Physics 451/551
Theoretical Mechanics

G. A. Krafft
Old Dominion University
Jefferson Lab

Lecture 9

Invariance of Symplectic Structure

- This result leads directly to the result that the symplectic structure is an invariant of the flow generated by any Hamiltonian *H*
- For any bounded surface in phase space σ, the phase flow generates a “volume” $V(t)$
- Note

$$\partial V(t) = g^t\sigma - \sigma + g^{t'}\partial\sigma \qquad 0 \le t' \le t$$

- Meaning

$$0 = \int_{V(t)} d\omega^2 = \int_{\partial V(t)} \omega^2 = \int_{g^t\sigma} \omega^2 - \int_{\sigma} \omega^2 + 0 \rightarrow \int_{g^t\sigma} \omega^2 = \int_{\sigma} \omega^2$$

- Arnold’s words

$$g^{t*}\omega^2 = \omega^2$$

Integral Invariants

- Absolute integral invariant: A differential k-form ω is called an absolute integral invariant of the map g if the integrals of ω on any k-chain and on its image are the same

$$\int_{gc} \omega = \int_{c} \omega$$

We have shown that ω^2 is an absolute integral invariant of the Hamiltonian flow g^t. So is $(\omega^2)^n$!

- Relative integral invariant: A differential k-form ω is called a relative integral invariant if the above equality holds on every *closed* k-chain
- Example $\quad g^t \qquad \omega^1 = \sum_{i=1}^{n} p_i dq^i$

Canonical Transformations

- Note: by Stoke's Theorem, d(relative inv)=absolute inv
- Definition: A transformation is called *canonical* if it has ω^2 as an (absolute) integral invariant. Then, it is true that all powers are also (absolute) integral invariants (Liouville).

$$g^{t*}\omega^2 = \omega^2$$

$$g^{t*}\omega^{2n} = g^{t*}\omega^2 \wedge \cdots \wedge g^{t*}\omega^2 = \left(g^{t*}\omega^2\right)^n = \omega^{2n}$$

- Another largely equivalent definition (we'll show in a bit): A transformation is canonical if it leaves the Hamilton equations of motion invariant (i.e. the motion is described by Hamilton's equations of motion both before and after the transformation)
- Conservation of energy:

$$dH\left(\vec{\xi}_H\right) = \omega^2\left(\vec{\xi}_H, I_A dH\right) = \omega^2\left(\vec{\xi}_H, \vec{\xi}_H\right) = 0$$

Poisson Bracket

- On a phase space with coordinates $(q^1,\ldots,q^n,p_1,\ldots,p_n)$ differentiate a function F along the phase flow given by the Hamiltonian function H

$$[F,H]\left(q^i,\cdots,q^n,p_1,\cdots,p_n\right)\equiv\left.\frac{d}{dt}\right|_{t=0}F\left(g_H^t\left(q^i,\cdots,q^n,p_1,\cdots,p_n\right)\right)$$

  - Corollary 1. F is a first integral (constant of the motion) iff $[F,H]=0$
  - Corollary 2+3.

$$\omega(\vec{\eta})=\omega^2\left(\vec{\eta},I_A\omega\right)\rightarrow[F,H]=dF\left(\xi_H\right)=dF\left(I_A dH\right)=\omega^2\left(I_A dH,I_A dF\right)$$

  - This gives Corollary 4. $[F,H]$ is a skew symmetric bi-linear function

Expressions in Coordinates

- Flipping the roles of *F* and *H* yields Nöther's Theorem: If the Hamiltonian *H* is not changed when displaced by $I_A\, dF$, then *F* is a constant of the motion given by *H*

$$dH\left(I_A dF\right) = 0 \rightarrow \omega^2\left(I_A dF, I_A dH\right) = 0$$

$$\rightarrow \omega^2\left(I_A dH, I_A dF\right) = 0 \rightarrow [F, H] = 0$$

- Poisson Brackets in coordinates

$$\left[F, H\right] = \omega^2\left(I_A dH, I_A dF\right)$$

$$= \sum_{i=1}^{n}\left[dp_i\left(I_A dH\right) dq^i\left(I_A dF\right) - dq^i\left(I_A dH\right) dp_i\left(I_A dF\right)\right]$$

$$= \sum_{i=1}^{n}\left[-\frac{\partial H}{\partial q^i}\frac{\partial F}{\partial p_i} + \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q^i}\right] = \sum_{i=1}^{n}\left[\frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i}\right]$$

- The same expression for any canonical coordinate set!

