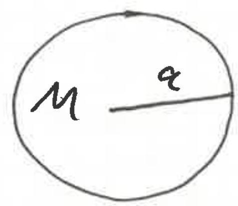
~~Rolling & Sliding Billiard ball~~

Sliding & Rolling Billiard ball

Slide - opposing sliding friction force $\vec{F}_f$

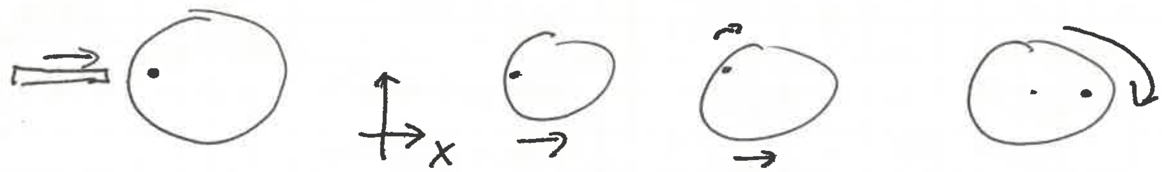
Roll - friction force neglected

Consider ~~homogeneous ball~~



$\rho(\vec{x}) = \frac{M}{4/3\,\pi a^3}$

Q: If struck (dead center), how long until pure rolling motion



$\hat{e}_1^{\,0} = \hat{x}$.

After initial hit: ~~μg~~ μ = coefficient of sliding friction

$$M\ddot{\vec{x}} = \vec{F}^{(ext)}$$

$$M\ddot{\vec{x}}\cdot\hat{x} = \vec{F}^{(ext)}\cdot\hat{e}_1^{\,0}$$

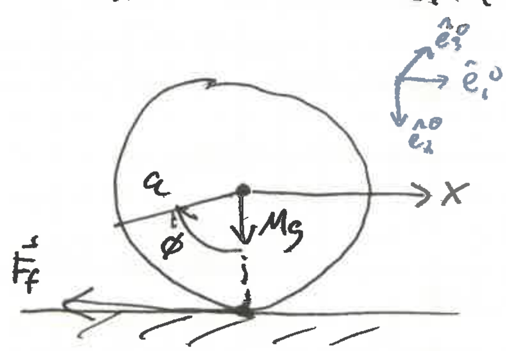
$$M\ddot{x} = \vec{F}_f\cdot\hat{e}_1^{\,0}$$

$$M\ddot{x} = -\mu g M$$

$$\ddot{x} = -\mu g$$ 28.17

$$\vec{F}_f = -\mu g M\hat{e}_1^{\,0}$$

Next consider ~~rate of~~ dL/dt



$$\frac{dL_3}{dt} = I_{33}\ddot{\phi} = \Gamma_3^{ext}$$

$$I_{33}\ddot{\phi} = \left[a\hat{e}_2^{\,0}\times\vec{F}_f\right]_3$$ 28.19

$$I_{33} = \int\rho(\vec{x})\left(\delta_{33}x^2 - x_3x_3\right)d^3x$$
$$= \int\rho(\vec{x})\left(x_1^2 + x_2^2\right)d^3x$$

evaluate I_{33} in spherical polar coordinates

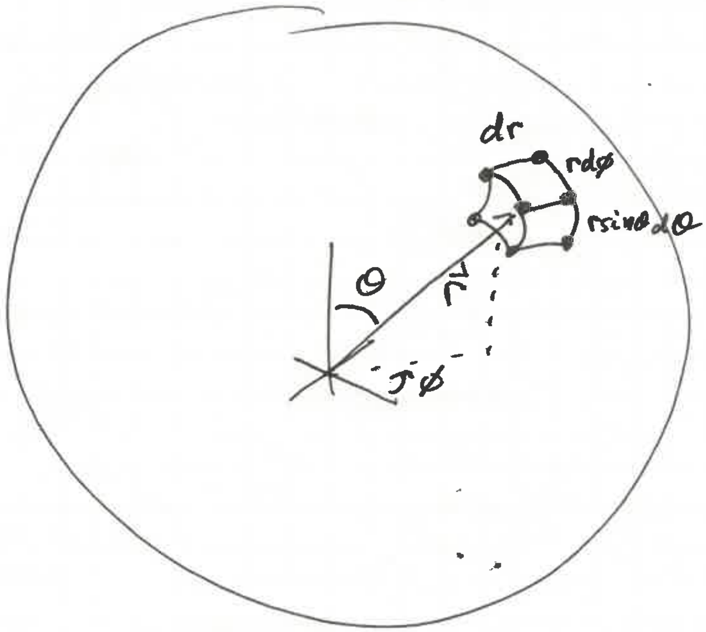
~~$I_{33} = \int_0^{\pi}\int_0^{2\pi}\int_0^{a}\left(\frac{M}{4/3\,\pi a^3}\right)$~~

