Physics 451/551
Theoretical Mechanics

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Lecture 10

Variational Principle via Arnold

- Integral invariant of Poincare-Cartan

$$S=\int L dt=\int\left[\vec{p}\cdot\dot{\vec{q}}\left(\vec{q},\vec{p}\right)-H\left(\vec{q},\vec{p}\right)\right]dt=\int\left[\vec{p}\cdot d\vec{q}-Hdt\right]$$

- Significance: when action integrated around a closed loop, it has constant value as loop moves according to the phase flow. If *dt*=0 on loop, value is sum of projected PS areas.
- Stokes Lemma in usual fluid mechanics. Let

$$\vec{r}=\vec{\nabla}\times\vec{v} \qquad \text{Note that } \nabla\cdot\vec{r}=0$$

and a loop γ flow along with the vector field $\vec{r}$. Then by considering the tube swept out in time

$$\oint_{\gamma}\vec{v}\cdot d\vec{l}-\oint_{\gamma'}\vec{v}\cdot d\vec{l}=\oiint_{\sigma}\vec{\nabla}\times\vec{v}\cdot\vec{n}dS=0$$

$$\int_{\partial\sigma}\omega^1_{\vec{v}}=\int_{\sigma}\omega^2_{\nabla\times\vec{v}}=\iint_{\sigma}\nabla\times\vec{v}\cdot\frac{\partial\vec{\sigma}}{\partial s}\times\frac{\partial\vec{\sigma}}{\partial t}dsdt=0 \qquad \vec{r}\cdot\frac{\partial\vec{\sigma}}{\partial s}\times\vec{r}=0$$

Extended Phase Space

- Use same idea on 2*n*+1 dimensional extended phase spaces where the Poincare-Cartan invariant 1-form and 2-form the generalized symplectic structure.

$$\int_{\gamma(t=0)} \vec{p}\cdot d\vec{q} - Hdt = \int_{\gamma(t)} \vec{p}\cdot d\vec{q} - Hdt + \int_{\sigma} d\vec{p}\wedge d\vec{q} - \frac{\partial H}{\partial \vec{q}} d\vec{q}\wedge dt - \frac{\partial H}{\partial \vec{p}} d\vec{p}\wedge dt$$

$$= 0 - \int_{\sigma}\left[-\frac{\partial H}{\partial \vec{q}}\frac{\partial H}{\partial \vec{p}} + \frac{\partial H}{\partial \vec{p}}\frac{\partial H}{\partial \vec{q}}\right] ds dt = 0$$

- Matrix for 2-form

$$A = \begin{bmatrix} 0 & I & \partial H / \partial \vec{q} \\ -I & 0 & \partial H / \partial \vec{p} \\ -\partial H / \partial \vec{q} & -\partial H / \partial \vec{p} & 0 \end{bmatrix}$$

Canonical Transformations

- Canonical Transformations
  - Preserve the canonical symplectic structure
  - Preserve the sum of the projected areas
  - At equal time, preserve the relative integral invariant

$$\vec{p}\cdot d\vec{q}$$

  - Preserve Poisson Brackets and HEOM
- Physicists mainly use this idea for making coordinate transformations in phase space that preserve HEOM

$$Q^i = Q^i\left(\vec{q},\vec{p}\right) \qquad P_i = P_i\left(\vec{q},\vec{p}\right)$$

$$\vec{p}\cdot d\vec{q} - Hdt = \vec{Q}\cdot d\vec{P} - Kdt + dS$$

- An arbitrary S does not change the EOM as $ddS=0$

Generating Functions

- Suppose have two canonical coordinate sets

$$\left(\vec{q}, \vec{p}\right) \qquad \left(\vec{Q}, \vec{P}\right)$$

$$\vec{p} \cdot d\vec{q} - H\left(\vec{q}, \vec{p}\right) dt \qquad \vec{P} \cdot d\vec{Q} - K\left(\vec{Q}, \vec{P}\right) dt$$

on the same phase space related by transformation equations

$$Q^i = Q^i\left(\vec{q}, \vec{p}, t\right) \qquad P_i = P_i\left(\vec{q}, \vec{p}, t\right)$$

- Both *simultaneously* satisfy extremal principle (solve Hamilton's equations!) as long as

$$\vec{p} \cdot d\vec{q} - H\left(\vec{q}, \vec{p}\right) dt = \vec{P} \cdot d\vec{Q} - K\left(\vec{Q}, \vec{P}\right) dt + dS$$

$$S_1\left(\vec{q}, \vec{Q}, t\right), S_2\left(\vec{q}, \vec{P}, t\right), S_3\left(\vec{p}, \vec{Q}, t\right), S_4\left(\vec{p}, \vec{P}, t\right)$$

