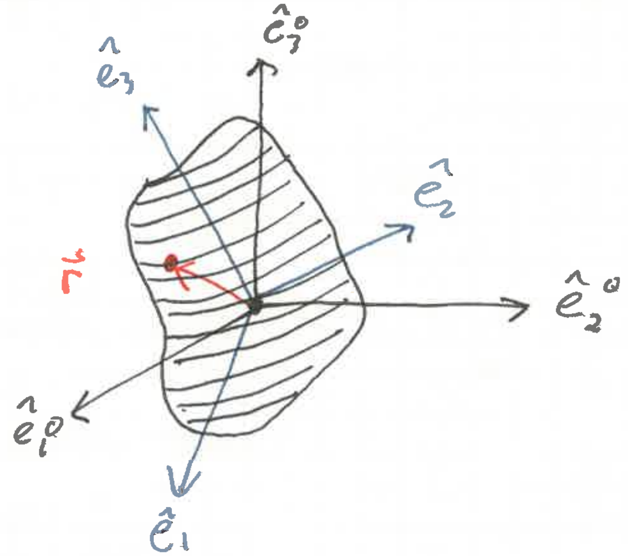
Consider N particles w/ relative separations fixed in magnitude

- 3 translational DOF
- 3 rotational DOF $\rightarrow$ ~~6 DOF total~~
- $3N-6$ holonomic constraints

$$\llcorner \quad f_i(x_1, \ldots, x_n, t) = c_i \qquad i = 1, 2, \ldots, 3N-6$$

* Elim translation DOF by fixing one point (take to be the origin)



$\{\hat{e}_i^{\,0}\}$ fixed in inertial frame

$\{\hat{e}_i\}$ body fixed frame

$\{x_i\} \equiv \{\vec{r}\cdot\hat{e}_i\}$

$\vec{r}$ is in inertial frame

x_i in body frame

$$\left(\frac{dr}{dt}\right)_{body} = \sum_{i=1}^{3} \hat{e}_i \frac{dx_i}{dt}$$ 26.1

It can be shown (w/ eq. 7.11)

$$\left(\frac{d\vec{r}}{dt}\right)_{inertial} = \left(\frac{d\vec{r}}{dt}\right)_{body} + \vec{\omega}\times\vec{r}$$

$\vec{\omega}$ = angular velocity of the body-fixed axes as seen in inertial frame

fix $\vec{r}$ in body i.e. $\left(\frac{d\vec{r}}{dt}\right)_{body} = \vec{0}$ then $\left(\frac{d\vec{r}}{dt}\right)_{inertial} = \vec{\omega}\times\vec{r}$

$$T = \tfrac{1}{2}\sum_{p=1}^{N} m_p v_p^2 = \tfrac{1}{2}\sum_p (\vec{\omega}\times\vec{r}_p)\cdot(\vec{\omega}\times\vec{r}_p)$$

TRIG ID: $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{B}\cdot\vec{C})(\vec{A}\cdot\vec{D})$

$$T = \tfrac{1}{2}\sum_p \left[(\vec{\omega}\cdot\vec{\omega})(\vec{r}_p\cdot\vec{r}_p) - (\vec{r}_p\cdot\vec{\omega})(\vec{\omega}\cdot\vec{r}_p)\right] m_p$$

$$= \tfrac{1}{2}\sum_p m_p\left[\omega^2 r_p^2 - (\vec{\omega}\cdot\vec{r}_p)^2\right] \qquad (26.56a)$$

$\llcorner$ sum over N particles, go to continuum limit

$$m_p \rightarrow dm = \rho(\vec{r})\,dV = \rho(\vec{r})\,d^3r$$

$$T = \tfrac{1}{2}\int \rho(\vec{r})\left[\omega^2 r^2 - (\vec{\omega}\cdot\vec{r})^2\right] d^3r \qquad (26.56)$$