$$\vec{x} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

$$\vec{x} = r\cos\phi\sin\theta\,\hat{x} + r\sin\phi\sin\theta\,\hat{y} + r\cos\theta\,\hat{z}$$

$x_1^2 = r^2 \cos^2\phi \sin^2\theta$, $x_2^2 = r^2 \sin^2\phi \sin^2\theta$

$d^3x = ??$



$$d^3x = dr \; r d\phi \; r\sin\theta d\theta = r^2 \sin\theta \, dr \, d\phi \, d\theta$$

$$I_{33} = \int_0^{\pi}\int_0^{2\pi}\int_0^{a} \left(\frac{M}{4/3 \pi a^3}\right)(r^2 \sin^2\theta)\, r^2 \sin\theta \, dr\, d\phi\, d\theta$$

$$= \frac{M}{4/3 \pi a^3} \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{2\pi} d\phi \int_0^a r^4$$

$$= \cancel{\frac{M}{4/3\pi a^3} \frac{1}{12} \cos}$$

$$= \frac{M}{4/3 \pi a^3} \frac{1}{12}\left(\cos 3\theta - 9\cos\theta\right)\Big|_0^{\pi} \; 2\pi \; \frac{1}{5} a^5$$

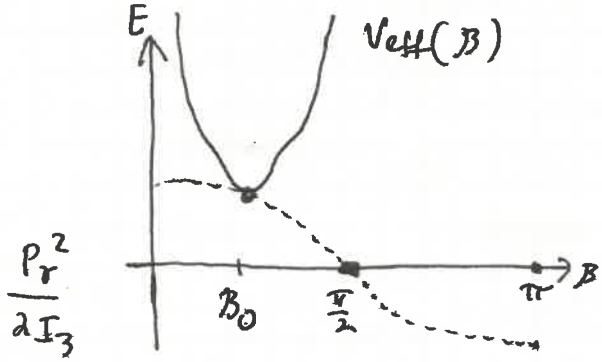
$$= \frac{3}{5} M a^2 \frac{1}{24}\left(\cos 3\pi - 9\cos\pi - \cos 0 + 9\cos 0\right)$$

$$= \frac{3}{5} M a^2 \frac{1}{24} \underbrace{\left(-1 - 9(-1) - 1 + 9\right)}_{16}$$

$$= \frac{3}{5}\frac{16}{24} M a^2 \qquad \frac{16}{24} = \frac{8\times 2}{12\times 2} = \frac{4\times 2}{4\times 3} = \frac{2}{3}$$

$$I_{33} = \frac{2}{5} M a^2$$

$$I_1 \ddot{\beta} = -\frac{\partial V_{eff}}{\partial \beta}$$



$$0 = \left.\frac{\partial V_{eff}}{\partial \beta}\right|_{\beta_0}$$

$$0 = \frac{(P_\alpha - P_\gamma \cos\beta_0)}{I_1 \sin^2\beta_0} P_\gamma \sin\beta_0 + \frac{(P_\alpha - P_\gamma \cos\beta_0)^2}{2 I_1}(-2)\sin^{-3}\beta_0 \cos\beta_0 - Mgl\sin\beta_0$$

$$0 = (P_\alpha - P_\gamma \cos\beta_0) P_\gamma \sin^2\beta_0 - (P_\alpha - P_\gamma \cos\beta_0)^2 \cos\beta_0 - I_1 Mgl \sin^4\beta_0$$

