Physics 451/551 Theoretical Mechanics

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Lecture 8

Hamiltonian Method

- Form generalized momenta and solve generalized velocity

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \qquad \text{obtain } \dot{q}_i\left(\vec{q}, \vec{p}\right)$$

- Form Hamiltonian from Lagrangian

$$H\left(\vec{q}, \vec{p}\right) = \sum_{i=1}^{n} p_i \dot{q}^i\left(\vec{q}, \vec{p}\right) - L\left(\vec{q}, \dot{\vec{q}}\left(\vec{q}, \vec{p}\right)\right)$$

- Dynamics are solutions to Hamilton's canonical equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}$$

- Interchangeable notations (for compact expressions)

$$\left(q^1, \cdots, q^n, p_1, \cdots, p_n\right) = \left(\vec{q}, \vec{p}\right) = \left(\boldsymbol{q}, \boldsymbol{p}\right) \qquad \dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}$$

Symplectic Structure

- A symplectic manifold is an even-dimensional manifold with a closed ($d\omega^2=0$) non-degenerate 2-form on it. ω^2 is called the *symplectic structure*. We will show that phase spaces are symplectic manifolds.
- Non-degenerate means

$$\forall \vec{v}_1 \ \exists \vec{v}_2 \text{ so that } \omega^2\left(\vec{v}_1, \vec{v}_2\right) \neq 0$$

This mainly technical requirement, allows one to exclude uninteresting cases where, e.g., $\omega^2=0$. Implies invertible matrix

- Example: the vector space $\mathbf{R}^{2n}$, call half the variables q^i, and half the variables p_i

$$\omega^2 = \sum_{i=1}^{n} dp_i \wedge dq^i$$

- Non-degenerate because for every vector with a component in q^i direction, associate p_i component and visa-versa

Co-tangent Bundle

- The potential values of the generalized coordinates of a mechanical system q^i form an n-dimensional manifold M.
- The co-tangent bundle T^*M is the set of all possible 1-forms on this manifold. In general, an element can be written as

$$\omega = \sum_{i=1}^{n} p_i(\vec{q})\, dq^i \qquad p_i = \omega\left(e_{q^i}\right)$$

- For each 1-form on M, one expresses it in a particular coordinate set by evaluating $p_i(q^1,\ldots,q^n)$, i.e. assigning an associated p_i set at each q-location.
- Because there are n dq^is, the cotangent bundle is a dimension $2n$ manifold. Next, regard p_i, q^i as the coordinates on the bundle.

Natural Symplectic Structure

- There is a natural (independent of coordinates) projection from the co-tangent bundle T^*M to M

$$f : T^*M \to M \qquad f\left(\sum_{i=1}^{n} p_i\left(\vec{q}\right)dq^i\right) = \vec{q}$$

- Using any coordinates on M, f projects out p_i components

$$f\left(\vec{q},\vec{p}\right) = \vec{q} \qquad f^i\left(\vec{q},\vec{p}\right) = q^i,\ i = 1,\cdots,n$$

$$\frac{\partial f^i}{\partial q^j} = I \qquad \frac{\partial f^i}{\partial p_j} = 0 \qquad D = \begin{pmatrix} I & 0 \end{pmatrix}$$

- For a tangent vector in the co-tangent bundle define the canonical one form

$$\omega^1_{(\vec{q},\vec{p})}\left(\vec{\xi}\right) = \omega\left(D\vec{\xi}\right) = \sum_{i=1}^{n} p_i dq^i\left(D\vec{\xi}\right)$$

Phase Space Symplectic Structure

- Take the exterior derivative to obtain the natural symplectic structure

$$\omega^2_{(\vec{q},\vec{p})} = d\omega^1 = \sum_{i=1}^{n} dp_i \wedge dq^i$$

Automatically closed!

- Analytic verification natural to coordinate transformations

$$\bar{v}^i e_{\bar{q}^i} = v^j e_{q^j} \rightarrow \bar{v}^i = v^j M^i{}_j \rightarrow \bar{v}^i M^{-1j}{}_i = v^j \qquad M^i{}_j = d\bar{q}^i\left(e_{q^j}\right)$$

$$d\bar{q}^i = \frac{\partial \bar{q}^i}{\partial q^j} dq^j \rightarrow M^i{}_j = \frac{\partial \bar{q}^i}{\partial q^j} \qquad \bar{p}_i = \omega\left(e_{\bar{q}^i}\right) = \omega\left(M^{-1j}{}_i e_{q^j}\right) = M^{-1j}{}_i p_j$$

$$\sum_{i=1}^{n} d\bar{p}_i \wedge d\bar{q}^i = \sum_{i=1}^{n} M^{-1j}{}_i dp_j \wedge M^i{}_k dq^k = \sum_{j=1}^{n} \delta^j{}_k dp_j \wedge dq^k = \sum_{j=1}^{n} dp_j \wedge dq^j$$

Canonical Isomorphism

- As in Riemannian geometry, one can make an association between the vectors and one-forms on the co-cotangent bundle, but now using the symplectic structure

$$\vec{\xi} \leftrightarrow \omega^1_{\vec{\xi}}\left(\vec{\eta}\right) \equiv \omega^2\left(\vec{\eta}, \vec{\xi}\right)$$

