

General Theory of Intense Beam Nonlinear Thomson Scattering

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Outline

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What's New in this Work

- Many of the the newer Thomson Sources are based on a **PULSED** Laser (e.g. all of the high-energy single-pulse lasers are pulsed by their very nature)
- Previously developed a general theory to cover the calculations in the general case of a pulsed, high field strength laser interacting with electrons in a Thomson backscatter arrangement. Have extended this theory to cover more general scattering geometries
- The new theory shows that in many situations the estimates people do to calculate flux and brilliance, based on a constant amplitude models, are just plain wrong.
- The new theory is general enough to cover all “1-D” undulator calculations and all pulsed laser Thomson scattering calculations.
- The main “new physics” that the new calculations include properly is the fact that the electron motion changes based on the local value of the field strength squared. Such ponderomotive forces (i.e., forces proportional to the field strength squared), lead to a detuning of the emission, angle dependent Doppler shifts of the emitted scattered radiation, and additional transverse dipole emission that this theory can calculate.



Ancient History

- Early 1960s: Laser Invented
- Brown and Kibble (1964): Earliest definition of the field strength parameters K and/or a in the literature that I'm aware of

$$a = \frac{eE_0\lambda_0}{2\pi mc^2} \text{ Thomson Sources} \quad K = \frac{eB_0\lambda_0}{2\pi mc^2} \text{ Undulators}$$

Interpreted frequency shifts that occur at high fields as a “relativistic mass shift”.

- Sarachik and Schappert (1970): Power into harmonics at high K and/or a . Full calculation for CW (monochromatic) laser. Later referenced, corrected, and extended by workers in fusion plasma diagnostics.
- Alferov, Bashmakov, and Bessonov (1974): Undulator/Insertion Device theories developed under the assumption of constant field strength. Numerical codes developed to calculate “real” fields in undulators.
- Coisson (1979): Simplified undulator theory, which works at low K and/or a , developed to understand the frequency distribution of “edge” emission, or emission from “short” magnets, i.e., including pulse effects



Coisson's Spectrum from a Short Magnet

Coisson low-field strength undulator spectrum*

$$\frac{dE}{d\nu d\Omega} = \frac{r_e^2 c}{\pi} \gamma^2 (1 + \gamma^2 \theta^2)^2 f^2 \left| \tilde{B}(\nu(1 + \gamma^2 \theta^2) / 2\gamma^2) \right|^2$$

$$f^2 = f_\sigma^2 + f_\pi^2$$

$$f_\sigma = \frac{1}{(1 + \gamma^2 \theta^2)^2} \sin \phi$$

$$f_\pi = \frac{1}{(1 + \gamma^2 \theta^2)^2} \left(\frac{1 - \gamma^2 \theta^2}{1 + \gamma^2 \theta^2} \right) \cos \phi$$

*R. Coisson, Phys. Rev. A **20**, 524 (1979)



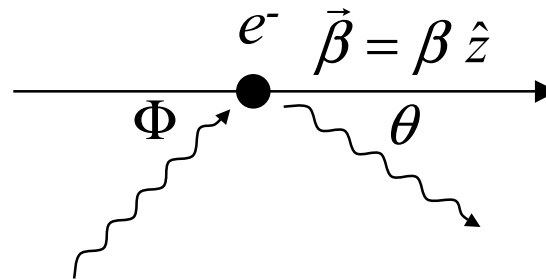
Thomson Scattering

- Purely “classical” scattering of photons by electrons
- Thomson regime defined by the photon energy in the electron rest frame being small compared to the rest energy of the electron, allowing one to neglect the quantum mechanical “Dirac” recoil on the electron
- In this case electron radiates at the same frequency as incident photon for low enough field strengths
- Classical dipole radiation pattern is generated in beam frame
- Therefore radiation patterns, at low field strength, can be largely copied from textbooks

- Note on terminology: Some authors call any scattering of photons by free electrons Compton Scattering. Compton observed (the so-called Compton effect) frequency shifts in X-ray scattering off (resting!) electrons that depended on scattering angle. Such frequency shifts arise only when the energy of the photon in the rest frame becomes comparable with 0.511 MeV.



Simple Kinematics



Beam Frame

$$p'_{e\mu} = (mc^2, 0)$$

$$p'_{p\mu} = (E'_L, \vec{E}'_L)$$

Lab Frame

$$p_{e\mu} = mc^2(\gamma, \gamma\beta \hat{z})$$

$$p_{p\mu} = E_L(1, \sin \Phi \hat{y} + \cos \Phi \hat{z})$$

$$p_e \cdot p_p = mc^2 E'_L = mc^2 E_L \gamma (1 - \beta \cos \Phi)$$



$$E'_L = E_L \gamma (1 - \beta \cos \Phi)$$

In beam frame scattered photon radiated with wave vector

$$k'_\mu = \frac{E'_L}{c} (1, \sin \theta' \cos \phi', \sin \theta' \sin \phi', \cos \theta')$$

Back in the lab frame, the scattered photon energy E_s is

$$E_s = E'_L \gamma (1 + \beta \cos \theta') = \frac{E'_L}{\gamma (1 - \beta \cos \theta)}$$

$$E_s = E_L \frac{(1 - \beta \cos \Phi)}{(1 - \beta \cos \theta)}$$



Cases explored

Backscattered

$$\Phi = \pi$$

$$E_s = E_L \frac{(1 + \beta_z)}{(1 - \beta_z \cos \theta)} \approx 4\gamma^2 E_L \quad \text{at } \theta = 0$$

Provides highest energy photons for a given beam energy, or alternatively, the lowest beam energy to obtain a given photon wavelength. Pulse length roughly the ELECTRON bunch length



Cases explored, contd.

Ninety degree scattering

$$\Phi = \pi / 2$$

$$E_s = E_L \frac{1}{(1 - \beta_z \cos \theta)} \approx 2\gamma^2 E_L \quad \text{at } \theta = 0$$

Provides factor of two lower energy photons for a given beam energy than the equivalent Backscattered situation. However, very useful for making short X-ray pulse lengths. Pulse length a complicated function of electron bunch length and transverse size.



Cases explored, contd.

Small angle scattered (SATS)

$$\Phi \ll 1$$

$$E_s = E_L \frac{\Phi^2}{2(1 - \beta_z \cos \theta)} \approx \Phi^2 \gamma^2 E_L \quad \text{at } \theta = 0$$

Provides much lower energy photons for a given beam energy than the equivalent Backscattered situation. Alternatively, need greater beam energy to obtain a given photon wavelength. Pulse length roughly the PHOTON pulse length.



Dipole Radiation

Assume a single charge moves in the x direction

$$\rho(x, y, z, t) = e\delta(x - d(t))\delta(y)\delta(z)$$

$$\vec{J}(x, y, z, t) = e\dot{d}(t)\hat{x}\delta(x - d(t))\delta(y)\delta(z)$$

Introduce scalar and vector potential for fields.

Retarded solution to wave equation (Lorenz gauge), $R = |\vec{r} - \vec{r}'(t')|$

$$\Phi(\vec{r}, t) = \int \frac{1}{R} \rho\left(\vec{r}', t - \frac{R}{c}\right) dx' dy' dz' = e \int \frac{\delta(t' - t + R/c)}{R} dt'$$

$$A_x(\vec{r}, t) = \int \frac{1}{Rc} J_x\left(\vec{r}', t - \frac{R}{c}\right) dx' dy' dz' = e \int \frac{\dot{d}(t')\delta(t' - t + R/c)}{Rc} dt'$$



Dipole Radiation

Perform proper differentiations to obtain field and integrate by parts the delta function.

Use far field approximation, $r = |\vec{r}| \gg d$ (velocity terms small)

“Long” wave length approximation, $\lambda \gg d$ (source smaller than λ)

Low velocity approximation, $\dot{d} \ll c$ (really a limit on excitation strength)

$$B_y = \partial A_x / \partial z \approx -\frac{e}{c^2} z \frac{\ddot{d}(t-r/c)}{r^2}$$

$$B_z = -\partial A_x / \partial y \approx \frac{e}{c^2} y \frac{\ddot{d}(t-r/c)}{r^2}$$



Dipole Radiation

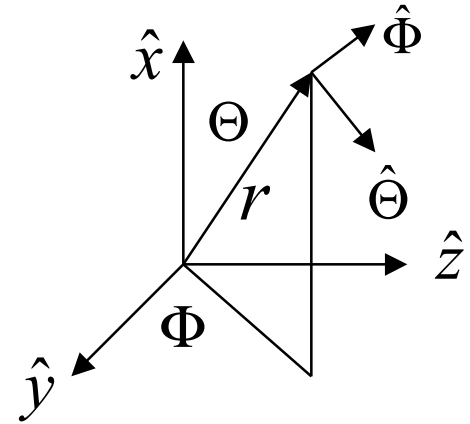
$$\vec{B} = \frac{e\ddot{d}(t-r/c)}{c^2 r} \sin \Theta \hat{\Phi}$$

$$\vec{E} = \frac{e\ddot{d}(t-r/c)}{c^2 r} \sin \Theta \hat{\Theta}$$

$$I = \frac{c\vec{E} \times \vec{B}}{4\pi} = \frac{1}{4\pi} \frac{e^2 \ddot{d}^2(t-r/c)}{c^3 r^2} \sin^2 \Theta \hat{r}$$

$$\frac{dI}{d\Omega} = \frac{1}{4\pi} \frac{e^2 \ddot{d}^2(t-r/c)}{c^3} \sin^2 \Theta$$

Polarized in the plane containing $\hat{r} = \vec{n}$ and \hat{x}



Dipole Radiation

Define the Fourier Transform

$$\tilde{d}(\omega) = \int d(t) e^{-i\omega t} dt \qquad d(t) = \frac{1}{2\pi} \int \tilde{d}(\omega) e^{i\omega t} d\omega$$

With these conventions Parseval's Theorem is

$$\int d^2(t) dt = \frac{1}{2\pi} \int |\tilde{d}|^2(\omega) d\omega$$

$$\frac{dE}{d\Omega} = \frac{e^2}{4\pi c^3} \int \ddot{d}^2(t - r/c) dt = \frac{e^2}{8\pi^2 c^3} \int \omega^4 |\tilde{d}|^2(\omega) d\omega$$

$$\frac{dE}{d\omega d\Omega} = \frac{1}{8\pi^2} \frac{e^2 \omega^4 |\tilde{d}(\omega)|^2}{c^3} \sin^2 \Theta$$

Blue Sky!

This equation does not follow the typical (see Jackson) convention that combines both positive and negative frequencies together in a single positive frequency integral. The reason is that we would like to apply Parseval's Theorem easily. By symmetry, the difference is a factor of two.



Dipole Radiation

For a motion in three dimensions

$$\frac{dE}{d\omega d\Omega} = \frac{1}{8\pi^2} \frac{e^2 \omega^4 \left| \vec{n} \times \vec{\tilde{d}}(\omega) \right|^2}{c^3}$$

Vector inside absolute value along the magnetic field

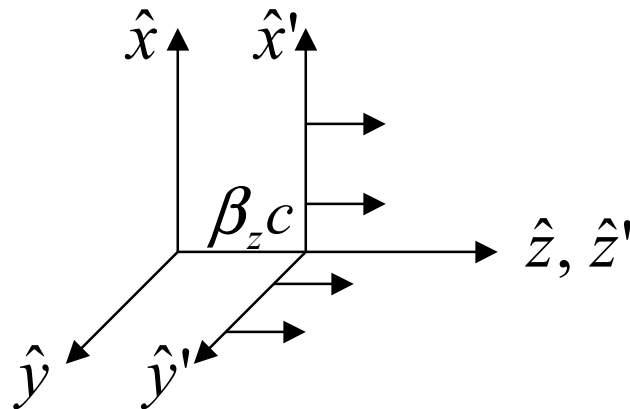
$$\frac{dE}{d\omega d\Omega} = \frac{1}{8\pi^2} \frac{e^2 \omega^4 \left| \left(\vec{n} \times \vec{\tilde{d}}(\omega) \right) \times \vec{n} \right|^2}{c^3} = \frac{1}{8\pi^2} \frac{e^2 \omega^4 \left| \vec{\tilde{d}}(\omega) - \left(\vec{n} \cdot \vec{\tilde{d}}(\omega) \right) \vec{n} \right|^2}{c^3}$$

Vector inside absolute value along the electric field. To get energy into specific polarization, take scalar product with the polarization vector