Jacobi Identity

- The Poisson Bracket satisfies Jacobi identity

$$\left[F_1,\left[F_2,F_3\right]\right]+\left[F_2,\left[F_3,F_1\right]\right]+\left[F_3,\left[F_1,F_2\right]\right]=0$$

- If F_1 and F_2 two first integrals, then so is their Poisson Bracket (Poisson's Theorem)

$$\left[F_1,H\right]=0 \qquad \left[F_2,H\right]=0$$

$$\left[F_1,\left[F_2,H\right]\right]+\left[F_2,\left[H,F_1\right]\right]+\left[H,\left[F_1,F_2\right]\right]=$$

$$=0+0-\left[\left[F_1,F_2\right],H\right]=0$$

- The Hamiltonian vector fields on a symplectic manifold form a sub-algebra of the algebra of all vector field.

Equations of Motion in PB Form

- Consider the time derivative of any function on phase space

$$F\left(\vec{q}(t),\vec{p}(t),t\right)$$

$$\frac{dF}{dt}=\frac{\partial F}{\partial \vec{q}}\frac{d\vec{q}}{dt}+\frac{\partial F}{\partial \vec{p}}\frac{d\vec{p}}{dt}+\frac{\partial F}{\partial t}$$

$$=\frac{\partial F}{\partial \vec{q}}\frac{\partial H}{\partial \vec{p}}-\frac{\partial F}{\partial \vec{p}}\frac{\partial H}{\partial \vec{q}}+\frac{\partial F}{\partial t}=\left[F,H\right]+\frac{\partial F}{\partial t}$$

- When applied to q_i or p_i (automatically!) get Hamilton's equations of motion back
- Fundamental Poisson Bracket relations

$$\left[p_i,p_j\right]=\left[p_i,q^j\right]=\left[q^i,q^j\right]=0 \quad \forall i,j, \text{ except } \left[q^i,p_i\right]=-\left[p_i,q^i\right]=1$$

- Become operator commutation relations in quantum mechanics

Back to Dynamics

ODU

- Recall Hamilton's Principle (variation of the action is extremal)

$$\delta \int L\left(\vec{q}, \dot{\vec{q}}\right) dt = 0$$

- Variation gives *n* second order Euler-Lagrange equations of motion
- Recall for time independent systems, the Hamiltonian function (energy) is a constant of motion

$$H = \frac{\partial L}{\partial \dot{\vec{q}}} \cdot \dot{\vec{q}} - L$$

- Change viewpoint: double the number of independent variables to 2*n*

$$\vec{p} \equiv \frac{\partial L}{\partial \dot{\vec{q}}} \qquad H\left(\vec{q}, \vec{p}\right) = \vec{p} \cdot \dot{\vec{q}}\left(\vec{q}, \vec{p}\right) - L\left(\vec{q}, \dot{\vec{q}}\left(\vec{q}, \vec{p}\right)\right)$$

Hamilton's Equations of Motion

- Variational principle still applies, now to variations of 2*n* independent variables

$$\delta\int L\left(\vec{q},\dot{\vec{q}}\right)dt=\delta\int\left[\vec{p}\cdot\dot{\vec{q}}\left(\vec{q},\vec{p}\right)-H\left(\vec{q},\vec{p}\right)\right]dt$$

$$=\int\left[\delta\vec{p}\cdot\dot{\vec{q}}+\vec{p}\cdot\delta\dot{\vec{q}}-\frac{\partial H}{\partial\vec{q}}\cdot\delta\vec{q}-\frac{\partial H}{\partial\vec{p}}\cdot\delta\vec{p}\right]dt$$

$$=\int\left[\delta\vec{p}\cdot\dot{\vec{q}}-\dot{\vec{p}}\cdot\delta\vec{q}-\frac{\partial H}{\partial\vec{q}}\cdot\delta\vec{q}-\frac{\partial H}{\partial\vec{p}}\cdot\delta\vec{p}\right]dt$$