$$\cancel{(P_\alpha - P_\gamma \cos\beta_0)^2 = P_\alpha P_\gamma \sin^2\beta_0}$$

$$(P_\alpha - P_\gamma \cos\beta_0)^2 \cos\beta_0 = P_\alpha P_\gamma \sin^2\beta_0 - P_\gamma^2 \cos\beta_0 \sin^2\beta_0 - I_1 Mgl \sin^4\beta_0$$

$$\boxed{P_\gamma^2 - 2 P_\gamma P_\alpha \cos\beta_0 + P_\alpha^2 = \frac{P_\gamma P_\alpha \sin^2\beta_0}{\cos\beta_0} - I_1 Mgl \frac{\sin^4\beta_0}{\cos\beta_0}}$$

Look at small oscillations about β_0

$$\beta(t) = \beta_0 + \varepsilon(t)$$

$$E = \tfrac{1}{2} I_1 \dot{\varepsilon}^2 + V_{eff}(\beta_0) + \tfrac{1}{2}\varepsilon^2 \left.\left(\frac{\partial^2 V_{eff}}{\partial \beta^2}\right)\right|_{\beta_0}$$

$\vdots$

$$\ddot{\varepsilon} = -\Omega^2 \varepsilon \quad \text{where} \quad \Omega^2 = \frac{1}{I_1}\left.\left(\frac{\partial^2 V_{eff}}{\partial \beta^2}\right)\right|_{\beta_0} = \frac{P_\alpha P_\gamma - I_1 Mgl(4 - 3\sin^2\beta_0)}{I_1^2 \cos\beta_0}$$

- stable motion when $\Omega^2 > 0$
- subbing ~~soln~~ solution back into $\dot{\alpha}(t)$ & $\dot{\gamma}(t)$ find,

$$\dot{\alpha}(t) \approx \left(\frac{P_\alpha - P_\gamma \cos\beta_0}{I_1 \sin^2\beta_0}\right) + \varepsilon(t)\left(\frac{\partial}{\partial\beta}\left\{\sim\right\}\Big|_{\beta_0}\right)$$

$$\approx \dot{\alpha}_0 + \varepsilon(t)\, \dot{\alpha}_1 \qquad \text{(similarly for } \gamma(t)\text{)}$$

$$I_{33}\ddot{\phi} = [(a\hat{e}_2^{(0)}) \times (-\mu g M \hat{e}_1^{(0)})]_3$$

$$2/5\, Ma^2 \ddot{\phi} = +a\mu g M$$

EOM

$$a\ddot{\phi} = 5/2\,\mu g \quad\Rightarrow\quad a\dot{\phi} = 5/2\,\mu g\, t + \cancel{\omega_0}\ 0$$

$$\ddot{x} = -\mu g \qquad\qquad \dot{x} = -\mu g t + V_0$$

Consider ICs: $x_0 = 0$, $\dot{x}_0 = V_0$, $\phi_0 = 0$, $\dot{\phi}_0 = 0$

~~Slip~~ Pure rolling occurs when

$$\dot{x}(t_c) = a\dot{\phi}(t_c)$$

$$-\mu g t_c + V_0 = 5/2\,\mu g\, t_c$$

~~$V_0 = 7/2$~~ $t_c = \frac{2}{7}\frac{V_0}{\mu g}$ 2.833

$$x(t_c) = \frac{12}{49}\frac{V_0^2}{\mu g}, \qquad V(t_c) = \frac{5}{7}V_0$$

~~for~~ $\lim_{\mu\to 0} t_c = \infty$, $\lim_{\mu\to 0} x(t_c) = \infty$

i.e. If no friction the ball will never start rolling

~~Torque free Asymmetric Top~~

~~Euler's Eqn~~ ~~$I_1\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3)$~~
~~$(1\to2,\ 2\to3,\ 3\to1)$~~
~~$(1\to3,\ 2\to1,\ 3\to2)$~~ ~~w/ $I_1 = I_2 \neq I_3$~~

