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High-order Approximations of Green's function technique in forming and transport of intensive beams

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Outline

- Integral representations for surface & volume sources;
- Linear approximation in 3D;
- Analytical integration over cubic mesh cells;
- Singularity problems;
- Simple benchmarks;
- Adaptive integration scheme;
- Sheet-beam gun design;
- Summary

Integral representations for surface & volume sources

$$\varphi(x_0, y_0, z_0) = \int_S \sigma(x, y, z) G(x_0, y_0, z_0; x, y, z) dS$$

$$+ \int_V \rho(x', y', z') G(x_0, y_0, z_0; x', y', z') dV, \quad (x, y, x) \in S, \quad (x', y', z') \in V.$$

Green's Gunction of point source in free space

$$G(x_0, y_0, z_0; x, y, z) = \frac{1}{4\pi\varepsilon_0 R} = \frac{1}{4\pi\varepsilon_0 \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}.$$

Linear approximation in 3D

Space charge density

$$\rho(x,y,z) = \{[(\rho_{i+1,j,k}(x-x_i) + \rho_{i,j,k}(x_{i+1}-x))(y_{i+1}-y) +$$
$$(\rho_{i+1,j+1,k}(x-x_i) + \rho_{i,j+1,k}(x_{i+1}-x))(y-y_j)](z_{k+1}-z) +$$
$$[(\rho_{i+1,j,k+1}(x-x_i) + \rho_{i,j,k+1}(x_{i+1}-x))(y_{i+1}-y) +$$
$$(\rho_{i+1,j+1,k+1}(x-x_i) + \rho_{i,j+1,k+1}(x_{i+1}-x))(y-y_j)](z-z_k)\}/(h_x h_y h_z).$$

Mesh

$\{x_i\}$ x $\{y_j\}$ x $\{z_k\}$, i=1,…,N_{x+1}, j=1,…,N_{y+1}, k=1,…,N_{z+1}

Potential

$$\varphi(x_0,y_0,z_0) = \sum_{i,j,k} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \int_{z_k}^{z_{k+1}} \rho(x,y,z) G(x_0,y_0,z_0;x,y,z)\,dxdydz,$$

Field

$$E_z(x,y,z) \equiv -\frac{\partial}{\partial z}\varphi(x,y,z) = \sum_{i,j,k} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \int_{z_k}^{z_{k+1}} \rho(x',y',z') \frac{\partial}{\partial z} G(x,y,z;x',y',z')\,dx'dy'dz'.$$

Analytical integration over cubic mesh cells

$$J_1 = \iiint \frac{x\, dxdydz}{\sqrt{x^2+y^2+z^2}} = \iint \sqrt{x^2+y^2+z^2}\, dydz =$$

$$\frac{1}{2}\int\left[y\sqrt{x^2+y^2+z^2} + (x^2+z^2)\ln\left| y+\sqrt{x^2+y^2+z^2}\right|\right] dz =$$

$$\frac{r}{4}yz + \frac{y}{4}(x^2+y^2)\ln|z+r| + \frac{x^2}{2}\int \ln|y+r|\,dz + \frac{1}{2}\int z^2\ln|y+r|\,dz,$$

$$\int \ln|y+r|\,dz = z\ln|y+r| - z + x\ tg^{-1}\left(\frac{z}{x}\right) - x\ tg^{-1}\left(\frac{zy}{xr}\right) + y\ln|z+r|,$$

$$\int z^2\ln|y+r|\,dz = \frac{z^3}{3}\ln|y+r| +$$

$$\frac{1}{18}\left[6x^2z - 2z^3 + 3zyr - 6x^3 tg^{-1}\left(\frac{z}{x}\right) + 6x^3 tg^{-1}\left(\frac{zy}{xr}\right) - 3y(y^2+3x^2)\ln|z+r|\right]$$

