Physics 451/551
Theoretical Mechanics

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Lecture 7

Vector Analysis Equivalents

- Integration of a 3-form normal volume integral

$$\int \omega_h^3 = \int_V h(x,y,z)\,dxdydz$$

- Integral of 1-form the line integral

$$\int_{C(s)} \omega_{\vec{f}}^1 = \oint_{C(s)} \vec{f} \cdot d\vec{l}$$

- Integral of 2-form the flux integral

$$\frac{\partial \vec{x}}{\partial s} \times \frac{\partial \vec{x}}{\partial t}\,dsdt = \vec{n}dA \rightarrow \int_{S(s,t)} \omega_{\vec{g}}^2 = \oint_S \vec{g} \cdot \vec{n}dA$$

- Note: *k*-form should be integrated over a *k* dimensional object

Some Interesting Results

- 3-D vector product of two vectors hidden in the rules

$$\omega^1_{\vec f}\wedge\omega^1_{\vec g}\left(\vec v_1,\vec v_2\right)=\omega^2_{\vec f\times\vec g}\left(\vec v_1,\vec v_2\right)$$

- 3-D scalar product of two vectors hidden in the rules

$$\omega^1_{\vec f}\wedge\omega^2_{\vec g}\left(\vec v_1,\vec v_2,\vec v_3\right)=\left(\vec f\cdot\vec g\right)\omega^3\left(\vec v_1,\vec v_2,\vec v_3\right)$$

Show both of these in HW problem

- Given a *k*-form and a linear map on a vector space, one can form a new *k*-form as follows

$$L:R^m\to R^n \quad \text{linear} \quad \omega^k \text{ a } k\text{-form on } R^n$$

$$L^*\omega^k\left(\vec v_1,\cdots,\vec v_k\right)\equiv\omega^k\left(L\vec v_1,\cdots,L\vec v_k\right)$$

$$\left(L^*\omega^k\right)\left(\vec v_{\xi_1},\cdots,\vec v_{\xi_k}\right)=\omega^k\left(L\vec v_{\xi_1},\cdots,L\vec v_{\xi_k}\right)=\left(-1\right)^\nu\omega^k\left(L\vec v_1,\cdots,L\vec v_k\right)$$

$$=\left(-1\right)^\nu\left(L^*\omega^k\right)\left(\vec v_1,\cdots,\vec v_k\right) \quad \text{is a } k\text{-form on } R^m\,!$$

- This important operation is called *pullback*

Differentiation of Differential Forms

- There is a derivative of forms, called the *exterior derivative d*, which is uniquely specified by the rules

$$d \quad \text{takes a } k\text{-form to a } k+1\text{-form}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$dd = 0$$

$$d\left(\alpha \wedge \beta\right) = d\alpha \wedge \beta + \left(-1\right)^p \alpha \wedge d\beta \quad \alpha \text{ a } p\text{-form}$$

- Operationally, treat functions as 0-forms and repeatedly use rules three and four. Rule 3 automatic from equality of mixed partial derivatives
- *d* encompasses the gradient operation!

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \omega^1_{\nabla f}$$

- *d* encompasses the curl operation!

$$d\omega^1_{\vec{f}} = d\left(f_x dx + f_y dy + f_z dz \right)$$

$$= \left[\frac{\partial f_x}{\partial x} dx + \frac{\partial f_x}{\partial y} dy + \frac{\partial f_x}{\partial z} dz \right] \wedge dx + \left[\frac{\partial f_y}{\partial x} dx + \frac{\partial f_y}{\partial y} dy + \frac{\partial f_y}{\partial z} dz \right] \wedge dx$$

$$+ \left[\frac{\partial f_z}{\partial x} dx + \frac{\partial f_z}{\partial y} dy + \frac{\partial f_z}{\partial z} dz \right] \wedge dz = \omega^2_{\nabla \times \vec{f}}$$

- *d* encompasses the divergence operation

$$d\omega^2_{\vec{g}} = d\left(g_x dy \wedge dz + g_y dz \wedge dx + g_z dx \wedge dy \right)$$

$$= \left[\frac{\partial g_x}{\partial x} dx + \frac{\partial g_x}{\partial y} dy + \frac{\partial g_x}{\partial z} dz \right] \wedge dy \wedge dz + \left[\frac{\partial g_y}{\partial x} dx + \frac{\partial g_y}{\partial y} dy + \frac{\partial g_y}{\partial z} dz \right] \wedge dz \wedge dx$$

$$+ \left[\frac{\partial g_z}{\partial x} dx + \frac{\partial g_z}{\partial y} dy + \frac{\partial g_z}{\partial z} dz \right] \wedge dx \wedge dy = \left(\nabla \cdot \vec{g} \right) \omega^3$$

- More than this: it allows similar operations to be defined *beyond* 3-D Euclidean space, e.g., in phase space!