The discrete & continuous forms of T can be rewritten as

$$T = \tfrac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} I_{ij}\,\omega_i\omega_j \qquad 26.6$$

inertia tensor : $I_{ij} \equiv \sum_p m_p\left(\delta_{ij} r_p^2 - x_{pi}x_{pj}\right)$

$$I_{ij} \equiv \int \rho(\vec r)\left(\delta_{ij} r^2 - x_i x_j\right) d^3r$$

* T is a scalar, $\therefore$ independent of coordinate system choosen
  Hence why we prefer the instantaneous body-fixed frame

$$\vec L = \sum_p m_p \vec r_p \times \vec v_p$$
$$= \sum_p m_p\left[\vec r_p \times (\vec\omega \times \vec r_p)\right]$$

Vector triple product $\vec A\times\vec B\times\vec C = (\vec A\cdot\vec C)\vec B - (\vec A\cdot\vec B)\vec C$

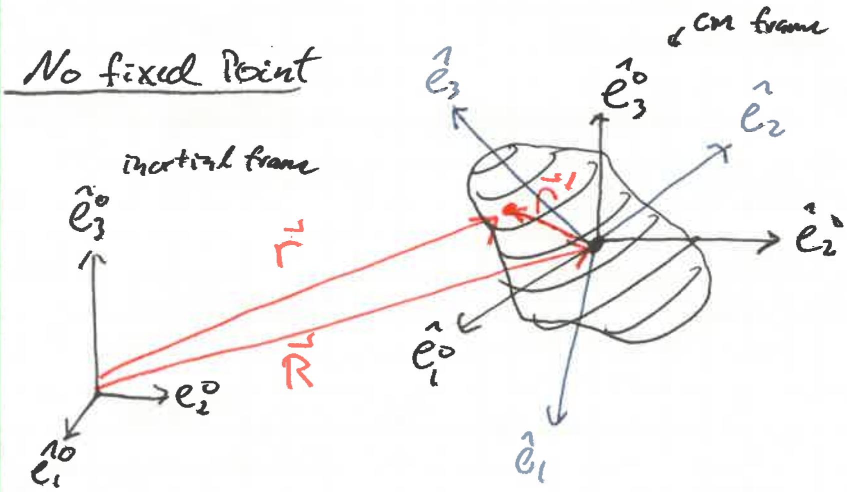
$$= \sum_p m_p\left[r_p^2\,\vec\omega - (\vec\omega\cdot\vec r_p)\vec r_p\right]$$
$$= \int \rho(\vec r)\left[r^2\vec\omega - (\vec\omega\cdot\vec r)\vec r\right]d^3r$$

26.8

26.9 & 26.10

$$\boxed{L_i = \sum_{j=1}^{3} I_{ij}\,\omega_j \implies T = \tfrac{1}{2}\sum_{i=1}^{3} L_i\omega_i = \tfrac{1}{2}\vec L\cdot\vec\omega}$$

No fixed Point



$$\vec R + \vec r' = \vec r$$

$$\left(\frac{d\vec r}{dt}\right)_{inertial} = \dot{\vec R} + \left(\frac{d\vec r'}{dt}\right)_{inertial}$$

CM frame $\rightsquigarrow$ inertial frame

$$\left(\frac{d\vec r'}{dt}\right)_{inertial} = \left(\frac{d\vec r'}{dt}\right)_{cm} = \sum_{i=1}^{3}\hat e_i^{\,o}\frac{d}{dt}\left(\vec r'\cdot\hat e_i^{\,o}\right)$$

$$\left(\frac{d\vec r'}{dt}\right)_{cm} = \left(\frac{d\vec r'}{dt}\right)_{body} + \vec\omega\times\vec r'$$

ω is angular velocity of the whole body as seen in the CM frame

$$\therefore \left(\frac{d\vec r}{dt}\right)_{inertial} = \dot{\vec R} + \left(\frac{d\vec r'}{dt}\right)_{body} + \vec\omega\times\vec r'$$