$$J_2 = \iiint \frac{x\ y\ dxdydz}{\sqrt{x^2+y^2+z^2}} = \iint y\sqrt{x^2+y^2+z^2}\,dydz = \frac{1}{3}\int r^3 dz =$$

$$\frac{r^3}{12}z + \frac{3}{24}(x^2+y^2)\left[zr + (x^2+y^2)\ln|z+r|\right].$$

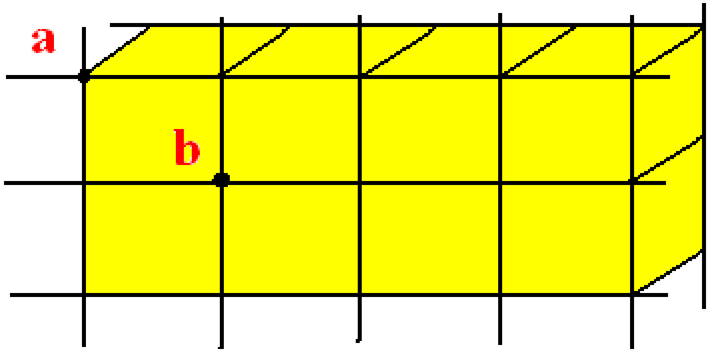
$$J_3 = \iiint \frac{x\ y\ z\ dxdydz}{\sqrt{x^2+y^2+z^2}} = \iint yz\sqrt{x^2+y^2+z^2}\,dydz = \frac{1}{3}\int r^3 z dz = \frac{r^5}{15}.$$

$$J_4 = \iiint \frac{dxdydz}{\sqrt{x^2+y^2+z^2}} =$$

$$xy\ln\left(z+\sqrt{x^2+y^2+z^2}\right) + yz\ln\left(z+\sqrt{x^2+y^2+z^2}\right) + xz\ln\left(y+\sqrt{x^2+y^2+z^2}\right)$$

$$-\frac{1}{2}\left[x^2 tg^{-1}\left(\frac{zy}{x\sqrt{x^2+y^2+z^2}}\right) + y^2 tg^{-1}\left(\frac{xz}{y\sqrt{x^2+y^2+z^2}}\right) + z^2 tg^{-1}\left(\frac{xy}{z\sqrt{x^2+y^2+z^2}}\right)\right].$$

Singularity problems



•The potential is regular limited function in all space, but the analytical formulae have some artificial singularities, which should be eliminated in numerical implementations.

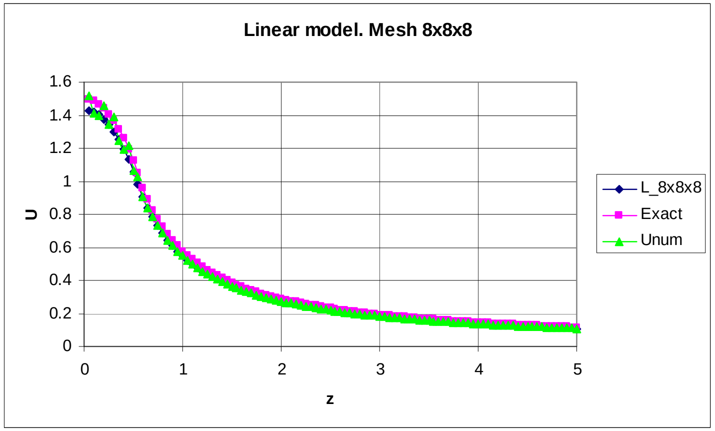
•The field gradients have as artificial singularities in the inner cells as real singularities on the beam boundary. Those singularities can generate a numerical noise, which lower the accuracy computation in using of piecewise-constant approximation for space charge density.