S_1 and S_2

- S_1

$$\vec{p}\cdot d\vec{q} - Hdt = \vec{P}\cdot d\vec{Q} - Kdt + \frac{\partial S_1}{\partial \vec{q}} d\vec{q} + \frac{\partial S_1}{\partial \vec{Q}} d\vec{Q} + \frac{\partial S_1}{\partial t} dt$$

$$\vec{p} = \frac{\partial S_1}{\partial \vec{q}} \qquad -\vec{P} = \frac{\partial S_1}{\partial \vec{Q}} \qquad K = H + \frac{\partial S_1}{\partial t}$$

- S_2

$$\vec{p}\cdot d\vec{q} - Hdt = d\left(\vec{Q}\cdot\vec{P}\right) - \vec{Q}\cdot d\vec{P} - Kdt + \frac{\partial S_2}{\partial \vec{q}} d\vec{q} + \frac{\partial S_2}{\partial \vec{P}} d\vec{P} + \frac{\partial S_2}{\partial t} dt$$

$$\vec{p} = \frac{\partial S_2}{\partial \vec{q}} \qquad \vec{Q} = \frac{\partial S_2}{\partial \vec{P}} \qquad K = H + \frac{\partial S_2}{\partial t}$$

S_3 and S_4

- S_3

$$d\left(\vec{q}\cdot\vec{p}\right)-\vec{q}\cdot d\vec{p}-Hdt=\vec{P}\cdot d\vec{Q}-Kdt+\frac{\partial S_3}{\partial\vec{p}}d\vec{p}+\frac{\partial S_3}{\partial\vec{P}}d\vec{P}+\frac{\partial S_3}{\partial t}dt$$

$$-\vec{q}=\frac{\partial S_3}{\partial\vec{p}}\qquad -\vec{P}=\frac{\partial S_3}{\partial\vec{Q}}\qquad K=H+\frac{\partial S_4}{\partial t}$$

- S_4

$$d\left(\vec{q}\cdot\vec{p}\right)-\vec{q}\cdot d\vec{p}-Hdt=d\left(\vec{Q}\cdot\vec{P}\right)-\vec{Q}\cdot d\vec{P}-Kdt+\frac{\partial S_4}{\partial\vec{p}}d\vec{p}+\frac{\partial S_4}{\partial\vec{P}}d\vec{P}+\frac{\partial S_4}{\partial t}dt$$

$$-\vec{q}=\frac{\partial S_4}{\partial\vec{p}}\qquad \vec{Q}=\frac{\partial S_4}{\partial\vec{P}}\qquad K=H+\frac{\partial S_4}{\partial t}$$

Examples

- Identity transformation

$$S_2 = \vec{q} \cdot \vec{P}$$

$$\vec{p} = \frac{\partial S_2}{\partial \vec{q}} = \vec{P} \qquad \vec{Q} = \frac{\partial S_2}{\partial \vec{P}} = \vec{q} \qquad K = H$$

- Coordinate-momenta swap

$$S_4 = \vec{p} \cdot \vec{P}$$

$$-\vec{q} = \frac{\partial S_4}{\partial \vec{p}} = \vec{P} \qquad \vec{Q} = \frac{\partial S_4}{\partial \vec{P}} = \vec{p} \qquad K = H$$

- Coordinate transformations on configuration space

$$S_2 = \sum_{i=1}^{n} Q^i\left(\vec{q}\right)\cdot \vec{P}_i$$

$$p_i = \frac{\partial S_2}{\partial q^i} = \frac{\partial Q^j}{\partial q^i} P_j \quad Q^i = \frac{\partial S_2}{\partial P_i} = Q^i\left(\vec{q}\right) \quad K = H$$

Momentum is indeed covariant

- Oscillator transformation (to action-angle coordinates)

$$q = A\cos Q \quad p = -m\omega A \sin Q = -m\omega q \tan Q$$

$$p = \frac{\partial S_1}{\partial q} \rightarrow S_1 = \frac{-m\omega q^2 \tan Q}{2} \quad P = -\frac{\partial S_1}{\partial Q} = \frac{m\omega A^2}{2}$$

$$H\left(Q,P\right) = \left(m\omega^2 A^2 \sin^2 Q + m\omega^2 A^2 \cos^2 Q\right)/2 = m\omega^2 A^2/2 = \omega P$$

Hamilton-Jacobi Equation

- Can we transform a difficult Hamiltonian into a simple one? Use S_2 also the classical action

$$\vec{p}\cdot d\vec{q} - Hdt = d\left(\vec{Q}\cdot\vec{P}\right) - \vec{Q}\cdot d\vec{P} - Kdt + \frac{\partial S_2}{\partial \vec{q}}d\vec{q} + \frac{\partial S_2}{\partial \vec{P}}d\vec{P} + \frac{\partial S_2}{\partial t}dt$$

$$\vec{p} = \frac{\partial S_2}{\partial \vec{q}} \qquad \vec{Q} = \frac{\partial S_2}{\partial \vec{P}} \qquad K = H + \frac{\partial S_2}{\partial t}$$

