Physics 451/551
Theoretical Mechanics

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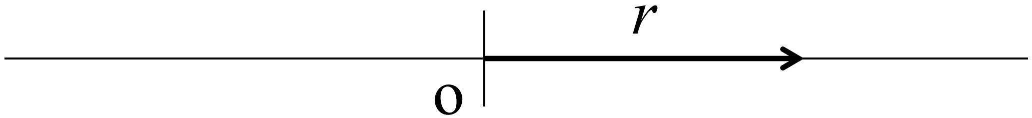
Lecture 6

Differential Forms

- Motions described by Hamiltonian Equations of Motion are very special. They preserved a whole series of quantities (an example, total phase space volume by Liouville's Theorem)
- The most efficient way to understand these quantities is in terms of differential forms; a special type of tensor.
- To understand, we must first define differential forms, and explore some of their more important algebraic and differential geometrical properties
- This exploration terminates in a discussion of a hugely general form of Stoke's Theorem, frequently called *the fundamental theorem of multidimensional integration*

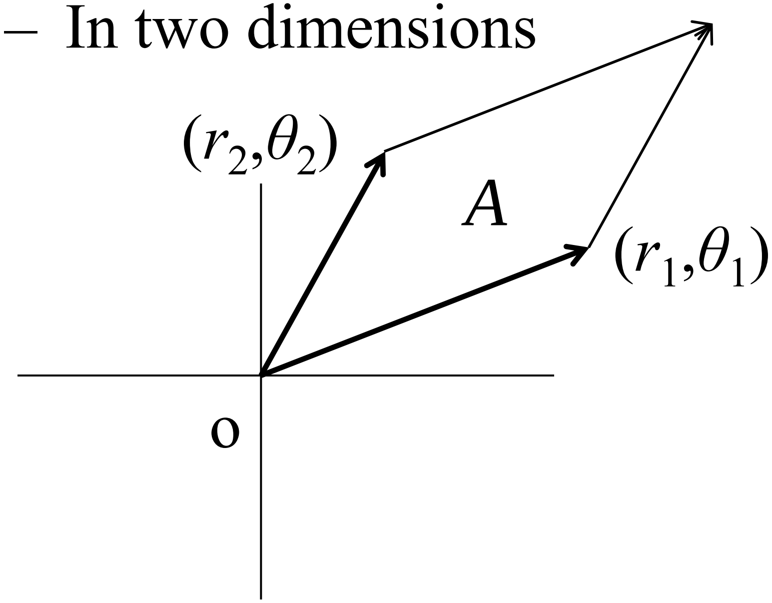
Simple Problem

- How should one compute the "extension" of a (set of) vector(s)
  - In one dimension



$$L = r = x_1$$

  - In two dimensions

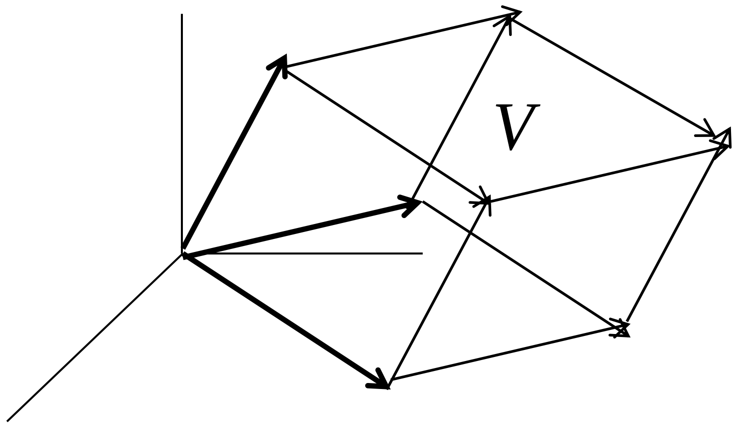


$$A = r_1 r_2 \sin\left(\theta_2 - \theta_1\right)$$

$$= r_1 r_2 \left(\sin\theta_2 \cos\theta_1 - \sin\theta_2 \cos\theta_1\right)$$

$$= x_1 y_2 - y_1 x_2 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

– In three dimensions



$$V = (base \times height) = \vec{v}_1 \times \vec{v}_2 \cdot \vec{v}_3$$
$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \cdot (x_3\hat{x} + y_3\hat{y} + z_3\hat{z})$$
$$= \begin{pmatrix} y_1 z_2 x_3 + z_1 x_2 y_3 + x_1 y_2 z_3 \\ -y_2 z_1 x_3 - z_2 x_1 y_3 - x_2 y_1 z_3 \end{pmatrix}$$
$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

- Conclusion: quantities defining “extensions” (classical language here), are best described mathematically using anti-symmetrical products of the projected length of the vectors on each coordinate axis. Generalization: should always use anti-symmetrical products for D > 3 too!