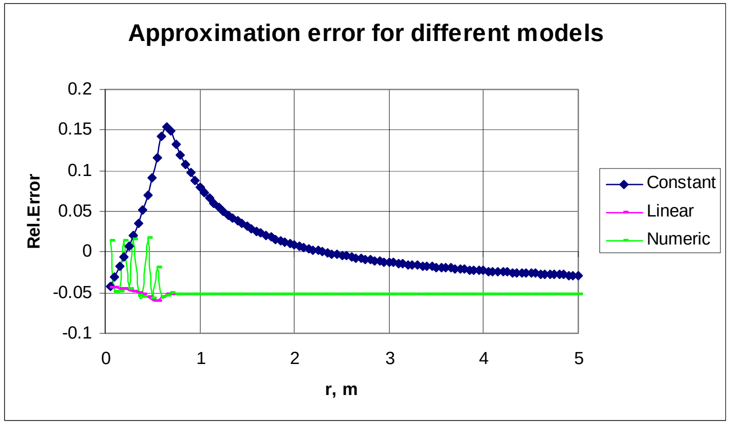
•Linear approximation for space charge density can eliminate all those singularities.

Simple benchmark

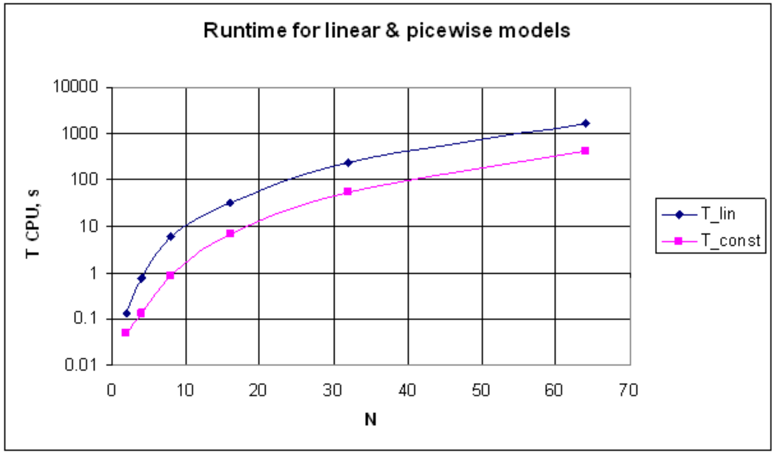
Potential of a sphere with charge density $\rho = \dfrac{3}{4\pi R^3}$ is

$$\varphi(r) = \begin{cases} (R^2 - r^2)/2 + 1/R, & r < R, \\ 1/R, & r > R. \end{cases}$$