$T = \frac{1}{2} M \dot{\vec{R}}^2 + T'$ (T' is kinetic energy of internal motion)

$L = \vec{R} \times (M \dot{\vec{R}}) + L'$

Rigid body acts like point mass, M, w/ coordinate $\vec{R}$.

$$M \ddot{\vec{R}} = \cancel{\sum_p \vec{F}_p^{(ext)}} = \vec{F}^{(ext)} \qquad (26.14)$$

CM contribution to T & L are easily determine

Repeat previous analysis to get T' & L' !! kinematic portions of

Only internal dynamics remain to be determine

$$\vec{L}' = \sum_p m_p \vec{r}_p' \times \left(\frac{d\vec{r}_p'}{dt}\right)_{inertial} = \sum_p m_p \vec{r}_p' \times \left(\frac{d\vec{r}_p'}{dt}\right)_{CM} \qquad 26.15a$$

$$\vec{\Gamma}^{(ext)\prime} \equiv \left(\frac{dL'}{dt}\right)_{inertial} = \left(\frac{dL'}{dt}\right)_{cm} = \sum_p \vec{r}_p' \times \vec{F}_p^{(ext)} \qquad 26.15b$$

26.14 & 26.15 specify dynamics of rigid bodies!

Inertia Tensor

In body-fixed frame $\overline{\overline{I}}$ components are invariants (depend only on distribution of matter)

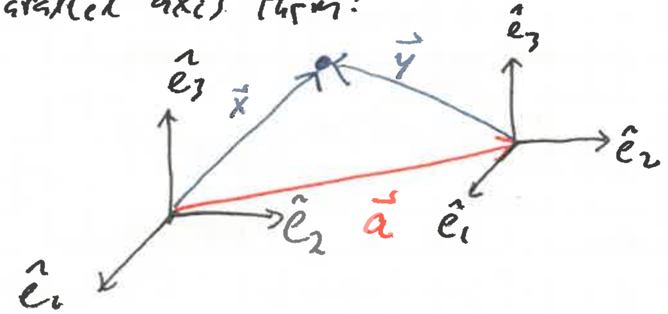
Consider body-fixed frame w/ origin at the CM

$$\overline{\overline{I}}_{ij} = \sum_p m_p \left(\delta_{ij} x_p^2 - x_{p_i} x_{p_j} \right)$$
$$= \int \rho(\vec{x}) \left(\delta_{ij} x^2 - x_i x_j \right) d^3x$$

$$\overline{\overline{I}}_{ij}^{\,T} = \overline{\overline{I}}_{ij} \quad \text{(real symmetric matrix)}$$

Body fixed^{CM} $\overline{\overline{I}}$ is easiest to calculate but not neccessarily the easiest to use in calculations.

Parallel axes thrm!



$$\vec{y} = \vec{x} - \vec{a}$$

The inertia tensor, I_{ij}, in this new frame is

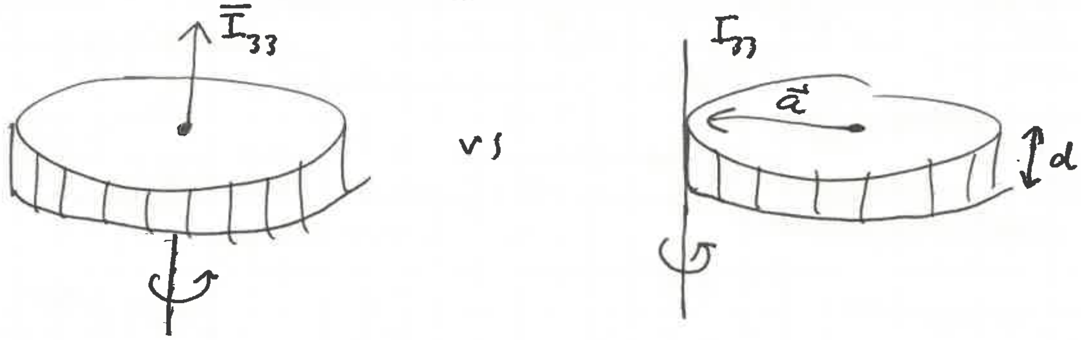
$$I_{ij} = \sum_p m_p \left(\delta_{ij} y_p^2 - y_{p_i} y_{p_j} \right) \qquad (\text{sub in for } \vec{y})$$