Forms

- A linear map on a vector space (in particular on the usual three dimensional Euclidean space) is called a one form

$$\omega_1\left(a\vec{v}_1 + b\vec{v}_2\right) = a\omega_1\left(\vec{v}_1\right) + b\omega_1\left(\vec{v}_2\right)$$

- In 3-D Euclidean space can always be written

$$\omega_1\left(\vec{v}\right) = \omega_1\left(\hat{x}\right)dx + \omega_1\left(\hat{y}\right)dy + \omega_1\left(\hat{z}\right)dz$$

$$dx\left(\vec{v}\right) = dx\left(v_x\hat{x} + v_y\hat{y} + v_z\hat{z}\right) \equiv v_x, \;\; dy\left(\vec{v}\right) \equiv v_y, \;\; dz\left(\vec{v}\right) \equiv v_z$$

The space of one forms is also three dimensional

- Two forms are bi-linear functions, with the additional property that the function is anti-symmetrical in the two arguments

$$\omega_2\left(a\vec{v}_1 + b\vec{v}_2, \vec{v}_3\right) = a\omega_2\left(\vec{v}_1, \vec{v}_3\right) + b\omega_1\left(\vec{v}_2, \vec{v}_3\right)$$

$$\omega_2\left(\vec{v}_1, \vec{v}_2\right) = -\omega_2\left(\vec{v}_2, \vec{v}_1\right)$$

- Ex: oriented area of parallelogram formed from two vectors

k-forms

- A *k*-form is a *k*-linear totally antisymmetric function

$$\omega_k\left(a\vec{v}_1' + b\vec{v}_1'', \vec{v}_2, \cdots, \vec{v}_k\right) = a\omega_k\left(\vec{v}_1', \vec{v}_2, \cdots, \vec{v}_k\right) + b\omega_k\left(\vec{v}_1'', \vec{v}_2, \cdots, \vec{v}_k\right)$$

$$\omega_k\left(\vec{v}_{\xi_1}, \cdots, \vec{v}_{\xi_k}\right) = \left(-1\right)^{\nu}\omega_k\left(\vec{v}_1, \cdots, \vec{v}_k\right)$$

- Essentially only one 3-form in 3-D Euclidean space: the triple vector product giving the oriented volume of a parallelipiped

$$\omega_3\left(\vec{v}_1, \vec{v}_2, \vec{v}_3\right) \propto \vec{v}_1 \cdot \vec{v}_2 \times \vec{v}_3$$

There are no 4-forms or higher in 3-D Euclidean space

- Wedge product of two one forms

$$\omega_1 \wedge \xi_1\left(\vec{v}_1, \vec{v}_2\right) = \begin{vmatrix} \omega_1\left(\vec{v}_1\right) & \xi_1\left(\vec{v}_1\right) \\ \omega_1\left(\vec{v}_2\right) & \xi_1\left(\vec{v}_2\right) \end{vmatrix}$$

- Note

$$\omega_1 \wedge \omega_1\left(\vec{v}_1, \vec{v}_2\right) = 0$$

Wedge Product in General

- The wedge product of 3 one-forms is

$$\omega_1 \wedge \xi_1 \wedge \sigma_1(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \begin{vmatrix} \omega_1(\vec{v}_1) & \xi_1(\vec{v}_1) & \sigma_1(\vec{v}_1) \\ \omega_1(\vec{v}_2) & \xi_1(\vec{v}_2) & \sigma_1(\vec{v}_2) \\ \omega_1(\vec{v}_3) & \xi_1(\vec{v}_3) & \sigma_1(\vec{v}_3) \end{vmatrix}$$

- There is a general definition that is operationally difficult to apply. For monomials, and forms made from sums of monomials the definition reduces to applying three rules

$$(\omega_1 \wedge \cdots \wedge \xi_1) \wedge (\sigma_1 \wedge \cdots \wedge \tau_1) = \omega_1 \wedge \cdots \wedge \xi_1 \wedge \sigma_1 \wedge \cdots \wedge \tau_1$$

$$(\omega_1 \wedge \cdots \wedge \xi_1) \wedge (a\omega_k + b\xi_k) = a\omega_1 \wedge \cdots \wedge \xi_1 \wedge \omega_k + b\omega_1 \wedge \cdots \wedge \xi_1 \wedge \xi_k$$

$$\omega_1 \wedge \xi_1 = -\xi_1 \wedge \omega_1 \qquad \text{including } \omega^1 \wedge \omega^1 = 0$$