~~$I_3\dot{\omega}_3 = \omega_1\omega_2(\cancel{I_1 - I_2})\ 0$~~

~~$I_3\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{const.}$~~

~~$I_1\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3)$~~

~~$I_2\omega$~~

Torque free ~~A~~symmetric Top

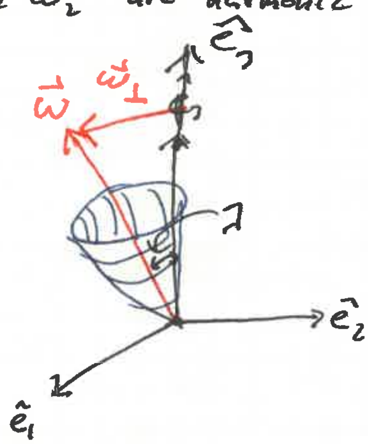
$$I_1 \dot\omega_1 = \omega_2 \omega_3 (I_2 - I_3) \qquad\qquad I_1 = I_2 \neq I_3 \qquad I_1 \dot\omega_1 = -\omega_2 \omega_3 (I_3 - I_1)$$
$$I_2 \dot\omega_2 = \omega_3 \omega_1 (I_3 - I_1) \qquad \Longrightarrow \qquad I_2 \dot\omega_2 = \omega_1 \omega_3 (I_3 - I_1)$$
$$I_3 \dot\omega_3 = \omega_1 \omega_2 (I_1 - I_2) \qquad\qquad\qquad I_3 \dot\omega_3 = 0$$

~~w/ $I_1 = I_2 \neq I_3$~~ let $\Omega = \omega_3 \dfrac{I_3 - I_1}{I_1}$

$$\dot\omega_1 = -\Omega\, \omega_2$$
$$\dot\omega_2 = +\Omega\, \omega_1$$

let $I_3 > I_1$ then $\Omega > 0$

ω_1 & ω_2 are harmonic oscillators



IC: $\omega_1 = \omega \sin\lambda$
$\omega_2 = 0$
$\omega_3 = \omega \cos\lambda$

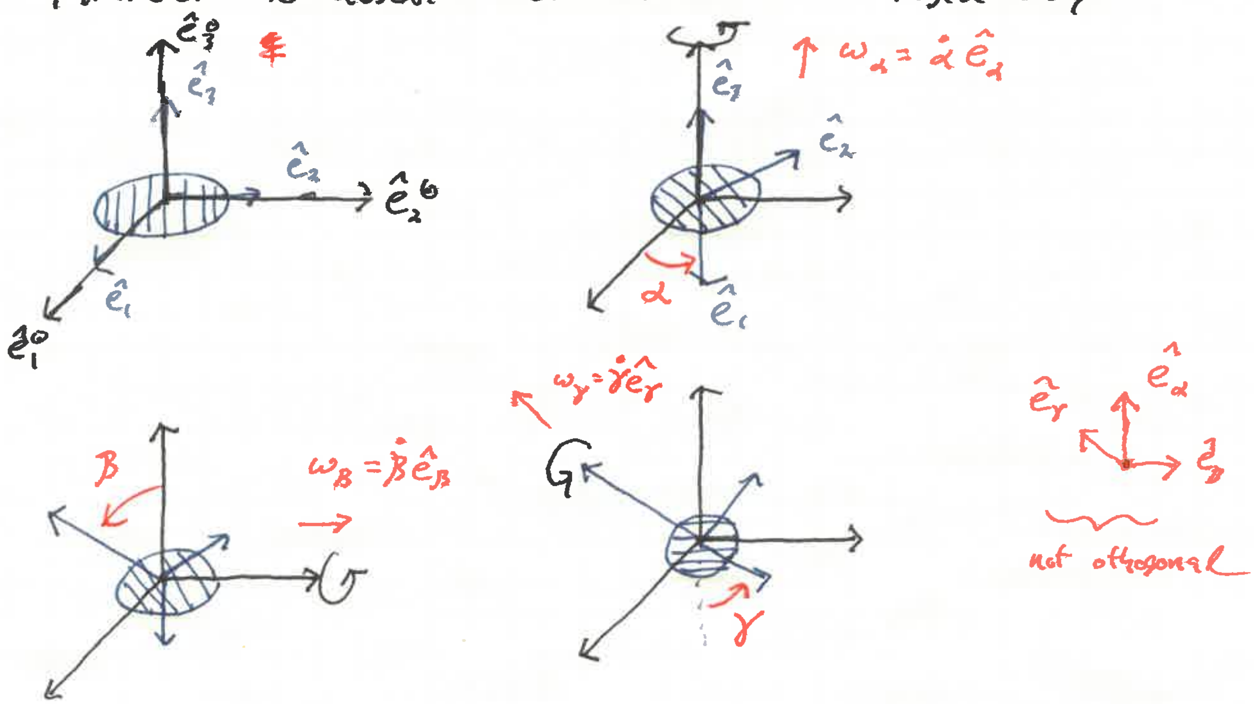
$$\omega_1 = \omega \sin\lambda \cos\Omega t$$
$$\omega_2 = \omega \sin\lambda \sin\Omega t$$
$$\omega_3 = \omega \cos\lambda$$