- Because must vanish for variations in all directions, get Hamilton's equations of motion. 2*n* first order equations!

$$\dot{\vec{q}}=\frac{\partial H}{\partial\vec{p}}\qquad\qquad\dot{\vec{p}}=-\frac{\partial H}{\partial\vec{q}}$$

Phase Flow Canonical

- Using Hamilton's equations of motion, the phase space flow is a canonical transformation

$$q'^i(\vec{q},\vec{p}) = q^i + \frac{\partial H}{\partial p_i}\Delta t + \cdots$$

$$p'_i(\vec{q},\vec{p}) = p_i - \frac{\partial H}{\partial q^i}\Delta t + \cdots$$

$$dq'^i = dq^i + \left[\sum_{j=1}^{n}\frac{\partial^2 H}{\partial q^j \partial p_i}dq^j + \sum_{j=1}^{n}\frac{\partial^2 H}{\partial p_j \partial p_i}dp_j\right]\Delta t + \cdots$$

$$dp'_i = dp_i - \left[\sum_{j=1}^{n}\frac{\partial^2 H}{\partial q^j \partial q^i}dq^j + \sum_{j=1}^{n}\frac{\partial^2 H}{\partial p_j \partial q^i}dp_j\right]\Delta t + \cdots$$

$$\sum_{i=1}^{n} dp'_i \wedge dq'^i = \sum_{i=1}^{n} dp_i \wedge dq^i + \sum_{i=1}^{n}\left[-\sum_{j=1}^{n}\frac{\partial^2 H}{\partial q^j \partial q^i}dq^j \wedge dq^i\right]\Delta t + \sum_{i=1}^{n}\left[\sum_{j=1}^{n}\frac{\partial^2 H}{\partial p_j \partial p_i}dp_i \wedge dp_j\right]\Delta t$$

$$+\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\frac{\partial^2 H}{\partial q^j \partial p_i}dp_i \wedge dq^j - \sum_{j=1}^{n}\frac{\partial^2 H}{\partial p_j \partial q^i}dp_j \wedge dq^i\right]\Delta t + O\left(\Delta t^2\right) = \sum_{i=1}^{n} dp_i \wedge dq^i$$

Variational Principle via Arnold

- Integral invariant of Poincare-Cartan

$$S=\int L dt=\int\left[\vec{p}\cdot\dot{\vec{q}}\left(\vec{q},\vec{p}\right)-H\left(\vec{q},\vec{p}\right)\right]dt=\int\left[\vec{p}\cdot d\vec{q}-Hdt\right]$$

- Significance: when action integrated around a closed loop, it has constant value as loop moves according to the phase flow. If *dt*=0 on loop, value is sum of projected PS areas.
- Stokes Lemma in usual fluid mechanics. Let

$$\vec{r}=\vec{\nabla}\times\vec{v} \qquad \text{Note that } \nabla\cdot\vec{r}=0$$

and a loop γ flow along with the vector field $\vec{r}$. Then by considering the tube swept out in time

$$\oint_{\gamma}\vec{v}\cdot d\vec{l}-\oint_{\gamma'}\vec{v}\cdot d\vec{l}=\oiint_{\sigma}\vec{\nabla}\times\vec{v}\cdot\vec{n}dS=0$$

$$\int_{\partial\sigma}\omega^1_{\vec{v}}=\int_{\sigma}\omega^2_{\nabla\times\vec{v}}=\iint_{\sigma}\nabla\times\vec{v}\cdot\frac{\partial\vec{\sigma}}{\partial s}\times\frac{\partial\vec{\sigma}}{\partial t}dsdt=0 \qquad \vec{r}\cdot\frac{\partial\vec{\sigma}}{\partial s}\times\vec{r}=0$$