- This association is a mathematical isomorphism (i.e., the two vector spaces are algebraically interchangeable)
- In a particular coordinate system

$$\vec{\eta} = \begin{pmatrix} \vec{q}_\eta \\ \vec{p}_\eta \end{pmatrix} \qquad \vec{\xi} = \begin{pmatrix} \vec{q}_\xi \\ \vec{p}_\xi \end{pmatrix}$$

$$\omega^2_{(\vec{q},\vec{p})}\left(\vec{\eta}, \vec{\xi}\right) = \sum_{i=1}^{n}\left(p_{\eta i} q^i_\xi - p_{\xi i} q^i_\eta\right) = \vec{\eta}^{\,t} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \vec{\xi} \equiv \vec{\eta}^{\,t} J \vec{\xi}$$

Hamiltonian Vector Fields

- To match up

$$\omega^1_{\vec{\xi}}(\vec{\eta}) = \sum_{i=1}^{n} w_i dq^i(\vec{\eta}) + w^i dp_i(\vec{\eta}) = \omega^2(\vec{\eta},\vec{\xi})$$

$$\vec{\xi} \leftrightarrow \omega^1_{\vec{\xi}}(\vec{\eta}) = \sum_{i=1}^{n} q^i_{\xi} dp_i - p_{\xi i} dq^i = \vec{q}_{\xi}\cdot d\vec{p} - \vec{p}_{\xi}\cdot d\vec{q}$$

- Denote by I_A the map that takes an arbitrary one-form to its associated vector field, and let H be a real-valued function on phase space. When

$$dH = \sum_{i=1}^{n}\left[\frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i\right] \qquad \vec{\xi}_H = I_A dH = \begin{pmatrix} \dfrac{\partial H}{\partial \vec{p}} \\ -\dfrac{\partial H}{\partial \vec{q}} \end{pmatrix}$$

for some function H, then $\vec{\xi}_H$ is called a Hamiltonian vector field with Hamiltonian H

- Picture "flow" as a mapping that takes all initial points in phase space at $t = 0$ into their positions at $t = \tau$

$$g^t : M^{2n} \to M^{2n} \qquad 0 \le t \le \tau$$

$$\left.\frac{dg^t}{dt}\right|_t = \vec{\xi}_H\left(\vec{q},\vec{p}\right) = I_A\ dH = \begin{pmatrix} \partial H / \partial \vec{p} \\ -\partial H / \partial \vec{q} \end{pmatrix}$$

$$q^i\left(t\right) = q_0^i + \int_0^t \left(I_A\ dH\right)^i dt \quad p_i\left(t\right) = p_{i0} + \int_0^t \left(I_A\ dH\right)^{i+n} dt \quad i = 1,\cdots,n$$

- Can think of as a matrix-like function

$$q^i\left(t\right) = g^{ti}\left(\vec{q}_0,\vec{p}_0,t\right)$$

$$p_i\left(t\right) = g^t{}_i\left(\vec{q}_0,\vec{p}_0,t\right)$$

$$\left.\frac{dg^t}{dt}\right|_t = \vec{\xi}_H\left(\vec{q}\left(t\right),\vec{p}\left(t\right)\right)$$

- Let $S(s)$ be a string in phase space and let

$$S^i(s,t) \equiv g^t(S)$$

be the surface the string traces out in phase space under a Hamiltonian flow given by its components in time. Then

$$\xi_H^i = \frac{\partial S^i}{\partial t} \qquad \eta^i = \frac{\partial S^i}{\partial s}$$ is the component of the tangent to string

$$\int_{S(s,t)} \omega^2 = \int_0^t \int_0^{s_0} \omega^2\left(\vec{\eta}, \vec{\xi}_H\right) ds dt = \int_0^t \int_{g^t S} \left[\frac{\partial H}{\partial \vec{q}} \frac{d\vec{q}}{ds} + \frac{\partial H}{\partial \vec{p}} \frac{d\vec{p}}{ds}\right] ds dt = \int_0^t \int_{g^t S} dH dt$$

- If the string is closed

$$\int_{g^t S} dH = \int_{\partial g^t S} H = 0 \rightarrow \int_{S(s,t)} \omega^2 = 0$$

Invariance of Symplectic Structure

- This result leads directly to the result that the symplectic structure is an invariant of the flow generated by any Hamiltonian H
- For any bounded surface in phase space σ, the phase flow generates a "volume" $V(t)$
- Note

$$\partial V(t) = g^t \sigma - \sigma + g^{t'} \partial \sigma \qquad 0 \le t' \le t$$

- Meaning

$$0 = \int_{V(t)} d\omega^2 = \int_{\partial V(t)} \omega^2 = \int_{g^t \sigma} \omega^2 - \int_{\sigma} \omega^2 + 0 \rightarrow \int_{g^t \sigma} \omega^2 = \int_{\sigma} \omega^2$$

- Arnold's words

$$g^{t*} \omega^2 = \omega^2$$