$|\vec\omega|$ remains constant

$\hat\omega$ precesses around $\hat{e}_3$

29 Euler Angles

3 parameters to describe orientation of a rigid body



- α & β are the polar & azimuthal angles
- α, β, γ are general coordinates

$$T = \tfrac{1}{2}\left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2\right) \qquad \text{(in body fixed frame)}$$

$$\omega_i = \vec{\omega}\cdot\hat{e}_i$$

$$\underbrace{\omega_\alpha = \dot{\alpha}\,\hat{e}_\alpha \;,\quad \omega_\beta = \dot{\beta}\,\hat{e}_\beta \;,\quad \omega_\gamma = \dot{\gamma}\,\hat{e}_\gamma}_{\text{not orthogonal (in } \hat{e}_i \; i=1,2,3 \text{ frame)}}$$

$$\hat{e}_\gamma = \hat{e}_3$$

$$\cancel{\hat{e}_\beta = \hat{e}_1 \cos(\tfrac{1}{2}\pi - \gamma) + \hat{e}_2 \sin(\tfrac{1}{2}\pi - \gamma)}$$

$$\hat{e}_\beta = \sin\gamma\, \hat{e}_1 + \cos\gamma\, \hat{e}_2$$

$$\hat{e}_\alpha = -\sin\beta\cos\gamma\, \hat{e}_1 + \sin\beta\sin\gamma\, \hat{e}_2 + \cos\beta\, \hat{e}_3$$

$$\therefore \quad \omega_1 = -\dot{\alpha}\sin\beta\cos\gamma + \dot{\beta}\sin\gamma$$

$$\omega_2 = \dot{\alpha}\sin\beta\sin\gamma + \dot{\beta}\cos\gamma$$

$$\omega_3 = \dot{\alpha}\cos\beta + \dot{\gamma}$$

analysis much harder ~~harder~~ if $I_1 \neq I_2$

Symmetric top (torque free) $\quad I_1 = I_2 \neq I_3$

$$L(\alpha,\beta,\gamma;\dot{\alpha},\dot{\beta},\dot{\gamma}) = \tfrac{1}{2} I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) + \tfrac{1}{2} I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2$$

$$\frac{\partial L}{\partial \alpha} = 0 \quad \& \quad \cancel{\frac{\partial L}{\partial \beta}} \quad \cancel{\frac{\partial L}{\partial \alpha}} \quad \frac{\partial L}{\partial \gamma} = 0 \;..$$

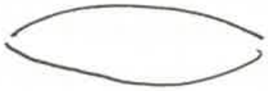
$$\left.\begin{aligned} p_\alpha &= I_1 \dot{\alpha}\sin^2\beta + I_3(\dot{\alpha}\cos\beta + \dot{\gamma})^{\cos\beta} = \text{const} \\ p_\gamma &= I_3(\dot{\alpha}\cos\beta + \dot{\gamma}) = \text{const} \end{aligned}\right\} \; 1^{st} \text{ Integrals}$$

$$\dot{p}_\beta = I\ddot{\beta} = \dot{\alpha}\sin\beta\left(I_1\dot{\alpha}\cos\beta - I_3\omega_3\right)$$