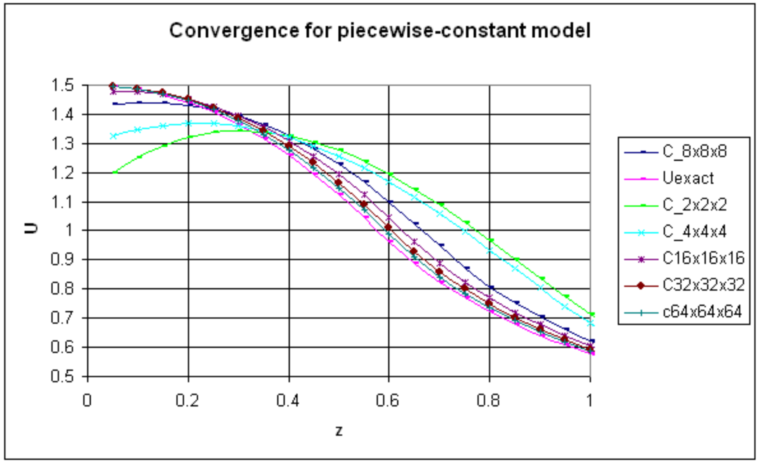


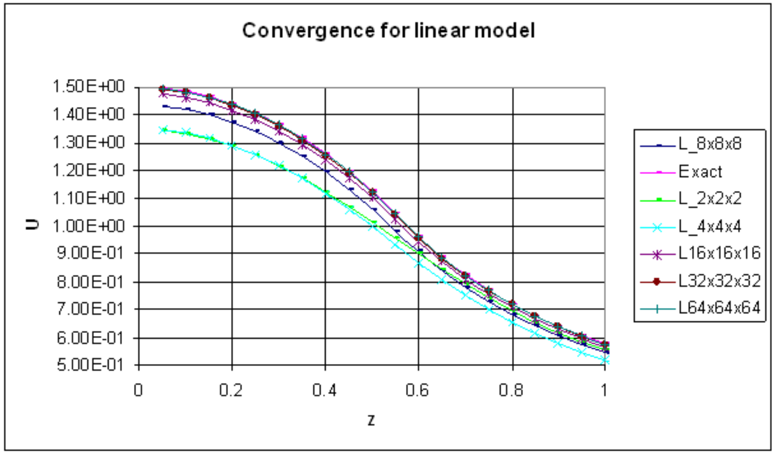


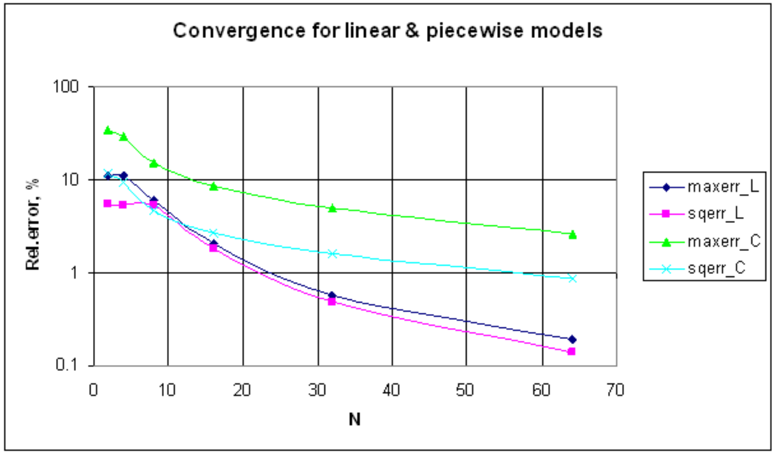
Piece-wise constant approximation gives unacceptable error on the boundary. Numerical integration produces oscillations near the field sources. Linear model has good accuracy in “near” and “far” zones.



