_{j=1}^{n} \frac{\partial P_i}{\partial q^j} dq^j + \sum_{j=1}^{n} \frac{\partial P_i}{\partial p_k} dp_k$$

$$\sum_{i=1}^{n} dP_i \wedge dQ^i = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial P_i}{\partial q^j}\frac{\partial Q^i}{\partial q^k} dq^j \wedge dq^k + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial P_i}{\partial q^j}\frac{\partial Q^i}{\partial p_k} dq^j \wedge dp_k$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial P_i}{\partial p_j}\frac{\partial Q^i}{\partial q^k} dp_j \wedge dq^k + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial P_i}{\partial p_j}\frac{\partial Q^i}{\partial p_k} dp_j \wedge dp_k$$

- Symplectic matrix definition

$$\begin{pmatrix} \dfrac{\partial \vec{Q}}{\partial \vec{q}}^t & \dfrac{\partial \vec{P}}{\partial \vec{q}}^t \\ \dfrac{\partial \vec{Q}}{\partial \vec{p}}^t & \dfrac{\partial \vec{P}}{\partial \vec{p}}^t \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial \vec{Q}}{\partial \vec{q}} & \dfrac{\partial \vec{Q}}{\partial \vec{p}} \\ \dfrac{\partial \vec{P}}{\partial \vec{q}} & \dfrac{\partial \vec{P}}{\partial \vec{p}} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \vec{Q}}{\partial \vec{q}}^t & \dfrac{\partial \vec{P}}{\partial \vec{q}}^t \\ \dfrac{\partial \vec{Q}}{\partial \vec{p}}^t & \dfrac{\partial \vec{P}}{\partial \vec{p}}^t \end{pmatrix} \begin{pmatrix} -\dfrac{\partial \vec{P}}{\partial \vec{q}} & -\dfrac{\partial \vec{P}}{\partial \vec{p}} \\ \dfrac{\partial \vec{Q}}{\partial \vec{q}} & \dfrac{\partial \vec{Q}}{\partial \vec{p}} \end{pmatrix}$$

$$
= \begin{pmatrix}
-\frac{\partial \vec{Q}}{\partial \vec{q}}^{t} \frac{\partial \vec{P}}{\partial \vec{q}} + \frac{\partial \vec{P}}{\partial \vec{q}}^{t} \frac{\partial \vec{Q}}{\partial \vec{q}} & -\frac{\partial \vec{Q}}{\partial \vec{q}}^{t} \frac{\partial \vec{P}}{\partial \vec{p}} + \frac{\partial \vec{P}}{\partial \vec{q}}^{t} \frac{\partial \vec{Q}}{\partial \vec{p}} \\
-\frac{\partial \vec{Q}}{\partial \vec{p}}^{t} \frac{\partial \vec{P}}{\partial \vec{q}} + \frac{\partial \vec{P}}{\partial \vec{p}}^{t} \frac{\partial \vec{Q}}{\partial \vec{q}} & -\frac{\partial \vec{Q}}{\partial \vec{p}}^{t} \frac{\partial \vec{P}}{\partial \vec{p}} + \frac{\partial \vec{P}}{\partial \vec{p}}^{t} \frac{\partial \vec{Q}}{\partial \vec{p}}
\end{pmatrix}
$$

- If fundamental form invariant C^tA, and D^tB are symmetric matrices. Also $D^tA\text{-}B^tC$ is the identity matrix (note the second term in the form becomes B^tC when the dummy indices are switched), as is its transpose $A^tD\text{-}C^tB$. This means the above matrix is J.
- If the above matrix is J, then clearly $C^tA = A^tC$ and $D^tB = B^tD$, and both $D^tA\text{-}B^tC$ and $A^tD\text{-}C^tB$ (which are transposes of each other) are the identity matrix. Therefore, the fundamental form is invariant.

Eigenvalues

- No zero eigenvalues. If λ an eigenvalue so is $1/\lambda$

$$\det\left(S-\lambda E\right)=\det\left(S^{t}-\lambda E\right)=\det\left(-JS^{-1}J-\lambda E\right)$$

$$=\det\left(-S^{-1}+\lambda E\right)=\det\left(-E+\lambda S\right)\qquad\left(\det\left(S\right)=1\right)$$

$$\det\left(S-\lambda E\right)=\lambda^{2n}\det\left(S-E/\lambda\right)$$

- Eigenvalue equation has real coefficients. Therefore if λ is an eigenvalue so is λ^{*}
- General picture

Strong Stability

- Stable Definition

- Strong Stability Definition

- Theorem: if all $2n$ eigenvalues of a symplectic transformation S are distinct and lie on the unit circle in the complex plane, then S is strongly stable.

Darboux's Theorem ("Arnold")

- Theorem (Darboux): Let ω^2 be any closed non-degenerate differential 2-form in a neighborhood of a point **x** in the space $\mathbf{R}^{2n}$. Then in some neighborhood of **x** one can choose a coordinate system $(q^1,\ldots,q^n,p_1,\ldots,p_n)$ such that the form has "the standard" form

$$\omega^2 = \sum_{i=1}^{n} dp_i \wedge dq^i$$

- This theorem allows us to extend to all symplectic manifolds any assertion of a local character which is invariant with respect to canonical transformations and is proven for the standard phase space $(\mathbf{R}^{2n}, \omega^2 = d\mathbf{p} \wedge d\mathbf{q})$
- For physicists and dynamics: ALL phase spaces have *canonical coordinates* where ω^2 is given as above