- $p_\alpha, p_\beta, p_\gamma$ are projections of L body axis

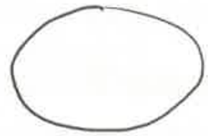
Motion in Inertial Frame (torque-free)

will describe torque-free motion of a symmetric body by an inertial observer



football

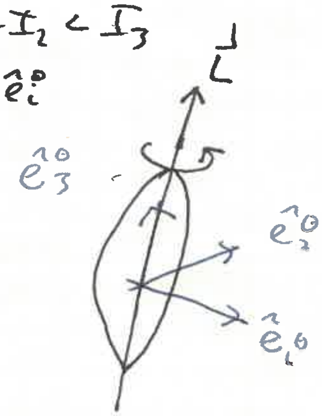
$I_1 = I_2 > I_3$



Earth

$I_1 = I_2 < I_3$

- $\vec{L}$ is const in inertial frame, $\hat{e}_i^0$
- choose $\hat{e}_3^0 \parallel \vec{L}$
- $p_\alpha = \vec{L} \cdot \hat{e}_3^0$



$$p_\alpha = |\vec{L}| = \text{const}$$

$$p_\gamma = \vec{L} \cdot \hat{e}_3$$
$$= |\vec{L}| \cos\beta = \text{const}$$

see figures on pg 14 of notes to clarify these vector products

$$p_\beta = \vec{L} \cdot \hat{e}_\beta = 0$$

Consider p_α 1st integral

$$p_\alpha = I_1 \dot{\alpha} \sin^2\beta + I_3(\dot{\alpha}\cos\beta + \dot{\gamma})\cos\beta = \text{const}$$

$$I_1 \dot{\alpha} \sin^2\beta + p_\gamma \cos\beta = \text{const}$$

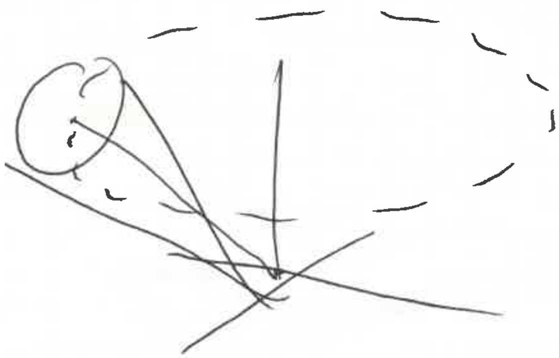
$I_1, p_\gamma,$ are constants

β ?? recall $p_\gamma = |\vec{L}|\cos\beta = \text{const}$ $\therefore$ $\beta = \text{const.}$

so $\dot{\alpha} = \text{constant!}$

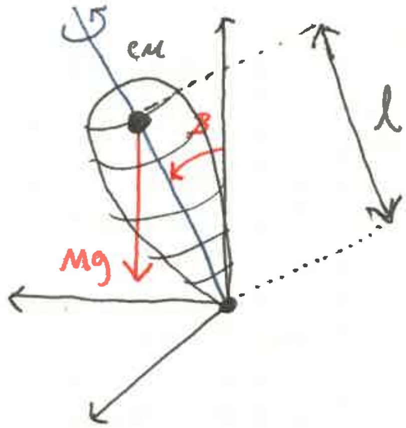
similarly $\dot{\gamma} = \text{constant}$

so rigid body has $\beta = \text{constant}$, $\gamma = \omega_\gamma t$, $\alpha = \omega_\alpha t$



rigid body precesses at constant rate

31: Symmetric Top w/ fixed point in grav field



$$L = \tfrac{1}{2} I_1 (\dot\alpha^2 \sin^2\beta + \dot\beta^2) + \tfrac{1}{2} I_3 (\dot\alpha \cos\beta + \dot\gamma)^2 - Mgl\cos\beta$$

$$V(\alpha,\beta,\gamma) = Mgl\cos\beta = V(\beta)$$

$\therefore$ p_α, p_γ are const.