Co-moving Coordinates

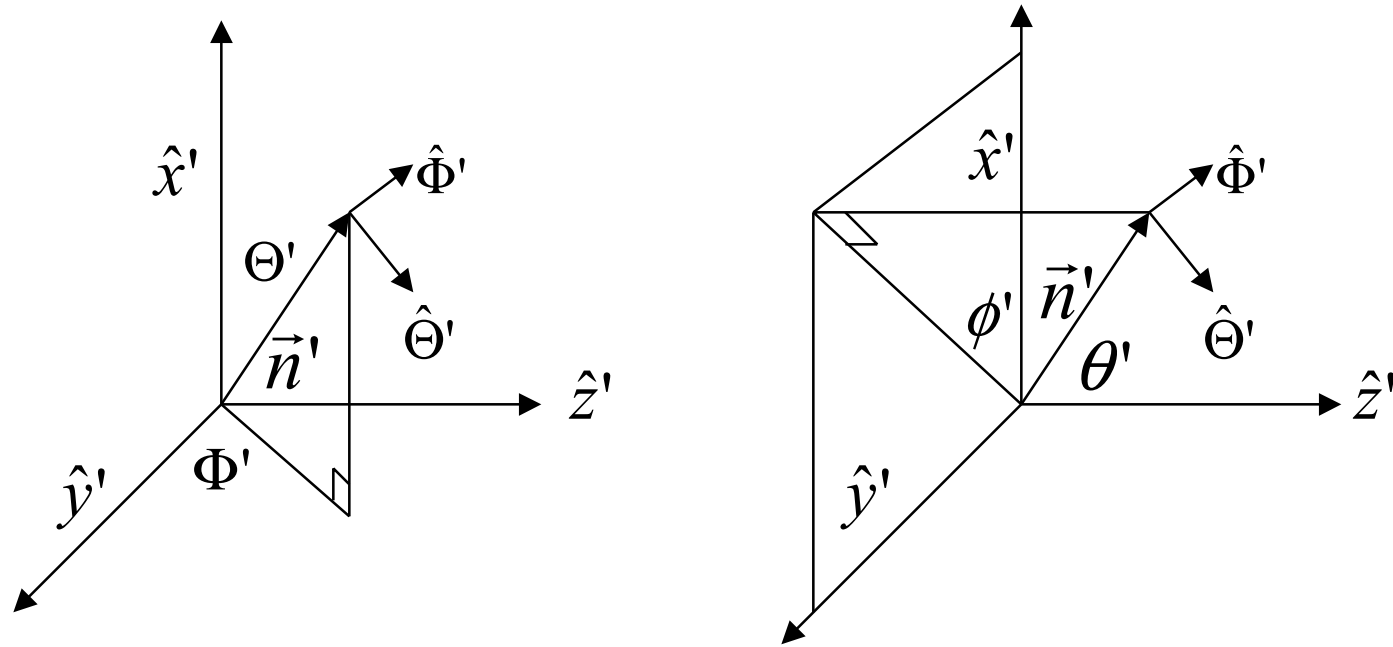
- Assume radiating charge is moving with a velocity close to light in a direction taken to be the z axis, and the charge is on average at rest in this coordinate system
- For the remainder of the presentation, quantities referred to the moving coordinates will have primes; unprimed quantities refer to the lab system



- In the co-moving system the dipole radiation pattern applies



New Coordinates



Resolve the polarization of scattered energy into that perpendicular (σ) and that parallel (π) to the scattering plane

$$\vec{n}' = \sin \theta' \cos \phi' \hat{x}' + \sin \theta' \sin \phi' \hat{y}' + \cos \theta' \hat{z}'$$

$$\hat{e}'_{\sigma} = \vec{n}' \times \hat{z}' / |\vec{n}' \times \hat{z}'| = \sin \phi' \hat{x}' - \cos \phi' \hat{y}' = -\hat{\phi}'$$

$$\hat{e}'_{\pi} = \vec{n}' \times \hat{e}'_{\sigma} = \cos \theta' \cos \phi' \hat{x}' + \cos \theta' \sin \phi' \hat{y}' - \sin \theta' \hat{z}' = \hat{\theta}'$$



Polarization

It follows that

$$\tilde{\mathbf{d}}'(\omega') \cdot \hat{\mathbf{e}}'_\sigma = \tilde{d}'_x(\omega') \sin \phi' - \tilde{d}'_y(\omega') \cos \phi'$$

$$\tilde{\mathbf{d}}'(\omega') \cdot \hat{\mathbf{e}}'_\pi = \tilde{d}'_x(\omega') \cos \theta' \cos \phi' + \tilde{d}'_y(\omega') \cos \theta' \sin \phi' - \sin \theta' \tilde{d}'_z(\omega')$$

So the energy into the two polarizations in the beam frame is

$$\frac{dE'_\sigma}{d\omega' d\Omega'} = \frac{1}{8\pi^2} \frac{e^2 \omega'^4}{c^3} \left| \tilde{d}'_x(\omega') \sin \phi' - \tilde{d}'_y(\omega') \cos \phi' \right|^2$$

$$\frac{dE'_\pi}{d\omega' d\Omega'} = \frac{1}{8\pi^2} \frac{e^2 \omega'^4}{c^3} \left| \tilde{d}'_x(\omega') \cos \theta' \cos \phi' + \tilde{d}'_y(\omega') \cos \theta' \sin \phi' - \sin \theta' \tilde{d}'_z(\omega') \right|^2$$



Comments/Sum Rule

- There is no radiation parallel or anti-parallel to the x -axis for x -dipole motion
- In the forward direction $\theta' \rightarrow 0$, the radiation polarization is parallel to the x -axis for an x -dipole motion
- One may integrate over all angles to obtain a result for the total energy radiated

$$\frac{dE'_\sigma}{d\omega'} = \frac{1}{8\pi^2} \frac{e^2 \omega'^4}{c^3} \left(\left| \tilde{d}'_x(\omega') \right|^2 + \left| \tilde{d}'_y(\omega') \right|^2 \right) 2\pi$$

$$\frac{dE'_\pi}{d\omega'} = \frac{1}{8\pi^2} \frac{e^2 \omega'^4}{c^3} \left[\left(\left| \tilde{d}'_x(\omega') \right|^2 + \left| \tilde{d}'_y(\omega') \right|^2 \right) \frac{2\pi}{3} + \left| \tilde{d}'_z(\omega') \right|^2 \frac{8\pi}{3} \right]$$

$$\frac{dE'_{tot}}{d\omega'} = \frac{1}{8\pi^2} \frac{e^2 \omega'^4}{c^3} \left| \tilde{\vec{d}}'(\omega') \right|^2 \frac{8\pi}{3}$$

Generalized Larmor



Sum Rule

Total energy sum rule

$$E'_{tot} = \int_{-\infty}^{\infty} \frac{1}{3\pi} \frac{e^2 \omega'^4}{c^3} \left| \tilde{\vec{d}}'(\omega') \right|^2 d\omega'$$

Parseval's Theorem again gives “standard” Larmor formula

$$P' = \frac{dE'_{tot}}{dt'} = \frac{2}{3} \frac{e^2 \ddot{\vec{d}}'^2(t')}{c^3} = \frac{2}{3} \frac{e^2 \vec{a}'^2(t')}{c^3}$$



Relativistic Invariances

To determine the radiation pattern for a “moving” oscillating charge we use this solution plus transformation formulas from relativity theory. As an example note **photon number invariance**: The total number of photons emitted must be independent of the frame where the calculation is done. In particular,

$$N_{tot} = \frac{1}{3\pi} \int_{-\infty}^{\infty} \frac{e^2 \left| \tilde{d}'(\omega') \right|^2 \omega'^4}{\hbar |\omega'| c^3} d\omega'$$

must be frame independent. Rewriting formulas in terms of relativistically invariant quantities can simplify formulas.



Wave Vector Transformation Law

Follows from relativistic invariance of wave phase, which implies $k^\mu = (\omega/c, k_x, k_y, k_z)$ is a 4-vector

$$\omega'/c = \gamma\omega/c - \beta\gamma k \cos\theta = (\omega/c)\gamma(1 - \beta \cos\theta)$$

$$k' \sin\theta' \cos\phi' = k \sin\theta \cos\phi$$

$$k' \sin\theta' \sin\phi' = k \sin\theta \sin\phi$$

$$k' \cos\theta' = -\beta\gamma\omega/c + \gamma k \cos\theta$$

and $k = \omega/c$ and $k' = \omega'/c$ are the magnitudes of the wave propagation vectors

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta \cos\theta'} \quad \phi = \phi'$$

Invert by reversing the sign of β



Solid Angle Transformation

$$d \cos \theta' \wedge d\phi' = d \left(\frac{\cos \theta - \beta}{1 - \beta \cos \theta} \right) \wedge d\phi$$

$$= \left(\frac{1 - \beta \cos \theta + \beta \cos \theta - \beta^2}{(1 - \beta \cos \theta)^2} \right) d \cos \theta \wedge d\phi$$

$$= \left(\frac{1}{\gamma^2 (1 - \beta \cos \theta)^2} \right) d \cos \theta \wedge d\phi$$

$$d\Omega' = \left(\frac{1}{\gamma^2 (1 - \beta \cos \theta)^2} \right) d\Omega$$



Energy Distribution in Lab Frame

$$\frac{dE_\sigma}{d\omega d\Omega} = \frac{e^2 \omega^4 \gamma^2 (1 - \beta \cos \theta)^2}{8\pi^2 c^3} \left| \begin{array}{l} \tilde{d}'_x (\omega\gamma(1 - \beta \cos \theta)) \sin \phi \\ - \tilde{d}'_y (\omega\gamma(1 - \beta \cos \theta)) \cos \phi \end{array} \right|^2$$

$$\frac{dE_\pi}{d\omega d\Omega} = \frac{e^2 \omega^4 \gamma^2 (1 - \beta \cos \theta)^2}{8\pi^2 c^3} \left| \begin{array}{l} \tilde{d}'_x (\omega\gamma(1 - \beta \cos \theta)) \frac{\cos \theta - \beta}{1 - \beta \cos \theta} \cos \phi \\ + \tilde{d}'_y (\omega\gamma(1 - \beta \cos \theta)) \frac{\cos \theta - \beta}{1 - \beta \cos \theta} \sin \phi \\ - \frac{\sin \theta}{\gamma(1 - \beta \cos \theta)} \tilde{d}'_z (\omega\gamma(1 - \beta \cos \theta)) \end{array} \right|^2$$

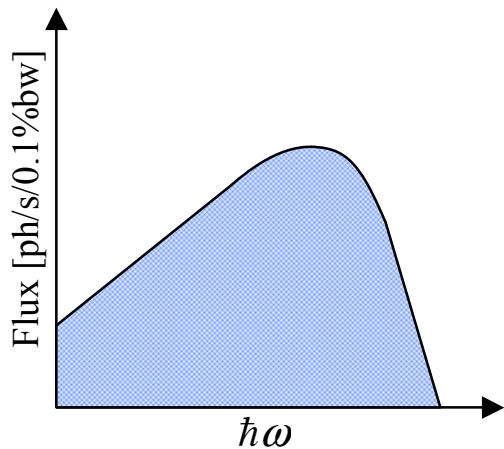
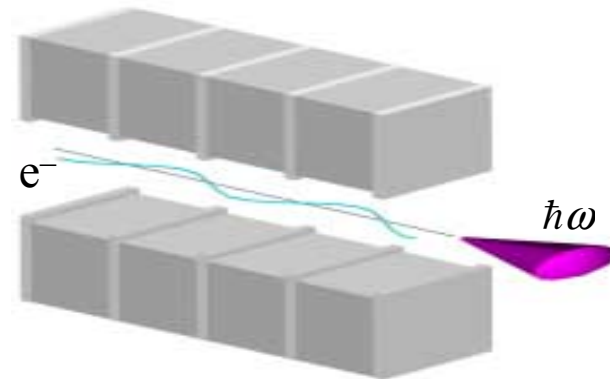
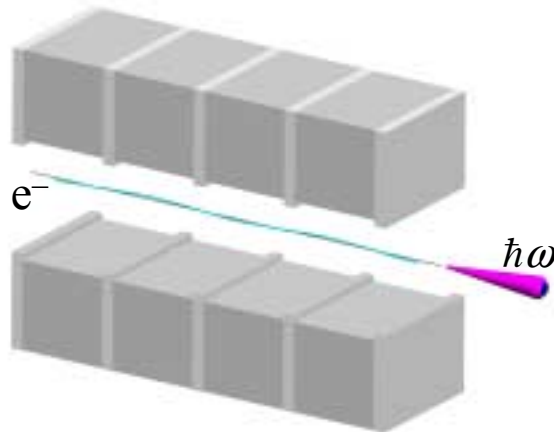
By placing the expression for the Doppler shifted frequency and angles inside the transformed beam frame distribution. Total energy radiated from d'_z is the same for same dipole strength.