- In general

$$\omega_k \wedge \xi_l = (-1)^{kl} \xi_l \wedge \omega_k$$

Differential Form

- A form "field", analogous to a vector field, has continuously differentiable "coefficients" multiplying the basic forms

$$\omega^1_{\vec{f}} = f_x(x,y,z)\,dx + f_y(x,y,z)\,dy + f_z(x,y,z)\,dz$$

$$\omega^2_{\vec{g}} = g_x(x,y,z)\,dy\wedge dz + g_y(x,y,z)\,dz\wedge dx + g_z(x,y,z)\,dx\wedge dy$$

$$\omega^3_h = h(x,y,z)\,dx\wedge dy\wedge dz$$

- Examples
  - Potential functions as line integrals of force fields
  - Directional derivatives

$$df(\vec{v}) = \frac{d}{dt} f(\vec{x}+\vec{v}t) = \frac{\partial f}{\partial x}v_x + \frac{\partial f}{\partial y}v_y + \frac{\partial f}{\partial z}v_z = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

  - Note formula works on the "coordinate" functions

$$dx = \frac{\partial x}{\partial x}dx \quad dy = \frac{\partial y}{\partial y}dy \quad dz = \frac{\partial z}{\partial z}dz$$

Integration of Differential Forms

- For 1-form is simply the line integral

$$\int_{C(s)} \omega_{\vec{f}}^1 \equiv$$

$$\int \left[f_x\left(x(s),y(s),z(s)\right)\frac{dx}{ds} + f_y\left(x(s),y(s),z(s)\right)\frac{dy}{ds} + f_z\left(x(s),y(s),z(s)\right)\frac{dz}{ds} \right] ds$$

For any parameterization of the curve $\vec{C}(s)$. Independent of parmetrization by calculus change of variables formula

- For 2-form is simply the surface (flux!) integral

$$\int_{S(s,t)} \omega_{\vec{g}}^2 \equiv \iint_{S(s,t)} \omega_{\vec{g}}^2 \left(\frac{\partial \vec{x}}{\partial s}, \frac{\partial \vec{x}}{\partial t} \right) ds dt$$

$$\int \vec{g}\left(x(s,t),y(s,t),z(s,t)\right) \cdot \frac{\partial \vec{x}}{\partial s} \times \frac{\partial \vec{x}}{\partial t} ds dt$$

$$\vec{x}(s,t) = x(s,t)\hat{x} + y(s,t)\hat{y} + z(s,t)\hat{z}$$

Independent of parameter choice by Jacobian formula

Vector Analysis Equivalents

- Integration of a 3-form normal volume integral

$$\int \omega_h^3 = \int_V h(x,y,z)\,dxdydz$$

- Integral of 1-form the line integral

$$\int_{C(s)} \omega_{\vec{f}}^1 = \oint_{C(s)} \vec{f} \cdot d\vec{l}$$

- Integral of 2-form the flux integral

$$\frac{\partial \vec{x}}{\partial s} \times \frac{\partial \vec{x}}{\partial t} dsdt = \vec{n} dA \rightarrow \int_{S(s,t)} \omega_{\vec{g}}^2 = \oint_S \vec{g} \cdot \vec{n} dA$$

- Note: *k*-form should be integrated over a *k* dimensional object

Some Interesting Results

- 3-D vector product of two vectors hidden in the rules

$$\omega^1_{\vec f} \wedge \omega^1_{\vec g}(\vec v_1, \vec v_2) = \omega^2_{\vec f \times \vec g}(\vec v_1, \vec v_2)$$

- 3-D scalar product of two vectors hidden in the rules

$$\omega^1_{\vec f} \wedge \omega^2_{\vec g}(\vec v_1, \vec v_2, \vec v_3) = (\vec f \cdot \vec g)\,\omega^3(\vec v_1, \vec v_2, \vec v_3)$$

Show both of these in HW problem

- Given a k-form and a linear map on a vector space, one can form a new k-form as follows

$$L : R^m \to R^n \quad \text{linear} \quad \omega^k \text{ a } k\text{-form on } R^n$$

$$L^*\omega^k(\vec v_1, \cdots, \vec v_k) \equiv \omega^k(L\vec v_1, \cdots, L\vec v_k)$$

$$(L^*\omega^k)(\vec v_{\xi_1}, \cdots, \vec v_{\xi_k}) = \omega^k(L\vec v_{\xi_1}, \cdots, L\vec v_{\xi_k}) = (-1)^\nu \omega^k(L\vec v_1, \cdots, L\vec v_k)$$

$$= (-1)^\nu (L^*\omega^k)(\vec v_1, \cdots, \vec v_k) \quad \text{is a } k\text{-form on } R^m!$$

- This important operation is called *pullback*