$$I_{ij} = \sum_p m_p \left[\delta_{ij} (\vec{x}_p - \vec{a})^2 - (x_{p_i} - a_i)(x_{p_j} - a_j) \right]$$

$$= \sum_p m_p \left[\delta_{ij} (x_p^2 + a^2 - 2\vec{x}_p \cdot \vec{a}) - (x_{p_i} x_{p_j} - x_{p_i} a_j - x_{p_j} a_i + a_i a_j) \right]$$

$$= \bar{I}_{ij} + \sum_p m_p \left[\delta_{ij} a^2 - a_i a_j \right] - \underbrace{\cancel{\sum_p m_p \left[\delta_{ij} 2\vec{x}_p \cdot \vec{a} + x_{p_i} a_j x_{p_j} a_i \right]}}_{2.12} \to 0$$

$$\boxed{I_{ij} = \bar{I}_{ij} + M(a^2 \delta_{ij} - a_i a_j)} \quad \text{parallel axis Thm}$$



$I_{33} = \bar{I}_{33} + Ma^2$ where

$$\bar{I}_{33} = \int \rho(\vec{r}) \left[\delta_{33} x^2 - x_3 x_3 \right] d^3x \qquad \text{assume } \rho(\vec{r}) = \frac{M}{(\pi a^2) d}$$

$$= \rho \int_0^a \int_0^{2\pi} \int_{-\frac{1}{2}d}^{\frac{1}{2}d} \left[(r^2 + z^2) - z^2 \right] (r\, dz\, d\phi\, dr)$$

$$= \rho\, 2\pi d \int_0^a r^3 dr$$

$$= \rho\, 2\pi d\, \tfrac{1}{3} a^3$$

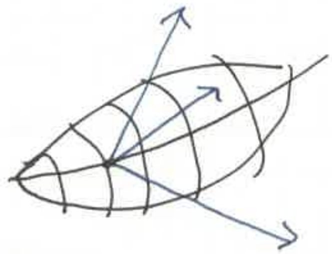
$$= \frac{M}{\pi a^2 d} 2\pi d \frac{a^4}{4}$$

$$= \cancel{\frac{2Ma}{3}} = \frac{1}{2} M a^2$$

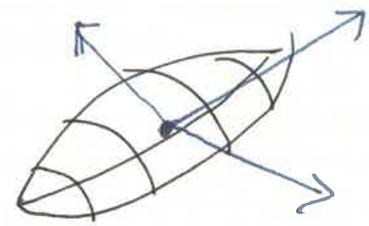
Principle Axes

I_{ij} is a real symmetric matrix in any body-fixed coordinate
Determine the principle axes where

$$I_{ij} = I_i \delta_{ij} \qquad \text{(diagonalize)}$$



vs



Let ξ_i denote components of ~~ω_i~~ $\vec{\omega}$ in principle axes

se 26.6
$$T = \tfrac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j$$
$$\cancel{= \tfrac{1}{2} \textstyle\sum_{i,j} I_{ij} \xi_i \xi_j}$$
$$= \tfrac{1}{2} \textstyle\sum_{i,j} I_i \delta_{ij} \xi_i \xi_j$$
$$= \tfrac{1}{2} \textstyle\sum_i I_i \xi_i^2 \qquad 26.22a$$

26.9
$$L_i = \textstyle\sum_j I_{ij} \omega_j$$
$$= \textstyle\sum_j I_i \delta_{ij} \xi_j$$
$$= I_i \xi_i \qquad 26.22b$$

Note that T ~~is positive definite~~ cannot be negative
$\therefore I_i \geq 0 \ \forall i$ (in this frame)

(skipping proof that principle axes $\exists$)

27 Euler Equations

Recall the DOF of a rigid body split into a set of 3 translational & 3 rotational coordinates

$M\ddot{\vec{R}} = \vec{F}^{(ext)}$ reduces translation to equiv 1 body problem.