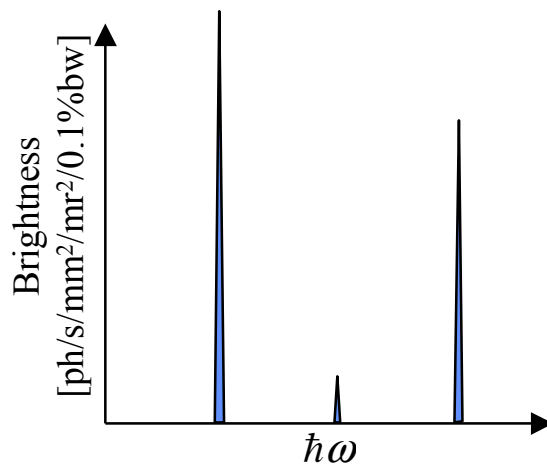
Bend

Undulator

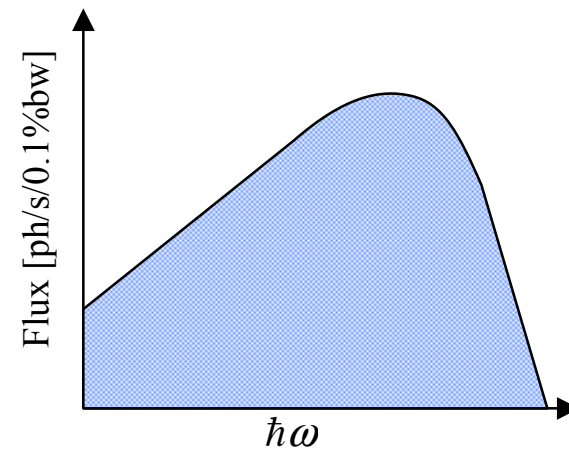
Wiggler



white source



partially coherent source



powerful white source



Weak Field Undulator Spectrum

$$\tilde{d}'(\omega') = \tilde{d}'(\omega')\hat{x} = -\frac{ec}{mc^2} \frac{\tilde{B}(\omega'/c\beta_z\gamma)}{\omega'^2} \hat{x} \quad \tilde{B}(k) = \int B(z)e^{-ikz} dz$$

$$\frac{dE_\sigma}{d\omega d\Omega} = \frac{1}{8\pi^2} \frac{e^4}{m^2 c^5} \frac{|\tilde{B}(\omega(1-\beta_z \cos\theta)/c\beta_z)|^2}{\gamma^2(1-\beta_z \cos\theta)^2} \sin^2 \phi$$

$$\frac{dE_\pi}{d\omega d\Omega} = \frac{1}{8\pi^2} \frac{e^4}{m^2 c^5} \frac{|\tilde{B}(\omega(1-\beta_z \cos\theta)/c\beta_z)|^2}{\gamma^2(1-\beta_z \cos\theta)^2} \left(\frac{\cos\theta - \beta_z}{1-\beta_z \cos\theta} \right)^2 \cos^2 \phi$$

$$r_e^2 \equiv \frac{e^4}{m^2 c^4} \quad \lambda = \frac{\lambda_0}{2\gamma^2} \quad (1-\beta_z \cos\theta)(1+\beta_z) \approx \frac{1}{\gamma^2} + \theta^2 + \dots \approx \frac{1+\gamma^2\theta^2}{\gamma^2}$$

Generalizes Coisson to arbitrary observation angles



Strong Field Case

$$\frac{d}{dt} \gamma = 0$$

$$\frac{d}{dt} \gamma m \vec{\beta} c = -e \vec{\beta} \times \vec{B}$$

$$\beta_x(z) = \frac{e}{\gamma m c^2} \int_{-\infty}^z B(z') dz'$$



High K

$$\beta_z(z) = \sqrt{1 - \frac{1}{\gamma^2} - \beta_x^2(z)}$$

$$\beta_z(z) = \sqrt{1 - \frac{1}{\gamma^2} - \left(\frac{e}{\gamma mc^2} \int_{-\infty}^z B(z') dz' \right)^2}$$

$$\beta_z(z) \approx 1 - \frac{1}{2\gamma^2} - \frac{1}{2} \left(\frac{e}{\gamma mc^2} \int_{-\infty}^z B(z') dz' \right)^2 = 1 - \frac{1}{2\gamma^2} \left(1 + \frac{K^2}{2} \right) - \frac{K^2}{4\gamma^2} \cos(2k_0 z)$$



High K

Inside the insertion device the average (z) velocity is

$$\beta^*_z = 1 - \frac{1}{2\gamma^2} \left(1 + \frac{K^2}{2} \right)$$

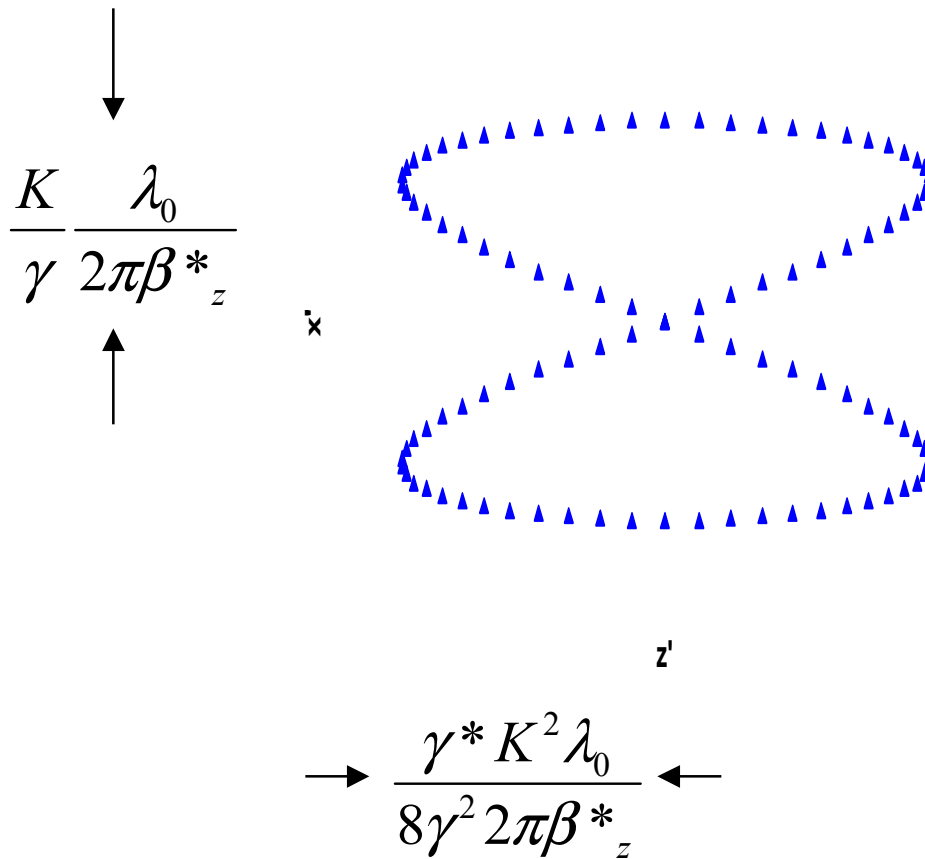
with corresponding

$$\gamma^* = \frac{1}{\sqrt{1 - \beta^{*2}_z}} = \frac{\gamma}{\sqrt{1 + K^2/2}}$$

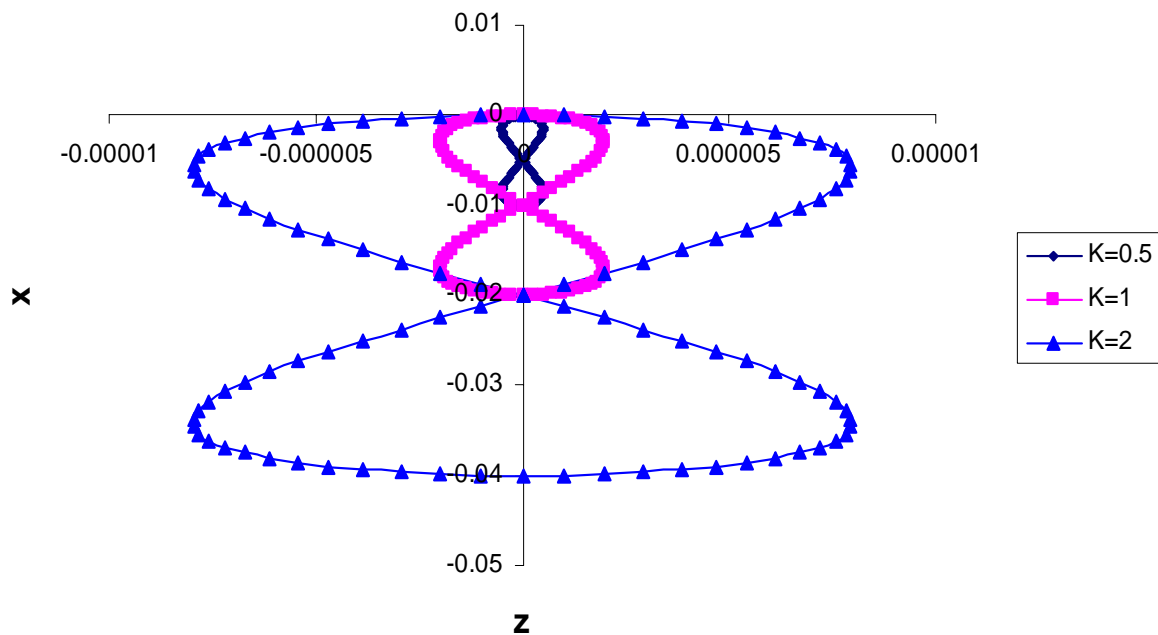
To apply dipole distributions, must be in this frame to begin with



Figure Eight



"Figure Eight" Orbits



$\gamma = 100$, distances are normalized by $\lambda_0 / 2\pi$



$$\frac{dE_{\sigma,n}}{d\omega d\Omega} = \frac{e^2}{2c} [S_{1n} + S_{2n}/n]^2 \frac{\sin^2 \phi}{\sin^2 \theta \cos^2 \phi} f_{nN}^2(\omega; n\omega(\theta))$$

$$\frac{dE_{\pi,n}}{d\omega d\Omega} = \frac{e^2}{2c} \left[\frac{S_{1n}(\cos \theta - \beta_z^*)}{(1 - \beta_z^* \cos \theta) \sin \theta} + \frac{S_{2n}}{n \sin \theta \cos \theta} \right]^2 f_{nN}^2(\omega; n\omega(\theta))$$

f_{nN} is highly peaked, with peak value nN , around angular frequency

$$n\omega(\theta) = \frac{\beta_z^* n\omega_0}{(1 - \beta_z^* \cos \theta)} \rightarrow 2\gamma^{*2} \beta_z^* n\omega_0 \approx \frac{2\gamma^2}{1 + K^2/2} n\omega_0 \text{ as } \theta \rightarrow 0$$



Energy Distribution in Lab Frame

$$\frac{dE_{\sigma,n}}{d\omega d\Omega} = \frac{e^2}{2c} [S_{1n} + S_{2n}/n]^2 \frac{\sin^2 \phi}{\sin^2 \theta \cos^2 \phi} f_{nN}^2(\omega; n\omega(\theta))$$

$$\frac{dE_{\pi,n}}{d\omega d\Omega} = \frac{e^2}{2c} \left[\frac{S_{1n}(\cos \theta - \beta_z^*)}{(1 - \beta_z^* \cos \theta) \sin \theta} + \frac{S_{2n}}{n \sin \theta \cos \theta} \right]^2 f_{nN}^2(\omega; n\omega(\theta))$$

The arguments of the Bessel Functions are now

$$\xi_x \equiv n \sin \theta' \cos \phi' d_x \omega'_0 / c = n \frac{\sin \theta \cos \phi}{(1 - \beta_z^* \cos \theta)} \frac{K}{\gamma}$$

$$\xi_z \equiv n(\beta_z^* + \cos \theta') d_z \omega'_0 / c = n \frac{\cos \theta}{(1 - \beta_z^* \cos \theta)} \frac{\beta_z^* K^2}{8\gamma^2 \beta^2}$$



In the Forward Direction

In the forward direction even harmonics vanish ($n+2k'$ term vanishes when “ x ” Bessel function non-zero at zero argument, and all other terms in sum vanish with a power higher than 2 as the argument goes to zero), and for odd harmonics only $n+2k'=1,-1$ contribute to the sum

$$\frac{dE_{\sigma,n}}{d\omega d\Omega} = \frac{e^2}{2c} \gamma^2 \left(\frac{F_n(K)}{n^2} \right) \sin^2 \phi f_{nN}^2(\omega; n\omega(\theta=0))$$

$$\frac{dE_{\pi,n}}{d\omega d\Omega} = \frac{e^2}{2c} \gamma^2 \left(\frac{F_n(K)}{n^2} \right) \cos^2 \phi f_{nN}^2(\omega; n\omega(\theta=0))$$

$$F_n(K) \approx \frac{1}{\gamma^2} \frac{n^2}{4(1-\beta_z^*)^2} \frac{K^2}{\gamma^2} \left[J_{\frac{n-1}{2}} \left(\frac{nK^2}{4(1+K^2/2)} \right) - J_{\frac{n+1}{2}} \left(\frac{nK^2}{4(1+K^2/2)} \right) \right]^2$$



Summary

- Coisson's Theory may be generalized to arbitrary observation angles by using the proper polarization decomposition
- Emission (in forward direction) is at ODD harmonics of the fundamental frequency, in addition to the fundamental frequency emission. The strength of the emission at the harmonics is quantified by a Bessel function factor.
- All kinematic parameters, including the angular distribution functions and frequency distributions, are just the same as before except unstarred quantities should be replaced by starred quantities
- In particular, the (FEL) resonance condition becomes

$$\lambda_n = \frac{n\lambda_0}{2\gamma^2} \left(1 + \frac{K^2}{2} \right)$$



Finite Pulse Thomson Scattering

Generalize the work done so far to cover cases with

1. High field strength lasers

And

2. Finite energy spread from the pulsed photon beam itself

Roughly speaking, the conclusion is that the energy spectra of the scattered photons is increased by a width of order of $1/N$, where N is the number of oscillations the electron makes for weak fields, but is considerably broader for strong fields.