Generalized Stoke's Theorem

- Fundamental Theorem of Multidimensional Integration

$$\int_{\partial S} \omega = \int_{S} d\omega$$

- Encompasses (and is an essential result from) the fundamental theorem of calculus. In 1-D

$$\int_{\partial S} f = f(b) - f(a) = \int_a^b \frac{df}{dx} dx = \int_S df$$

- In 3-D use the line integral to define things like the potential

$$\int_{\partial[a,b]} -dU = -U(b) + U(a) = -\int_a^b \left[\frac{dU}{dx}\frac{dx}{ds} + \frac{dU}{dy}\frac{dy}{ds} + \frac{dU}{dz}\frac{dz}{ds} \right] ds = \int \vec{f} \cdot d\vec{l}$$

Classical Vector Analysis Theorems

- Green's Theorem in 2-D

$$\int_{\partial S} Pdx + Qdy = \int_{S} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dxdy$$

- Gauss's Divergence Theorem

$$\int_{\partial V} \vec{V} \cdot \vec{n} dA = \int_{\partial V} \omega^2_{\vec{V}} = \int_{V} d\omega^2_{\vec{V}} = \int_{V} \left(\nabla \cdot \vec{V} \right) dV$$

- Classical Stoke's Theorem

$$\int_{\partial S} \vec{V} \cdot d\vec{l} = \int_{\partial S} \omega^1_{\vec{V}} = \int_{S} d\omega^1_{\vec{V}} = \int_{S} \omega^2_{\nabla \times \vec{V}} = \int_{S} \nabla \times \vec{V} \cdot \vec{n} dA$$

Closed and Exact Forms

- A *Closed form* is one whose exterior derivative vanishes

$$dC = 0$$

- An *Exact form* is one that is the exterior derivative of a lower-dimensional form

$$E = dF \quad \text{for some form } F$$

- Exact implies closed because $d^2 = 0$
- On a simply connected regions of 3-space (essentially no holes in the space) closed implies exact (Poincare's lemma)
- Conservative vector force fields are exact: they are the gradients of the potential function

Some Important Pullback Properties

- Pullback and exterior derivative commute

$$d\left(f^*\omega\right) = f^* d\omega$$

- Pullback preserves the exterior product (Arnold problem!)

$$f^*\left(\omega_1 \wedge \omega_2\right) = f^*\omega_1 \wedge f^*\omega_2$$

- Fundamental Theorem of Pullback Integration: for a diffeomorphism (usually a coordinate transformation)

$$f : M \to N$$

$$\int_c f^*\omega = \int_{fc} \omega$$

- Coordinate independence on multi-dimensional integrals.

Phase Spaces of Mechanical Systems

- We know from Physics 319 that we can set up a Lagrangian for a mechanical system. The state of the system is described by generalized coordinate values for the motion. The possible motions are a manifold (a mathematical object that can be described by a continuous coordinate system)
- From the Lagrangian we can derive the generalized momenta
- We imagine a $2n$ dimensional phase space associated with the problem, where n is the number of degrees of freedom (e.g. CM motion in 3 space is a 3 degree of freedom problem. Phase space has 6 dimensions)
- One can define a fundamental differential form involving phase space for every mechanical system.

Hamilton Method

- Form generalized momenta and solve generalized velocity

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \qquad \text{obtain } \dot{q}_i\left(\vec{q}, \vec{p}\right)$$

- Form Hamiltonian from Lagrangian

$$H\left(\vec{q}, \vec{p}\right) = \sum_{i=1}^{n} p_i \dot{q}^i\left(\vec{q}, \vec{p}\right) - L\left(\vec{q}, \dot{\vec{q}}\left(\vec{q}, \vec{p}\right)\right)$$

- Dynamics are solutions to Hamilton's canonical equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}$$

- Interchangeable notations (for compact expressions)

$$\left(q^1, \cdots, q^n, p_1, \cdots, p_n\right) = \left(\vec{q}, \vec{p}\right) = \left(\boldsymbol{q}, \boldsymbol{p}\right) \qquad \dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}$$