Fundamentally we are left solving

$$\left(\frac{d\vec{L}}{dt}\right)_{inertial} = \vec{\Gamma}^{(ext)}$$

which is valid in the 2 cases

1) Origin of coordinates fixed in an inertial frame

2) Origin located at the CM

In general $\left(\frac{d\vec{L}}{dt}\right) \neq \vec{\Gamma}^{ext}$ for coordinates in an accelerating origin

$$\left(\frac{d\vec{L}}{dt}\right)_{inertial} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L} = \vec{\Gamma}^{ext} \qquad \text{(see eq 7.11)}$$

Project onto the principle body axes

$$\Gamma_s^{(ext)} = \vec{\Gamma}^{(ext)} \cdot \hat{e}^{(s)}, \quad L_s = \vec{L} \cdot \hat{e}^{(s)}, \quad \omega_s = \vec{\omega} \cdot \hat{e}^{(s)}$$

$$L_s = I_s\,\omega_s \quad \text{(in principle frame)}$$

$$\cancel{\left(\frac{d\vec{L}}{dt}\right)_{body} = I_s \left(\frac{d\vec{\omega}}{dt}\right)} \qquad \left(\frac{dL_s}{dt}\right)_{body} = I_s \frac{d\omega_s}{dt}$$

$$\vec{\Gamma}^{(ext)} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L}$$

$$\Gamma_s^{(ext)} = I_s \frac{d\omega_s}{dt} + [\vec{\omega} \times \vec{L}]_s$$

$$\left.\begin{aligned} I_1 \frac{d\omega_1}{dt} &= \omega_2\omega_3(I_2 - I_3) + \Gamma_1^{(ext)} \\ I_2 \frac{d\omega_2}{dt} &= \omega_3\omega_1(I_3 - I_1) + \Gamma_2^{(ext)} \\ I_3 \frac{d\omega_3}{dt} &= \omega_1\omega_2(I_1 - I_2) + \Gamma_3^{(ext)} \end{aligned}\right\} \text{Euler Equations}$$

273

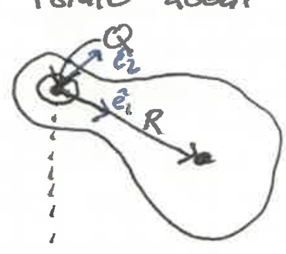
- $\omega_s(t)$ describes motion seen by observer in body-fixed frame
- more work need to get to an inertial observer