Convergence Study

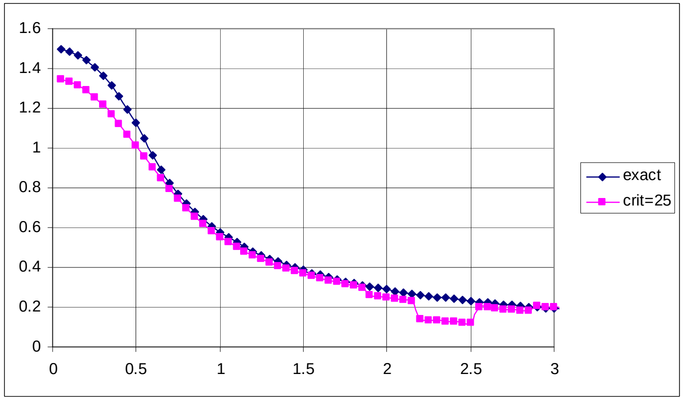


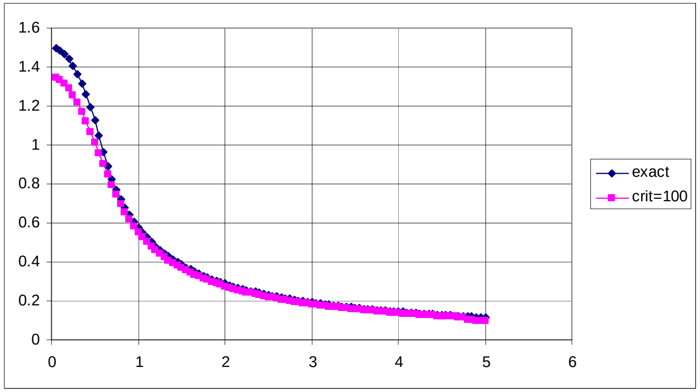




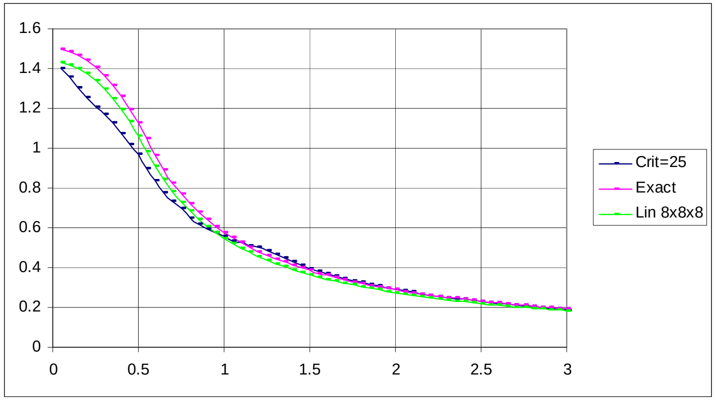
Maximal & mean-square errors for piecewise-constant and linear approximations of the field sources with analytic integration scheme

Adaptive integration scheme



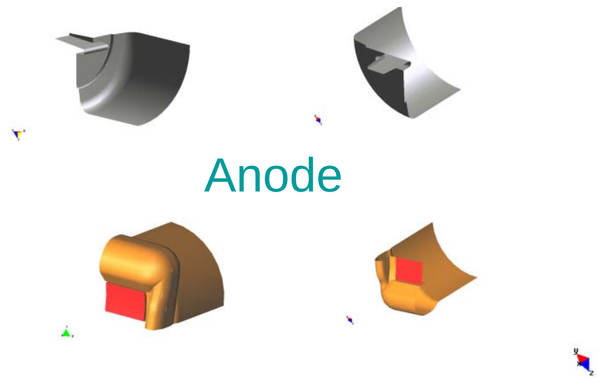


Worse mesh 2x2x2 cells shows clear the influence of the adaptation criterion

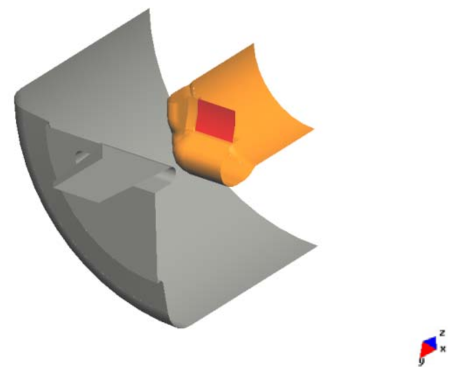


Adaptive integration algorithm can reduce the runtime dramatically (factor of 5-10) lowering the accuracy of calculations just for 0.5-1% comparing the piecewise-constant, linear approximation; linear & numerical integration scheme

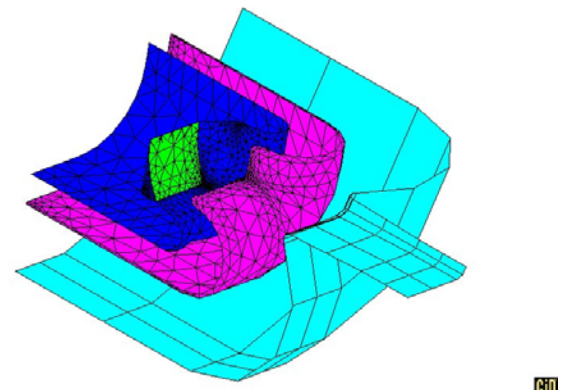
Sheet-beam gun design for SLAC X-band klystron