Electron in a Plane Wave

Assume linearly-polarized pulsed laser beam moving in the direction (electron charge is $-e$)

$$\vec{n}_{inc} = \sin \Phi \hat{y} + \cos \Phi \hat{z}$$

$$\vec{A}_{inc}(\vec{x}, t) = A_x (ct - \sin \Phi y - \cos \Phi z) \hat{x} \equiv A(\xi) \hat{x}$$

Polarization 4-vector

$$\varepsilon^\mu = (0, 1, 0, 0)$$

Light-like incident propagation 4-vector

$$n_{inc}^\mu = (1, 0, \sin \Phi, \cos \Phi)$$

$$\varepsilon \cdot n_{inc} = \varepsilon_\mu n_{inc}^\mu = \vec{\varepsilon} \cdot \vec{n}_{inc} = 0$$



Electromagnetic Field

$$\begin{aligned} F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu = \varepsilon^\nu \frac{\partial A}{\partial x_\mu} - \varepsilon^\mu \frac{\partial A}{\partial x_\nu} \\ &= \left(\varepsilon^\nu n_{inc}^\mu - \varepsilon^\mu n_{inc}^\nu \right) \frac{dA}{d\xi}(\xi) \end{aligned}$$

Our goal is to find $x^\mu(\tau) = (ct(\tau), x(\tau), y(\tau), z(\tau))$ when the 4-velocity $u^\mu(\tau) = (cdt/d\tau, dx/d\tau, dy/d\tau, dz/d\tau)(\tau)$ satisfies $du^\mu/d\tau = -eF^{\mu\nu}u_\nu/mc$ where τ is proper time. For any solution to the equations of motion.

$$\frac{d(n_{inc\mu} u^\mu)}{d\tau} = n_{inc\mu} F^{\mu\nu} u_\nu = 0 \quad \therefore n_{inc\mu} u^\mu = n_{inc\mu} u^\mu(-\infty)$$

Proportional to amount frequencies up-shifted going to beam frame



ξ is exactly proportional to the proper time!

On the orbit

$$\xi(\tau) = ct(\tau) - \vec{n}_{inc} \cdot \vec{x}(\tau) \qquad d\xi / d\tau = n_{inc\mu} u^\mu$$

Integrate with respect to ξ instead of τ . Now

$$\frac{d(\epsilon_\mu u^\mu)}{d\tau} = c \frac{df}{d\xi} n_{inc\mu} u^\mu = c \frac{d}{d\tau} f(\xi(\tau))$$

where the unitless vector potential is $f(\xi) = -eA(\xi) / mc^2$.

$$\therefore \epsilon_\mu u^\mu - cf = \epsilon_\mu u^\mu(-\infty)$$



Electron Orbit

$$u^\mu(\xi) = u^\mu(-\infty) + cf(\xi) \left\{ \frac{\mathcal{E}_\nu u^\nu(-\infty)}{n_{inc\nu} u^\nu(-\infty)} n_{inc}^\mu - \mathcal{E}^\mu \right\} + \frac{c^2 f^2(\xi)}{2(n_{inc\nu} u^\nu(-\infty))} n_{inc}^\mu$$

Direct Force from Electric Field

Ponderomotive Force

$$x^\mu(\xi) = \frac{u^\mu(-\infty)\xi}{n_{inc\nu} u^\nu(-\infty)} + \left\{ \frac{c\mathcal{E}_\nu u^\nu(-\infty)}{(n_{inc\nu} u^\nu(-\infty))^2} n_{inc}^\mu - \frac{c\mathcal{E}^\mu}{n_{inc\nu} u^\nu(-\infty)} \right\} \int_{-\infty}^{\xi} f(\xi') d\xi'$$

$$+ \frac{c^2 n_{inc}^\mu}{(n_{inc\nu} u^\nu(-\infty))^2} \int_{-\infty}^{\xi} \frac{f^2(\xi')}{2} d\xi'$$



In Rest Frame of Electron

$$\vec{A}'_{inc}(\vec{x}', t') = A'_x(ct' - (\sin \Phi / \gamma(1 - \beta \cos \Phi))y' - ((\cos \Phi - \beta) / (1 - \beta \cos \Phi))z')\hat{x}' \equiv A'(\xi')\hat{x}'$$

$$ct' = \xi' + \int_{-\infty}^{\xi'} \frac{e^2 A'^2(\xi'')}{2m^2 c^4} d\xi'' \quad y' = \frac{\sin \Phi}{\gamma(1 - \beta \cos \Phi)} \int_{-\infty}^{\xi'} \frac{e^2 A'^2(\xi'')}{2m^2 c^4} d\xi''$$

$$x' = \int_{-\infty}^{\xi'} \frac{eA'(\xi'')}{mc^2} d\xi'' \quad z' = \frac{\cos \Phi - \beta}{(1 - \beta \cos \Phi)} \int_{-\infty}^{\xi'} \frac{e^2 A'^2(\xi'')}{2m^2 c^4} d\xi''$$



Energy Distribution: Beam Frame

$$\frac{dE'_\sigma}{d\omega' d\Omega'} = \frac{e^2 \omega'^2}{8\pi^2 c^3} \left| D'_x(\omega'; \theta', \phi') \sin \phi' - D'_y(\omega'; \theta', \phi') \cos \phi' \right|^2$$

$$\frac{dE'_\pi}{d\omega' d\Omega'} = \frac{e^2 \omega'^2}{8\pi^2 c^3} \left| \begin{aligned} &D'_x(\omega'; \theta', \phi') \cos \theta' \cos \phi' \\ &+ D'_y(\omega'; \theta', \phi') \cos \theta' \sin \phi' \\ &- D'_z(\omega'; \theta', \phi') \sin \theta' \end{aligned} \right|^2$$



Effective Dipole Motions

$$D'_x(\omega'; \theta', \phi') = D'_t(\omega'; \theta', \phi') = \int \frac{eA'(\xi')}{mc^2} e^{i\varphi(\omega', \xi'; \theta', \phi')} d\xi'$$

$$D'_y(\omega'; \theta', \phi') = \frac{\sin \Phi}{\gamma(1 - \beta \cos \Phi)} D'_p(\omega'; \theta', \phi')$$

$$D'_z(\omega'; \theta', \phi') = \frac{\cos \Phi - \beta}{1 - \beta \cos \Phi} D'_p(\omega'; \theta', \phi')$$

$$D'_p(\omega'; \theta', \phi') = \int \frac{e^2 A'^2(\xi')}{2m^2 c^4} e^{i\varphi(\omega', \xi'; \theta', \phi')} d\xi'$$



Energy Distribution: Lab Frame

$$\frac{dE_\sigma}{d\omega d\Omega} = \frac{e^2 \omega^2}{8\pi^2 c^3} \left| \begin{array}{l} D_t(\omega; \theta, \phi) \sin \phi \\ - \frac{\sin \Phi}{\gamma(1 - \beta \cos \Phi)} D_p(\omega; \theta, \phi) \cos \phi \end{array} \right|^2$$

$$\frac{dE_\pi}{d\omega d\Omega} = \frac{e^2 \omega^2}{8\pi^2 c^3} \left| \begin{array}{l} D_t(\omega; \theta, \phi) \frac{\cos \theta - \beta}{1 - \beta \cos \theta} \cos \phi \\ + \frac{\sin \Phi}{\gamma(1 - \beta \cos \Phi)} D_p(\omega; \theta, \phi) \frac{\cos \theta - \beta}{1 - \beta \cos \theta} \sin \phi \\ + \frac{\beta - \cos \Phi}{1 - \beta \cos \Phi} D_p(\omega; \theta, \phi) \frac{\sin \theta}{\gamma(1 - \beta \cos \theta)} \end{array} \right|^2$$



Effective Dipole Motions: Lab Frame

$$D_t(\omega; \theta, \phi) = \frac{1}{\gamma(1 - \beta \cos \Phi)} \int \frac{eA(\xi)}{mc^2} e^{i\varphi(\omega, \xi; \theta, \phi)} d\xi$$

$$D_p(\omega; \theta, \phi) = \frac{1}{\gamma(1 - \beta \cos \Phi)} \int \frac{e^2 A^2(\xi)}{2m^2 c^4} e^{i\varphi(\omega, \xi; \theta, \phi)} d\xi$$

And the (Lorentz invariant!) phase is

$$\varphi(\omega, \xi; \theta, \phi) = \frac{\omega}{c} \left(\begin{array}{l} \xi \frac{(1 - \beta \cos \theta)}{(1 - \beta \cos \Phi)} - \frac{\sin \theta \cos \phi}{\gamma(1 - \beta \cos \Phi)} \int_{-\infty}^{\xi} \frac{eA(\xi')}{mc^2} d\xi' \\ + \frac{1 - \sin \theta \sin \phi \sin \Phi - \cos \theta \cos \Phi}{\gamma^2 (1 - \beta \cos \Phi)^2} \int_{-\infty}^{\xi} \frac{e^2 A^2(\xi')}{2m^2 c^4} d\xi' \end{array} \right)$$



Weak Field Thomson Backscatter

With $\Phi = \pi$ and $f \ll 1$ the result is identical to the weak field undulator result with the replacement of the magnetic field Fourier transform by the electric field Fourier transform

Undulator

Thomson Backscatter

Driving Field $\tilde{B}_y(\omega(1 - \beta_z \cos \theta) / c\beta_z)$ $\tilde{E}_x(\omega(1 - \beta_z \cos \theta) / (c(1 + \beta_z)))$

Forward Frequency $\lambda \approx \frac{\lambda_0}{2\gamma^2}$ $\lambda \approx \frac{\lambda_0}{4\gamma^2}$

Lorentz contract + Doppler

Double Doppler



Summary

- Overall structure of the distributions is very like that from the general dipole motion, only the effective dipole motion, including physical effects such as the relativistic motion of the electrons and retardation, must be generalized beyond the straight Fourier transform of the field
- At low field strengths ($f \ll 1$), the distributions reduce directly to the classic Fourier transform dipole distributions
- The effective dipole motion from the ponderomotive force involves a simple projection of the incident wave vector in the beam frame onto the axis of interest, times the general ponderomotive dipole motion integral
- The radiation from the two transverse dipole motions are compressed by the same angular factors going from beam to lab frame as appears in the simple dipole case. The longitudinal dipole radiation is also transformed between beam and lab frame by the same factor as in the simple longitudinal dipole motion. Thus the usual compression into a $1/\gamma$ cone applies



High Field Strength Thomson Backscatter

For a flat incident laser pulse the main results are very similar to those from undulators with the following correspondences

Undulator

Thomson Backscatter

Field Strength

K

a

Forward
Frequency

$$\lambda \approx \frac{\lambda_0}{2\gamma^2} \left(1 + \frac{K^2}{2} \right)$$

$$\lambda \approx \frac{\lambda_0}{4\gamma^2} \left(1 + \frac{a^2}{2} \right)$$

Transverse Pattern

$$\beta^*_z + \cos \theta'$$

$$1 + \cos \theta'$$

NB, be careful with the radiation pattern, it is the same at small angles, but quite a bit different at large angles



Realistic Pulse Distribution at High a

In general, it's easiest to just numerically integrate the lab-frame expression for the spectrum in terms of D_x , D_y , and D_z . A 10^5 to 10^6 point Simpson integration is adequate for most purposes. We've done two types of pulses, flat pulses to reproduce the previous results and to evaluate numerical error, and Gaussian Laser pulses.

One may utilize a two-timing approximation (i.e., the laser pulse is a slowly varying sinusoid with amplitude $a(\xi)$), and the fundamental expressions, to write the energy distribution at any angle in terms of Bessel function expansions and a ξ integral over the modulation amplitude. This approach actually has a limited domain of applicability ($K, a < 0.1$)