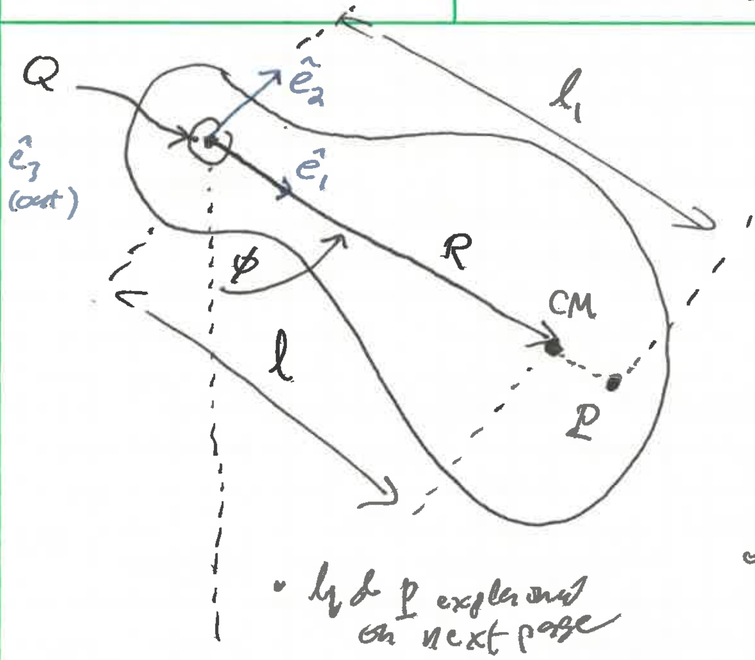
<u>Compound Pendulum</u>

rigid body constrained to rotate about a fixed axis

$\hat{e}_3^{\,0} = \hat{e}_3$

$\hat{e}_3$ (out of page)





$\vec{R} = l\,\hat{e}_1$

$\hat{e}_3$ out of page

$\hat{e}_2 = \hat{e}_3 \times \hat{e}_1$

ϕ determines entire motion

$\dot{\phi} = \omega_3$

only ~~ext~~ $\vec{\Gamma}^{(ext)}$ is gravity on CM

• l_1 & P explained on next page

$$\frac{dL_1}{dt} = \frac{dL_2}{dt} = 0 \quad , \quad \frac{dL_3}{dt} = \Gamma_3^{(ext)}$$

$$L_3 = \sum_j I_{3j}\,\omega_j = \underset{\substack{\uparrow \\ \text{principle axes}}}{I_{33}\,\omega_3} = I_{33}\,\dot{\phi}$$

$$I_{33} = \int \rho(\vec{x})\,(x^2 - x_3^2)\,d^3x \qquad (x^2 = x_1^2 + x_2^2 + x_3^2)$$

$$= \int \rho(\vec{x})\,(x_1^2 + x_2^2)\,d^3x$$

$$\equiv \int \rho(\vec{x})\,r_\perp^2\,d^3x \qquad r_\perp^2 = x_1^2 + x_2^2$$

Book labels this I

(don't care about choice of $\hat{e}_1$ & $\hat{e}_2$ just orthogonal distance from axis)

$$\vec{\Gamma}^{(ext)} = \sum_p m_p\,\vec{r}_p \times \vec{g} = M\vec{R} \times \vec{g}$$

$$\Gamma_3^{(ext)} = -Mgl\sin\phi$$

$$\Gamma_3^{(ext)} = \frac{dL_3}{dt}$$

$$-Mgl\sin\phi = \frac{d}{dt}\left[I_{33}\,\omega_3\right]$$

$$-Mgl\sin\phi = I_{33}\,\ddot{\phi}$$

$$I_{33}\,\ddot{\phi} \approx -Mgl\,\phi$$

$$\ddot{\phi} = -\Omega^2\phi \quad , \quad \Omega = \sqrt{\frac{Mgl}{I_{33}}} \qquad (28.7)$$

Appl parallel axis thm to $\bar{I}_{33}$

$$I_{33} = \bar{I}_{33} + M\ell^2$$

$\bar{I}_{33}$ is moment of inertia ~~through~~ along $\hat{e}_3$ through the CM.

Convention to write $\bar{I}_{33}$ as

$\bar{I}_{33} \equiv M\bar{K}^2$ where $\bar{K}$ is the radius of gyration about CM

$\bar{K}$ is the equivalent radial distance of a point Mass M with moment of inertia $\bar{I}_{33}$.

~~(Give ... (trying to write $\bar{I}_{33}$ like~~

(reason for this will become apparent at the end)

$$I_3 = \bar{I}_{33} + M\ell^2$$
$$= M\bar{K}^2 + M\ell^2$$
$$= M(\bar{K}^2 + \ell^2)$$

sub into Ω (28.7)

$$\Omega^2 = \frac{Mg\ell}{I_{33}}$$
$$= \frac{g\ell}{\bar{K}^2 + \ell^2}$$
$$= \frac{g}{\bar{K}^2/\ell + \ell}$$

let $\ell_1 \equiv \ell + \bar{K}^2/\ell$
$\Omega_Q^2 = \frac{g}{\ell_1}$ } 28.11

ℓ_1 is the length of the equivalent simple pendulum!!