α, γ are cyclic coordinates

$$p_\alpha = I_1 \dot\alpha \sin^2\beta + \overset{p_\gamma \cos\beta}{\cancel{I_3 \cos\beta(\dot\alpha\cos\beta + \dot\gamma)}} = \text{const.}$$

$$p_\gamma = I_3(\dot\alpha\cos\beta + \dot\gamma) = I_3\omega_3 = \text{const.}$$

rewrite p_α & p_γ relations

$$\dot\alpha = \frac{p_\alpha - p_\gamma\cos\beta}{I_1 \sin^2\beta}, \qquad \dot\gamma = p_\gamma\left(\frac{1}{I_3} + \frac{\cot^2\beta}{I_1}\right) - \frac{p_\alpha\cos\beta}{I_1\sin^2\beta}$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial\dot\beta}\right] = \frac{\partial L}{\partial\beta}$$

$$I_1\ddot\beta = I_1\dot\alpha^2\sin\beta\cos\beta - I_3(\dot\alpha\cos\beta + \dot\gamma)\dot\alpha\sin\beta + Mgl\sin\beta$$

* $\boxed{\text{Take derivatives of } L \text{ first, then substitute in constants}}$ *

We can reduce this to a 1D eq.

$$I_1\ddot\beta = I_1\sin\beta\cos\beta\left(\frac{p_\alpha - p_\gamma\cos\beta}{I_1\sin^2\beta}\right)^2 - p_\gamma\left(\frac{p_\alpha - p_\gamma\cos\beta}{I_1\sin^2\beta}\right)\sin\beta + Mgl\sin\beta$$

$\vdots$

$$I_1\ddot\beta = \frac{\cos\beta}{I_1\sin^3\beta}(p_\alpha^2 - 2p_\alpha p_\gamma\cos\beta + p_\gamma^2) - \frac{p_\alpha p_\gamma}{I_1\sin\beta} + Mgl\sin\beta$$ 3LS

- higly nonlinear eq
- use energy conservation to tackle

$$\mathcal{H} = T + V = \text{const.} = E$$

$$E = \tfrac{1}{2}I_1(\dot\alpha^2\sin^2\beta + \dot\beta^2) + \tfrac{1}{2}I_3(\dot\alpha\cos\beta + \dot\gamma)^2 + Mgl\cos\beta$$

$$= \tfrac{1}{2}I_1\dot\beta^2 + \tfrac{1}{2}I_1\dot\alpha^2\sin^2\beta + \frac{p_\gamma^2}{2I_3} + Mgl\cos\beta$$

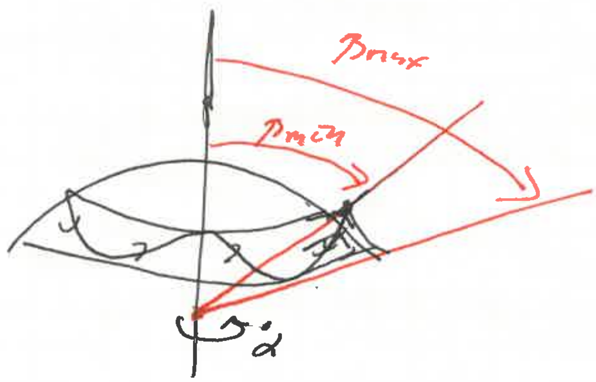
$$= \tfrac{1}{2}I_1\dot\beta^2 + \underbrace{\frac{(p_\alpha - p_\gamma\cos\beta)^2}{2I_1\sin^2\beta} + \frac{p_\gamma^2}{2I_3} + Mgl\cos\beta}_{V_{\text{eff}}(\beta)}$$

1) rate of precession, $\dot{\alpha}(t)$, & spin, $\dot{\gamma}(t)$ vary sinusoidally about initial spin values

2) if $\dot{\alpha}_1 > 0$ then $\dot{\alpha}$ is max for max β & z (symmetry axis furthest from $\hat{e}_3^{(0)}$)

~~[illegible]~~ $\dot{\alpha}_1$ smallest when $\hat{e}_3$ closest to $\hat{e}_3^{(0)}$

body nutates about $\hat{e}_3$



nutation