- Transform so K=0! Hamilton-Jacobi Equation

$$H\left(\vec{q},\frac{\partial S}{\partial \vec{q}}\right) + \frac{\partial S}{\partial t} = 0 \rightarrow S\left(\vec{q},\vec{\alpha}\right)$$

$$\dot{P}_i = 0 \rightarrow P_i = const,\ \text{ set to } \alpha_i \quad \dot{Q}^i = 0 \rightarrow Q^i = \beta^i$$

- Works best with separable Hamiltonian. Independent integrals

Hamiltonian not time-dependent

- One of the constants can be the Hamiltonian (energy)

$$S\left(\vec{q},\vec{P}\right)=W\left(\vec{q},\vec{P}\right)-\alpha t$$

$$H\left(\vec{q},\frac{\partial S}{\partial \vec{q}}\right)=\alpha \to S\left(\vec{q},\vec{\alpha}\right)$$

- Must get independent constants by

$$\beta_i=\frac{\partial S}{\partial \vec{P}}\left(\vec{q},\vec{P}\right)=\frac{\partial S}{\partial \alpha^i}$$

- Examples

$$H=\frac{p^2}{2m} \quad S=W\left(q,\alpha\right)-\alpha t=\sqrt{2m\alpha}\,q-\alpha t \to \beta=\frac{2m}{\sqrt{2m\alpha}}q-t$$

$$H=\frac{p^2}{2m}+\frac{kq^2}{2} \quad S=\pm\int\sqrt{2m}\left(\alpha-\frac{kq^2}{2}\right)^{1/2}dq-\alpha t \to$$

$$\beta=\pm\left(\frac{m}{2}\right)^{1/2}\left\{-\left(\frac{2}{k}\right)^{1/2}\cos^{-1}\left[\left(\frac{k}{2\alpha}\right)^{1/2}q\right]\right\}$$

Particle in EM Field

- Follow procedure using the Lagrangian (MKS)

$$L = \frac{m\left|\vec{v}\right|^2}{2} - \left[q\phi - \vec{v}\cdot\vec{A}\right]$$

$$\vec{E}\left(\vec{x},t\right) = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$$

$$\vec{B}\left(\vec{x},t\right) = \vec{\nabla}\times\vec{A}$$

- Generalized momentum (usually capitalized for this problem (F&W leave the generalized momentum small!))

$$\vec{P} = \frac{\partial L}{\partial\vec{v}} = m\vec{v} + q\vec{A}$$

$$m\vec{v} = \vec{P} - q\vec{A}$$

Hamiltonian for Particle

- Using the prescription defining the Hamiltonian

$$H = \vec{P} \cdot \vec{v} - L$$

$$\frac{\vec{P} \cdot \left(\vec{P} - q\vec{A}\right)}{m} - \frac{\left(\vec{P} - q\vec{A}\right) \cdot \left(\vec{P} - q\vec{A}\right)}{2m} + q\phi - q\frac{\left(\vec{P} - q\vec{A}\right)}{m} \cdot \vec{A}$$

$$H\left(\vec{x}, \vec{P}\right) = \frac{\left(\vec{P} - q\vec{A}\right) \cdot \left(\vec{P} - q\vec{A}\right)}{2m} + q\phi$$

- Equations of motion

$$\dot{\vec{x}} = \frac{\partial H}{\partial \vec{P}} = \frac{\vec{P} - q\vec{A}}{m}$$

$$\dot{\vec{P}} = -\frac{\partial H}{\partial \vec{x}} = \frac{\left(\vec{P} - q\vec{A}\right)}{m} \cdot \frac{\partial\left(q\vec{A}\right)}{\partial \vec{x}} - q\nabla\phi$$

- x component of the momentum equation

$$\dot{P}_x = m\dot{v}_x + q\dot{A}_x = q\vec{v}\cdot\frac{\partial\vec{A}}{\partial x} - q\frac{\partial\phi}{\partial x}$$

$$m\dot{v}_x = -q\left[v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}\right] + q\left[v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\right]$$

$$-q\frac{\partial\phi}{\partial x} - q\frac{\partial A_x}{\partial t}$$

$$= q\left(\vec{E} + \vec{v}\times\vec{B}\right)_x$$

- Relativistic Lagrangian

$$L = -\sqrt{1-\left|\vec{v}\right|^2/c^2}\,mc^2 - q\left(\phi - \vec{v}\cdot\vec{A}\right) \qquad H = \sqrt{\left(\vec{P}-q\vec{A}\right)\cdot\left(\vec{P}-q\vec{A}\right)c^2 + m^2c^4} + q\phi$$

$$\frac{d}{dt}\left(\gamma m\vec{v}\right) = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$