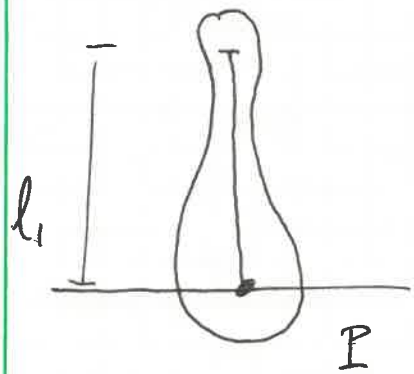
$\ell_1 > \ell$ because $\bar{K}^2$ is pos. definate

let P be a point a distance $\ell_1 \hat{e}_1$ from Q

P is the center of percussion (sweet spot on a bat)

Consider the inverted pendulum

(rotations fixed at P)



similarly to eq 26.7 $\Omega^2 = Mgl/I$

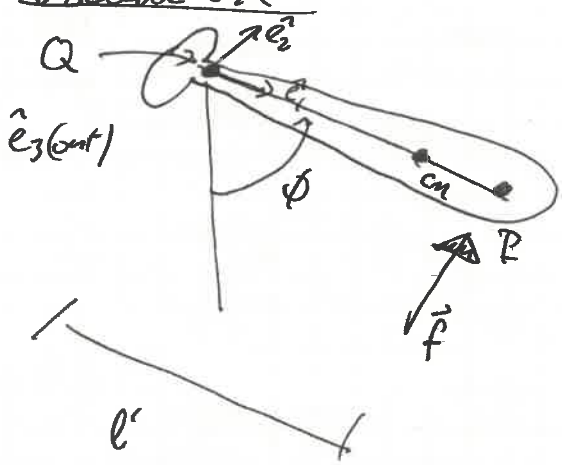
$$\Omega_P^2 = \frac{Mg(l_1-l)}{M[(l_1-l)^2+E^2]}$$

$$= \frac{g(l+E^2/l-l)}{[(l+E^2/l-l)^2+E^2]}$$

$$\Omega_P^2 = \frac{g}{\frac{l}{E^2}\left[\left(\frac{E^2}{l}\right)^2+E^2\right]} = \frac{g}{[E^2/l+l]} = \Omega_Q^2$$ 28.12

- experimenter can measure Ω_Q^2, then find corresponding P
- The measured Ω_Q^2 & l_1 give absolute value of $\underline{g}$!

Baseball bat



$\vec f = f\hat e_2$ at distance l'

$\vec f$ generates reaction force at Q, $\vec f_r$

$-f_r\hat e_2$

Compare magnitudes of $\vec f$ & $\vec f_r$

$M\ddot{\vec R} = \vec F^{(ext)}$

$M\ddot{\vec R}\cdot\hat e_2 = f - f_r$

$\ddot{\vec R}\cdot\hat e_2 = l\ddot\phi$

$Ml\ddot\phi = f - f_r$

$$\frac{dL_3}{dt} = I_3\ddot\phi = \Gamma_3^{(ext)} = fl'$$

previously we've shown $I_3 = M(l^2+E^2) = Ml(l+E^2/l)$

$I_3 = Mll_1$

thus $Mll_1\ddot\phi = fl'$ & $Ml\ddot\phi = f - f_r$

$$\frac{M l \ddot{\phi}}{M l l_1 \ddot{\phi}} = \frac{f - f_r}{f l'}$$

$$\frac{1}{l_1} = \frac{f - f_r}{f l'}$$

$$f \frac{l'}{l_1} = f - f_r$$

$$f_r = f\left(1 - \frac{l'}{l_1}\right)$$

$$\boxed{\frac{f_r}{f} = \frac{l_1 - l'}{l_1}}$$

~~lengths~~ length of equivalent simple pendulum

if $l' = l_1$ than $f_r = 0$

sweet spot of a bat!