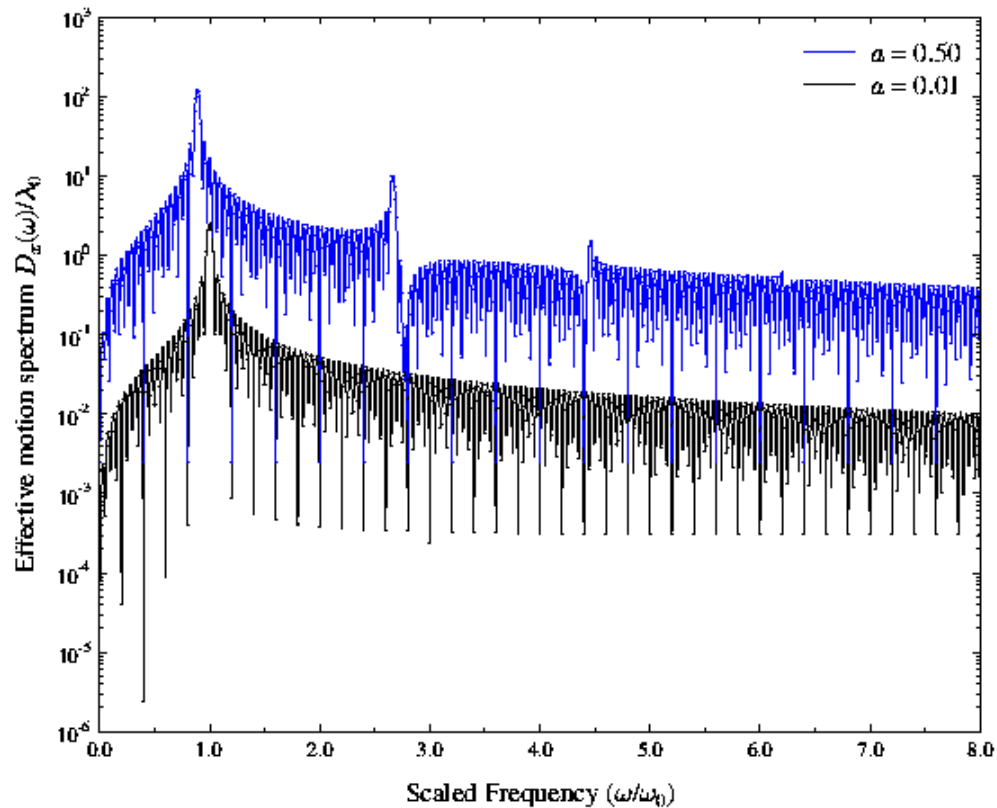


Forward Direction: Flat, Undulator-like Pulse

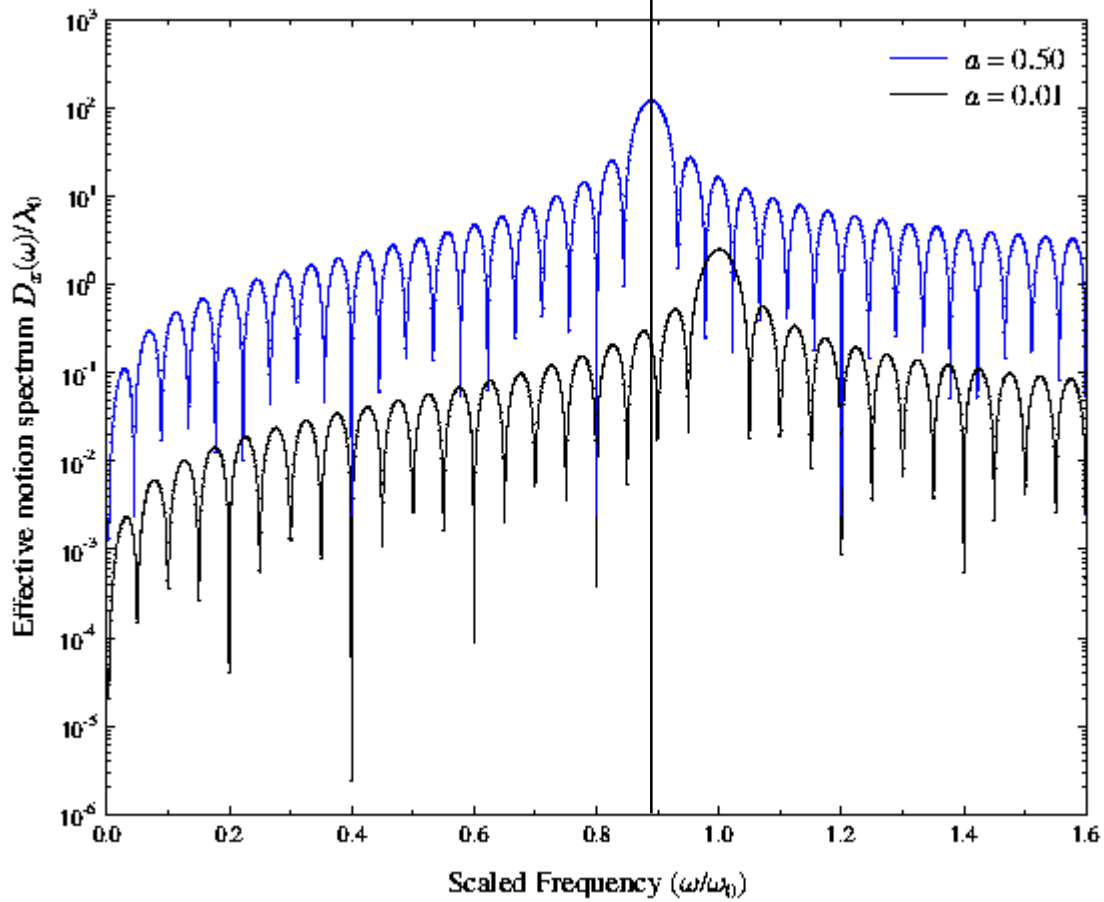
20-period

equivalent undulator: $A_x(\xi) = A_0 \cos(2\pi\xi / \lambda_0) [\Theta(\xi) - \Theta(\xi - 20\lambda_0)]$

$\omega_0 \equiv (1 + \beta_z)^2 \gamma^2 2\pi c / \lambda_0 \approx 4\gamma^2 2\pi c / \lambda_0$, $a = eA_0 / mc^2$



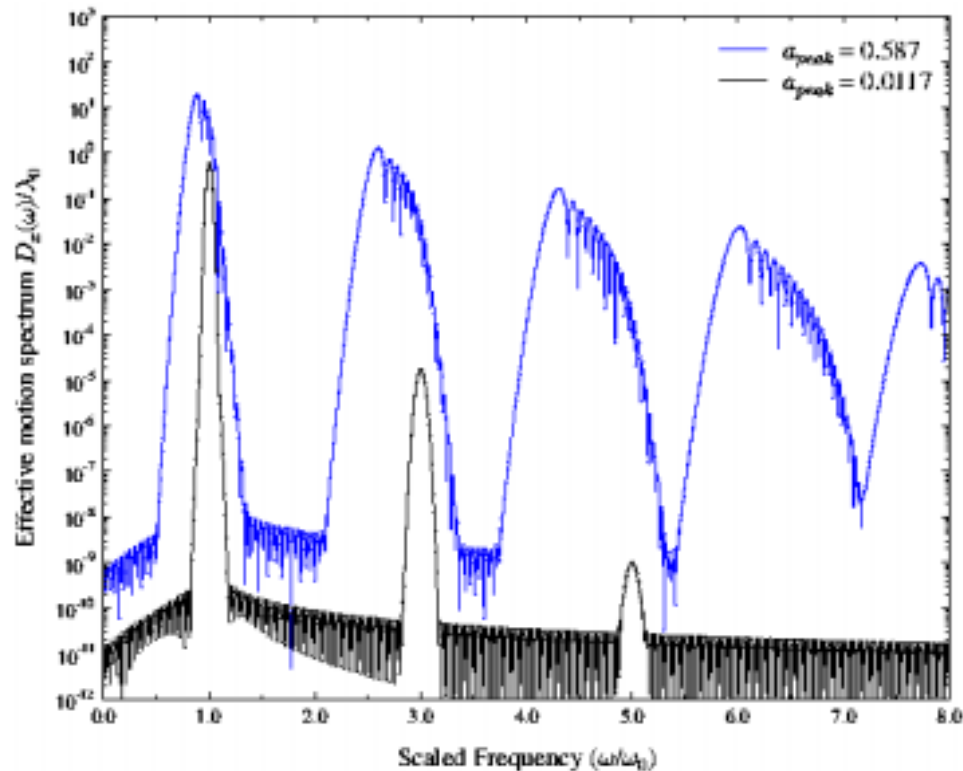
$$1/(1 + a^2 / 2)$$



Forward Direction: Gaussian Pulse

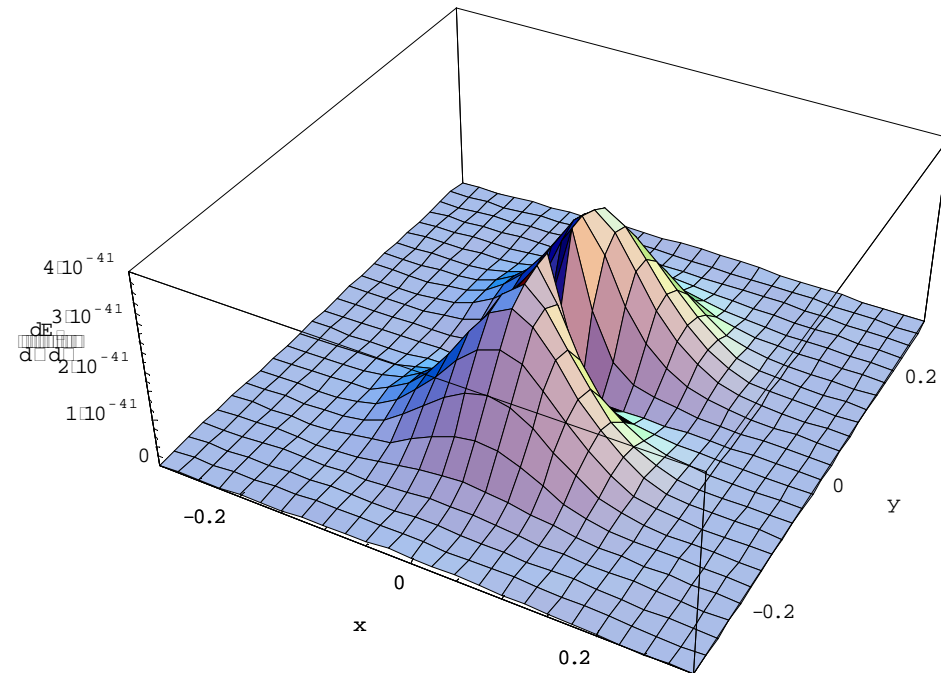
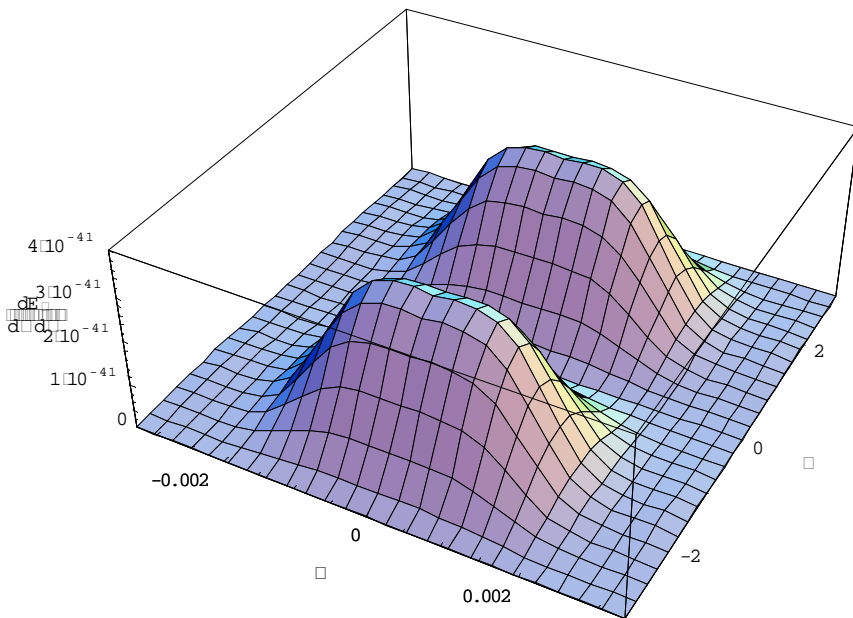
$$A_x(\xi) = A_{peak} \exp\left(-z^2 / 2(8.156\lambda_0)^2\right) \cos(2\pi\xi / \lambda_0) \quad a_{peak} = eA_{peak} / mc^2$$

A_{peak} and λ_0 chosen for same intensity and same *rms* pulse length as previous slide



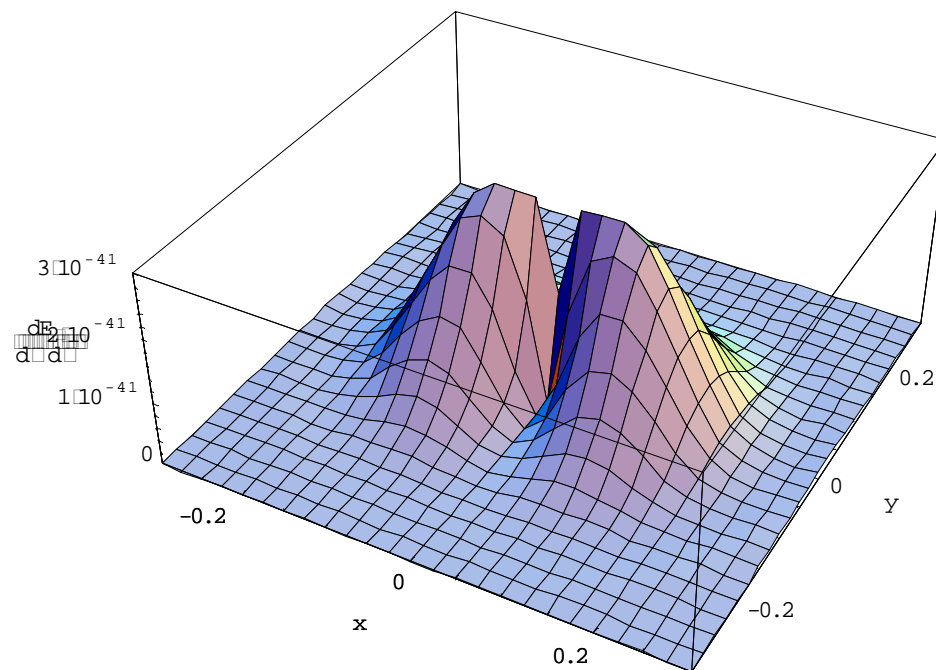
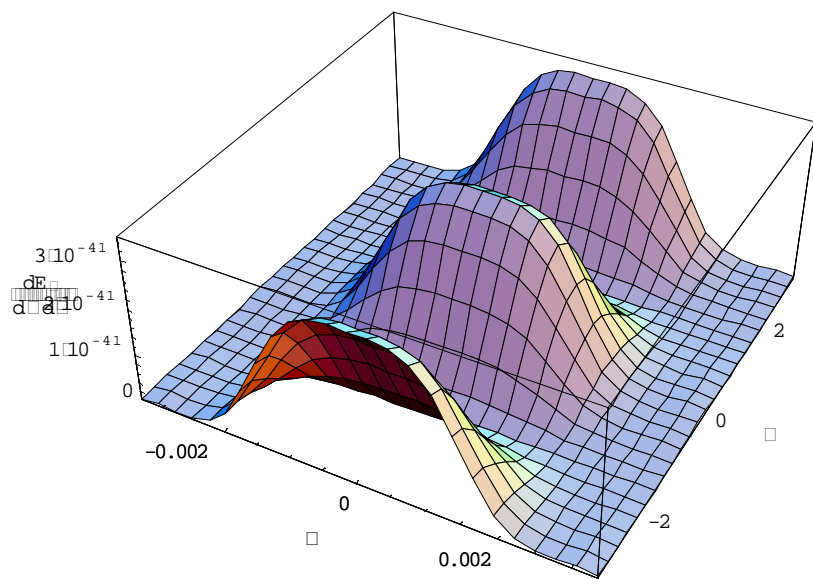
Radiation Distributions: Backscatter