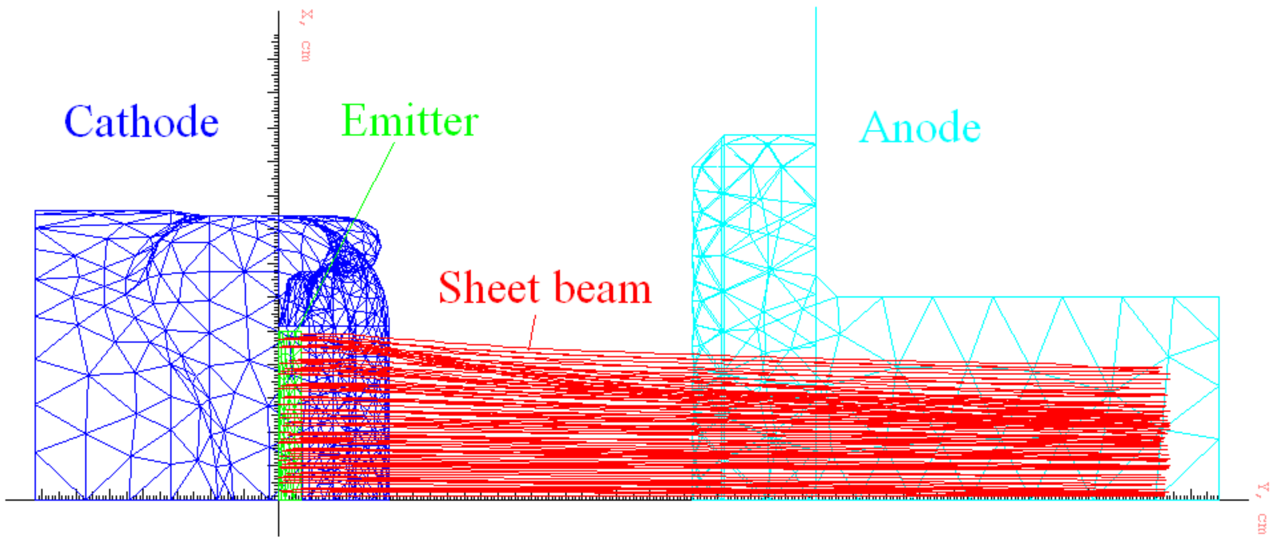
Cathode

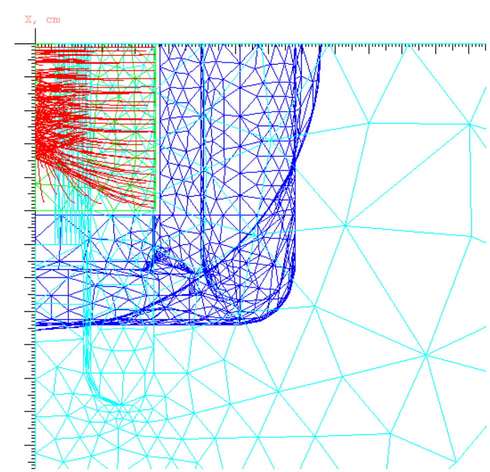


Gun assembly



Geometry model





Beam optics simulation using the POISSON-3 code

Summary

•The analytical model to take into account the space charge of the beam on 3D rectangular mesh is implemented;
•The efficiency of the analytical model was studied by comparing the numerical integration, piecewise-constant & tri-linear analytical models with the exact solution for simple test problems;
• The singularity problems have been studies for different integration schemes;
• The adaptive integration scheme is suggested to increase the speed of calculations;
•The algorithms of analytical integration have been implemented in the POISSON-3D code devoted to simulation of high-current relativistic beam optics in 3D;
• Green's function technique demonstrated its efficiency in the design of sheet-beam gun for SLAC X-band klystron.

References

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4. V. Ivanov, A. Krasnykh, G. Scheitrum, D. Sprehn, L. Ives, G. Miram, 3D Modeling Activity for Novel High Power Electron Guns at SLAC, PAC-2003, May 12-16, 2003, Portland, USA.