Canonical Transformations

- Canonical Transformations
  - Preserve the canonical symplectic structure
  - Preserve the sum of the projected areas
  - At equal time, preserve the relative integral invariant

$$\vec{p} \cdot d\vec{q}$$

  - Preserve Poisson Brackets and HEOM
- Physicists mainly use this idea for making coordinate transformations in phase space that preserve HEOM

$$Q^i = Q^i\left(\vec{q}, \vec{p}\right) \qquad P_i = P_i\left(\vec{q}, \vec{p}\right)$$

$$\vec{p} \cdot d\vec{q} - Hdt = \vec{Q} \cdot d\vec{P} - Kdt + dS$$

- An arbitrary S does not change the EOM as $ddS=0$

Generating Functions

- Suppose have two canonical coordinate sets

$$(\vec{q},\vec{p}) \qquad \left(\vec{Q},\vec{P}\right)$$

$$\vec{p}\cdot d\vec{q}-H(\vec{q},\vec{p})dt \qquad \vec{P}\cdot d\vec{Q}-K\left(\vec{Q},\vec{P}\right)dt$$

on the same phase space related by transformation equations

$$Q^i = Q^i(\vec{q},\vec{p},t) \qquad P_i = P_i(\vec{q},\vec{p},t)$$

- Both *simultaneously* satisfy extremal principle (solve Hamilton’s equations!) as long as

$$\vec{p}\cdot d\vec{q}-H(\vec{q},\vec{p})dt=\vec{P}\cdot d\vec{Q}-K\left(\vec{Q},\vec{P}\right)dt+dS$$

$$S_1\left(\vec{q},\vec{Q},t\right),S_2\left(\vec{q},\vec{P},t\right),S_3\left(\vec{p},\vec{Q},t\right),S_4\left(\vec{p},\vec{P},t\right)$$

S_1 and S_2

- S_1

$$\vec{p}\cdot d\vec{q} - Hdt = \vec{P}\cdot d\vec{Q} - Kdt + \frac{\partial S_1}{\partial \vec{q}} d\vec{q} + \frac{\partial S_1}{\partial \vec{Q}} d\vec{Q} + \frac{\partial S_1}{\partial t} dt$$

$$\vec{p} = \frac{\partial S_1}{\partial \vec{q}} \qquad -\vec{P} = \frac{\partial S_1}{\partial \vec{Q}} \qquad K = H + \frac{\partial S_1}{\partial t}$$

- S_2

$$\vec{p}\cdot d\vec{q} - Hdt = d\left(\vec{Q}\cdot\vec{P}\right) - \vec{Q}\cdot d\vec{P} - Kdt + \frac{\partial S_2}{\partial \vec{q}} d\vec{q} + \frac{\partial S_2}{\partial \vec{P}} d\vec{P} + \frac{\partial S_2}{\partial t} dt$$

$$\vec{p} = \frac{\partial S_2}{\partial \vec{q}} \qquad \vec{Q} = \frac{\partial S_2}{\partial \vec{P}} \qquad K = H + \frac{\partial S_2}{\partial t}$$

S_3 and S_4

- S_3

$$d\left(\vec{q}\cdot\vec{p}\right)-\vec{q}\cdot d\vec{p}-Hdt=\vec{P}\cdot d\vec{Q}-Kdt+\frac{\partial S_3}{\partial\vec{p}}d\vec{p}+\frac{\partial S_3}{\partial\vec{P}}d\vec{P}+\frac{\partial S_3}{\partial t}dt$$

$$-\vec{q}=\frac{\partial S_3}{\partial\vec{p}}\qquad -\vec{P}=\frac{\partial S_3}{\partial\vec{Q}}\qquad K=H+\frac{\partial S_4}{\partial t}$$

- S_4

$$d\left(\vec{q}\cdot\vec{p}\right)-\vec{q}\cdot d\vec{p}-Hdt=d\left(\vec{Q}\cdot\vec{P}\right)-\vec{Q}\cdot d\vec{P}-Kdt+\frac{\partial S_4}{\partial\vec{p}}d\vec{p}+\frac{\partial S_4}{\partial\vec{P}}d\vec{P}+\frac{\partial S_4}{\partial t}dt$$

$$-\vec{q}=\frac{\partial S_4}{\partial\vec{p}}\qquad \vec{Q}=\frac{\partial S_4}{\partial\vec{P}}\qquad K=H+\frac{\partial S_4}{\partial t}$$