Flat Pulse σ at first harmonic peak



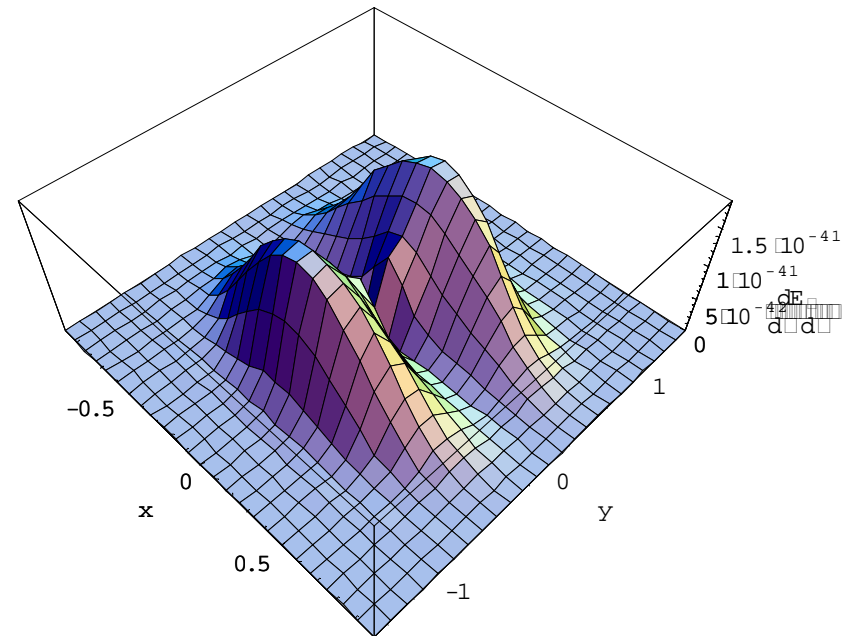
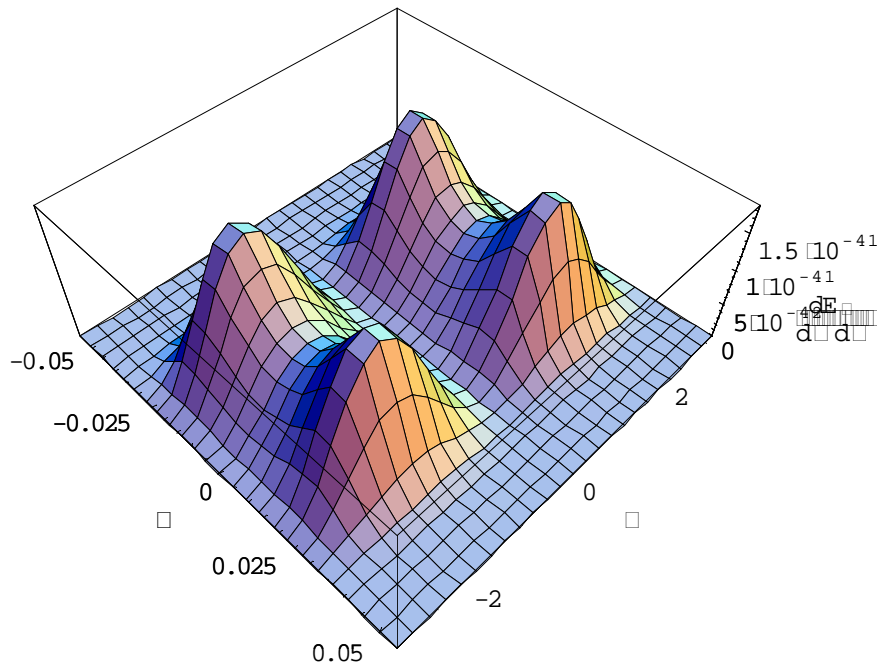
Radiation Distributions: Backscatter

Flat Pulse π at first harmonic peak



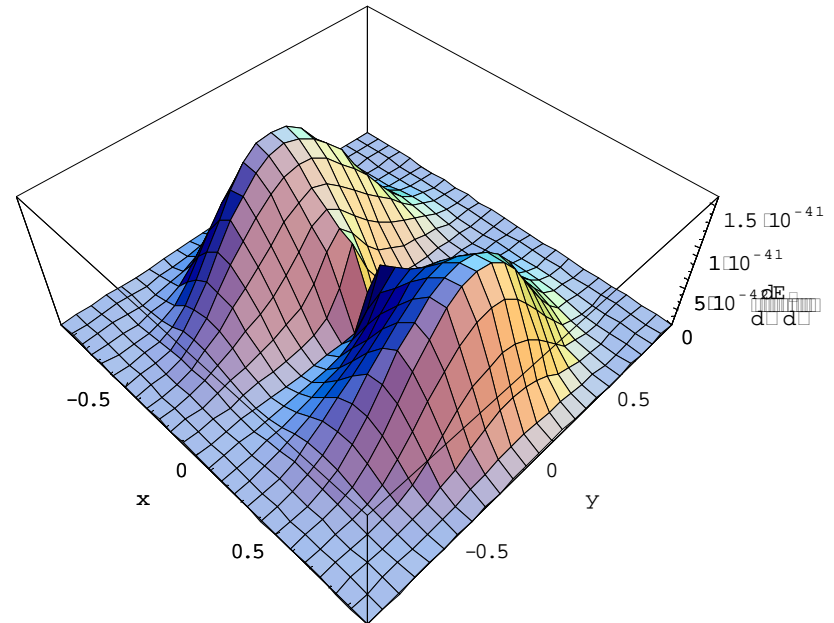
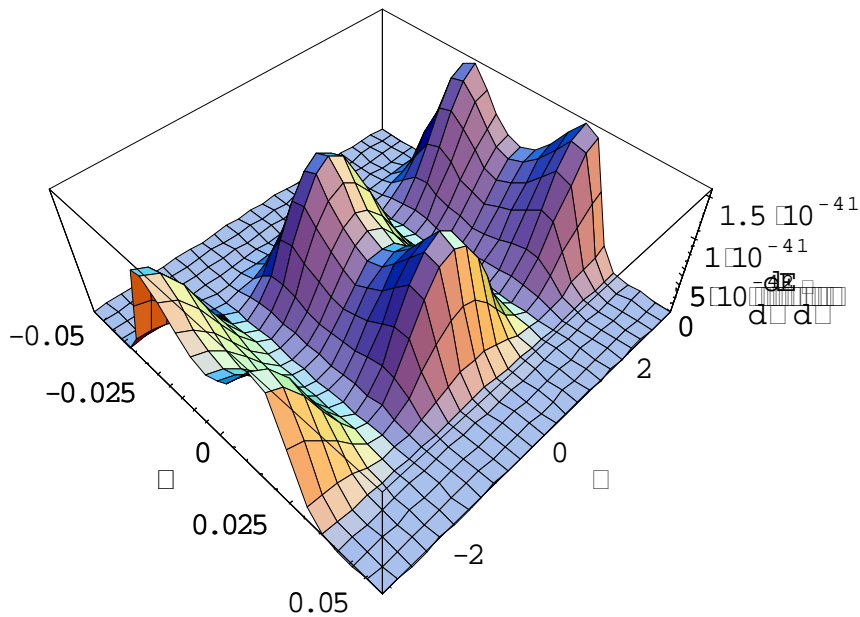
Radiation Distributions: Backscatter

Gaussian Pulse σ at first harmonic peak



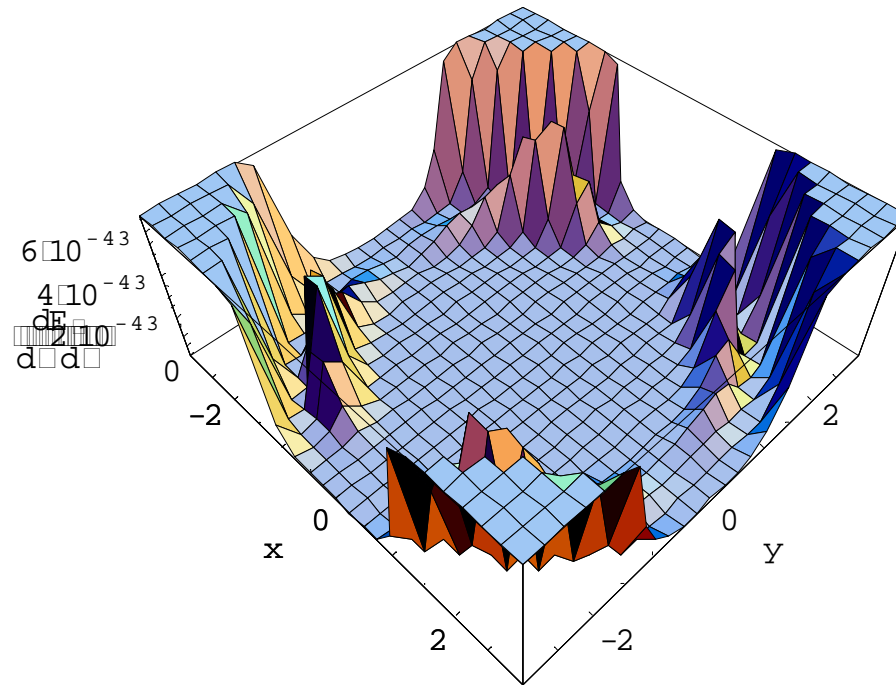
Radiation Distributions: Backscatter

Gaussian π at first harmonic peak



Radiation Distributions: Backscatter

Gaussian σ at second harmonic peak



90 Degree Scattering

$$\frac{dE_{\sigma}}{d\omega d\Omega} = \frac{e^2 \omega^2}{8\pi^2 c^3} \left| D_t(\omega; \theta, \phi) \sin \phi - \frac{1}{\gamma} D_p(\omega; \theta, \phi) \cos \phi \right|^2$$

$$\frac{dE_{\pi}}{d\omega d\Omega} = \frac{e^2 \omega^2}{8\pi^2 c^3} \left| \begin{aligned} & D_t(\omega; \theta, \phi) \frac{\cos \theta - \beta}{1 - \beta \cos \theta} \cos \phi \\ & + \frac{1}{\gamma} D_p(\omega; \theta, \phi) \frac{\cos \theta - \beta}{1 - \beta \cos \theta} \sin \phi \\ & + D_p(\omega; \theta, \phi) \frac{\beta \sin \theta}{\gamma(1 - \beta \cos \theta)} \end{aligned} \right|^2$$



90 Degree Scattering

$$D_t(\omega; \theta, \phi) = \frac{1}{\gamma} \int \frac{eA(\xi)}{mc^2} e^{i\varphi(\omega, \xi; \theta, \phi)} d\xi$$

$$D_p(\omega; \theta, \phi) = \frac{1}{\gamma} \int \frac{e^2 A^2(\xi)}{2m^2 c^4} e^{i\varphi(\omega, \xi; \theta, \phi)} d\xi$$

And the phase is

$$\varphi(\omega, \xi; \theta, \phi) = \frac{\omega}{c} \left(\begin{aligned} &\xi(1 - \beta \cos \theta) - \frac{\sin \theta \cos \phi}{\gamma} \int_{-\infty}^{\xi} \frac{eA(\xi')}{mc^2} d\xi' \\ &+ \frac{1 - \sin \theta \sin \phi}{\gamma^2} \int_{-\infty}^{\xi} \frac{e^2 A^2(\xi')}{2m^2 c^4} d\xi' \end{aligned} \right)$$



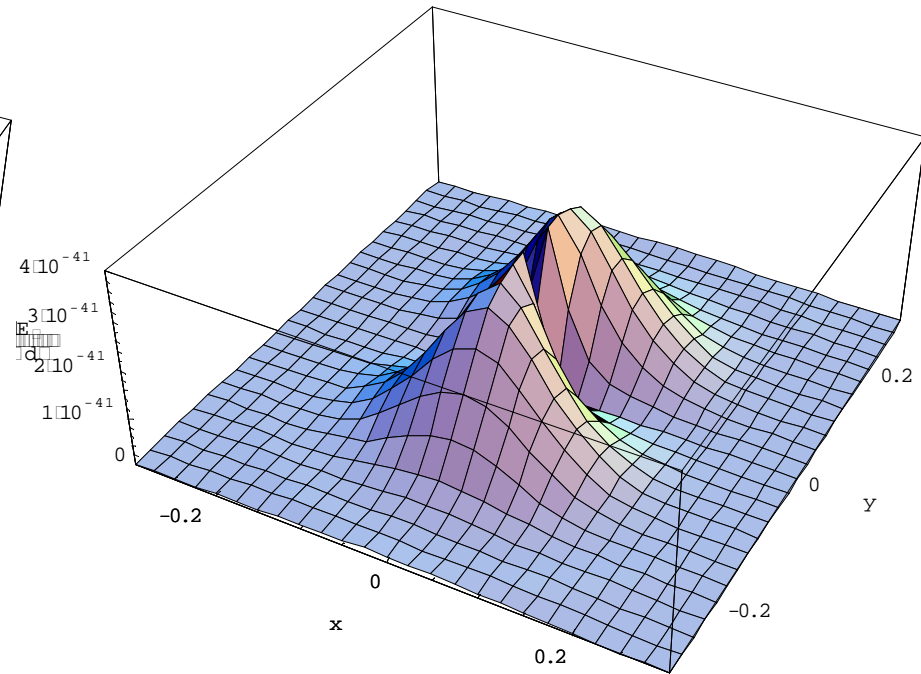
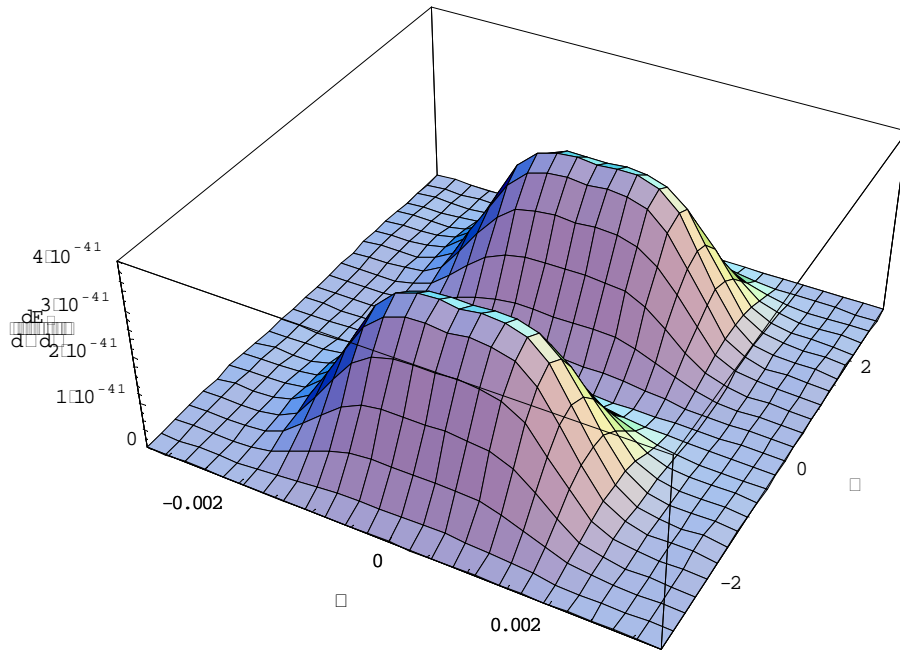
For Flat Pulse

$$\omega(\theta, \phi) = \frac{2\pi c / \lambda_0}{1 - \beta \cos \theta + (a^2 / 4\gamma^2)(1 - \sin \theta \sin \phi)}$$



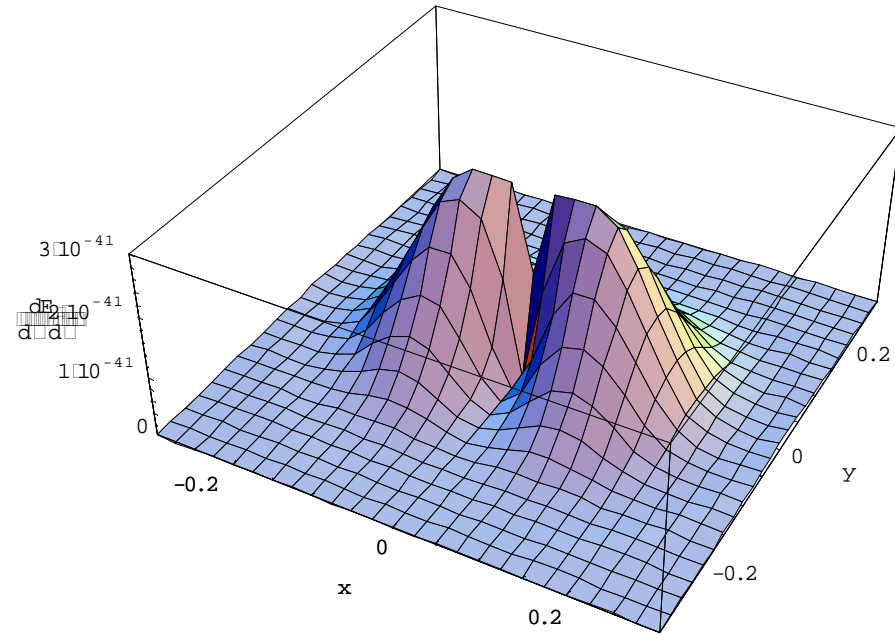
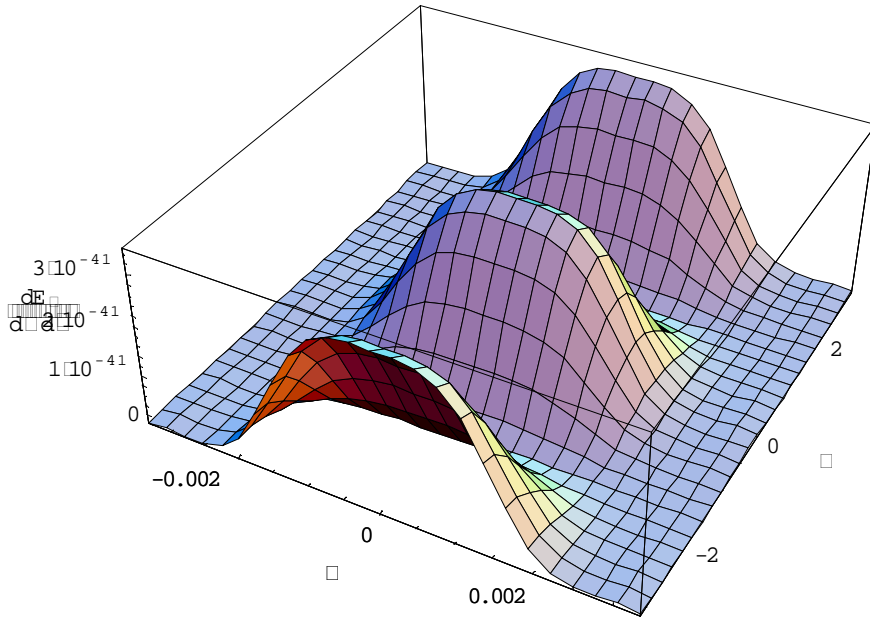
Radiation Distribution: 90 Degree

Flat Pulse σ at first harmonic peak



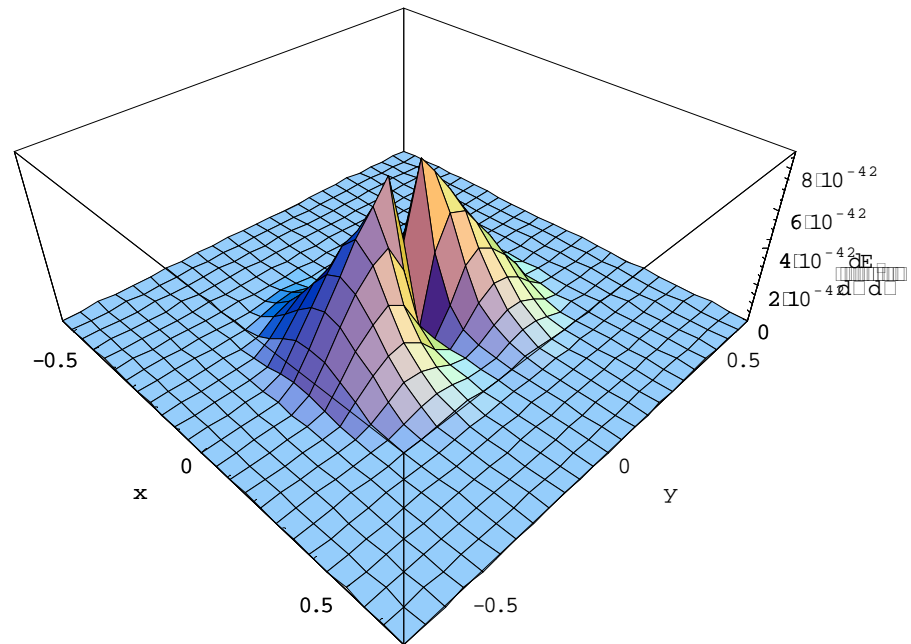
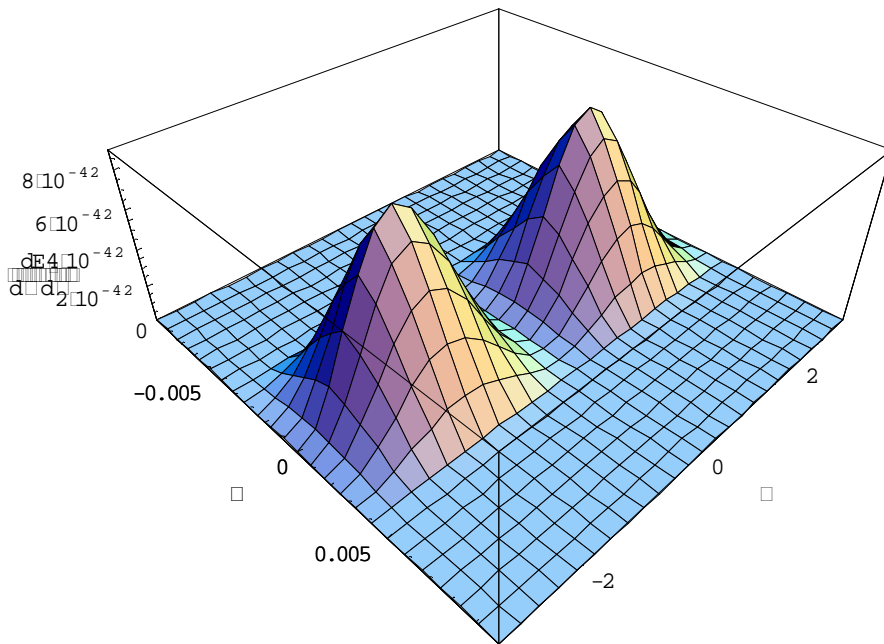
Radiation Distribution: 90 Degree

Flat Pulse π at first harmonic peak



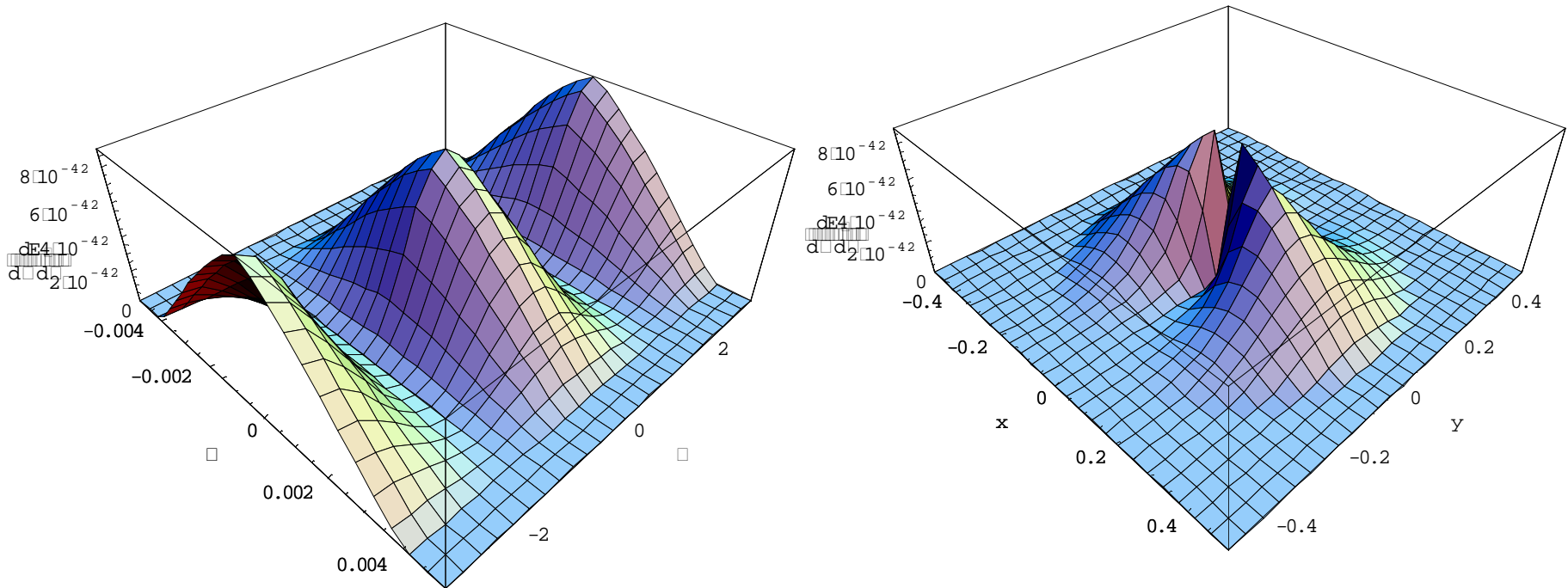
Radiation Distribution: 90 Degree

Gaussian Pulse σ at first harmonic peak

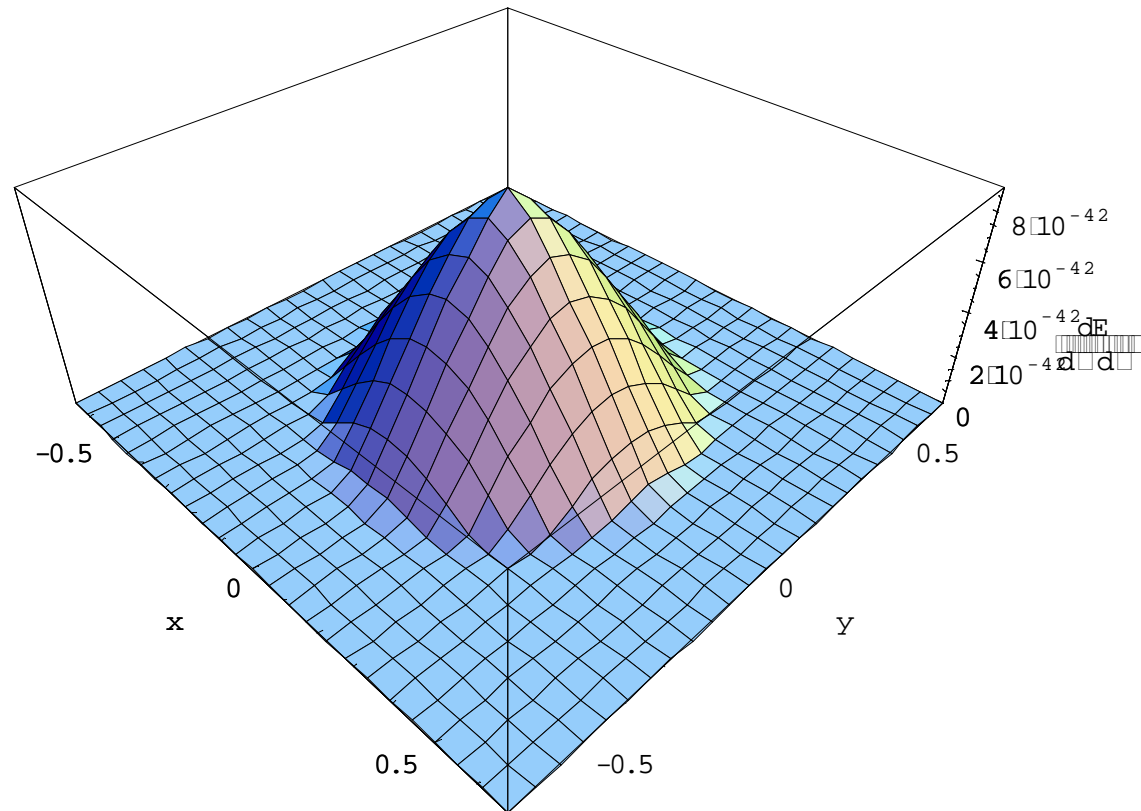


Radiation Distributions: 90 Degree

Gaussian Pulse π at first harmonic peak

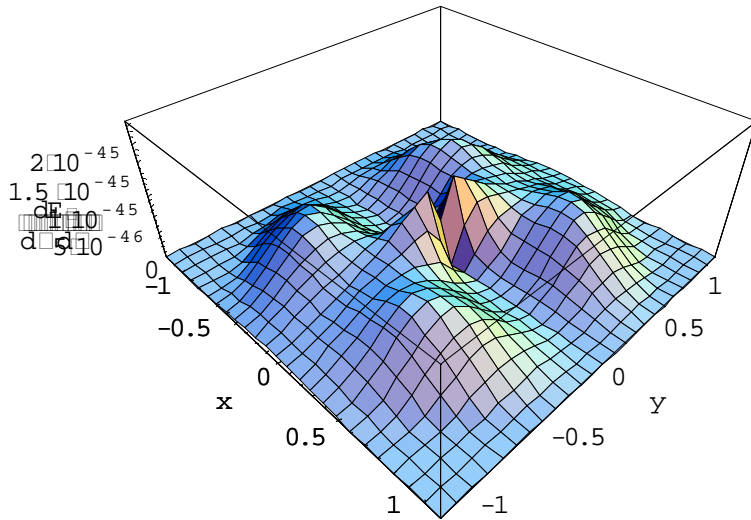


Polarization Sum: Gaussian 90 Degree

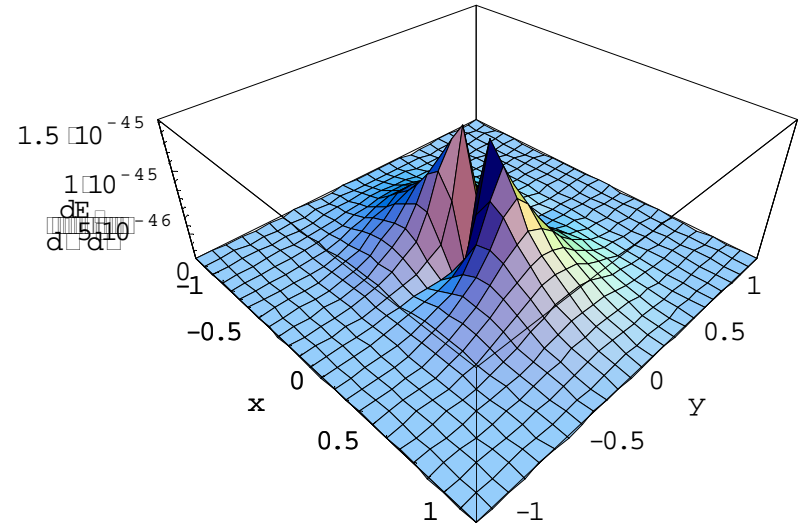


Radiation Distributions: 90 Degree

Gaussian Pulse second harmonic peak



σ

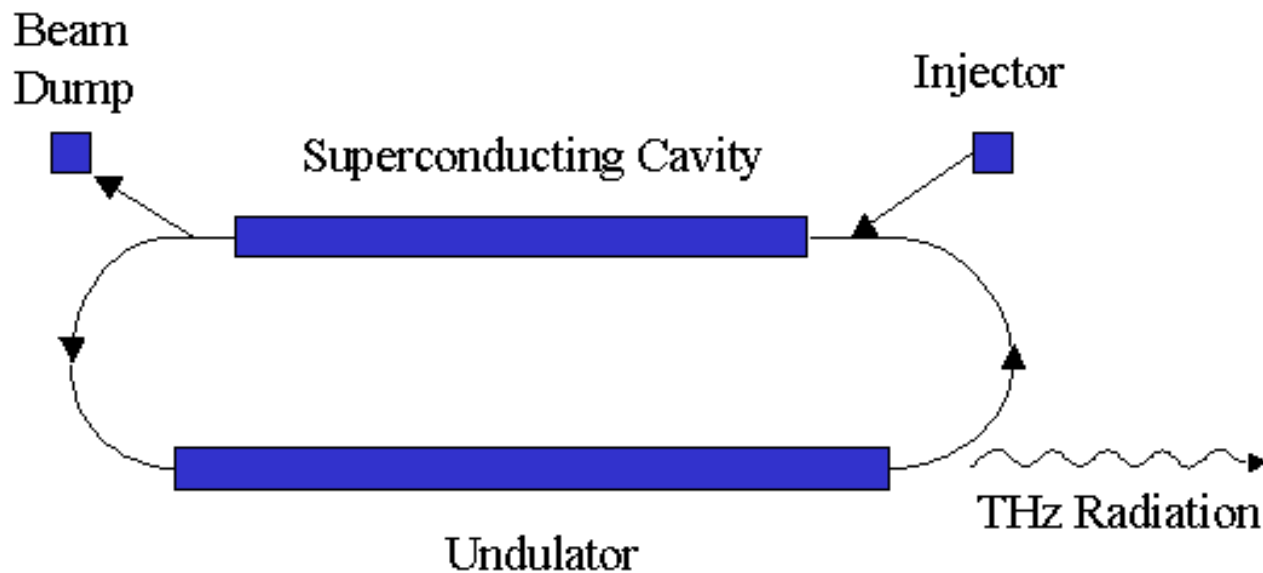


π

Second harmonic emission on axis from ponderomotive dipole!



THz Source



Wideband THz Undulator

Primary requirements: wide bandwidth and no motion and deflection. Implies generate A and B by simple motion. “One half” an oscillation is highest bandwidth!

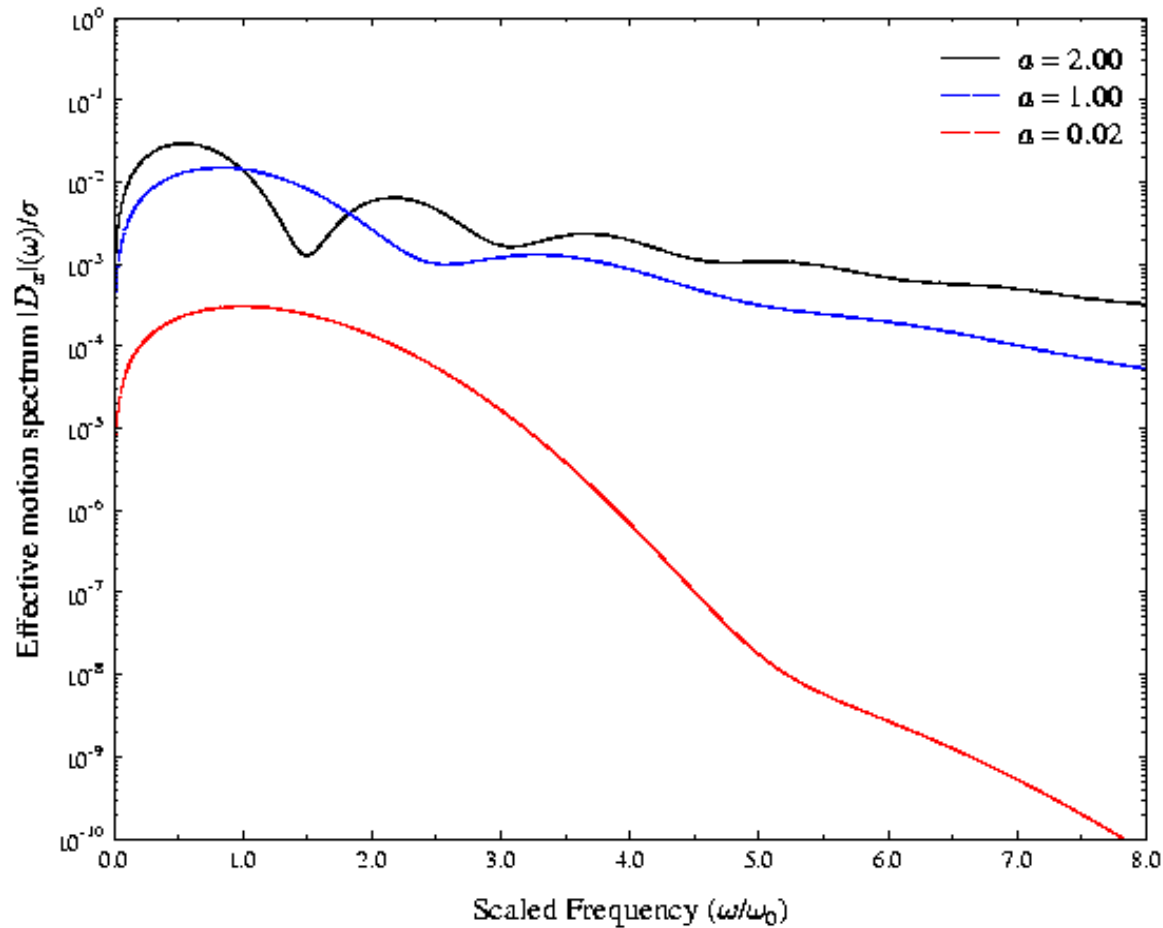
$$x(\xi) = -\sigma \exp(-\xi^2 / 2\sigma^2)$$

$$f(\xi) = \left(\frac{\xi}{\sigma}\right) \exp(-\xi^2 / 2\sigma^2)$$

$$B(\xi) \propto \frac{df}{d\xi} = B_{peak} \left(1 - \left(\frac{\xi}{\sigma}\right)^2\right) \exp(-\xi^2 / 2\sigma^2)$$



THz Undulator Motion Spectrum



Total Energy Radiated

Lienard's Generalization of Larmor Formula (1898!)

$$\frac{dE}{dt} = \frac{2}{3} \frac{e^2}{c} \gamma^6 \left[\left(\dot{\vec{\beta}} \right)^2 + \left(\vec{\beta} \times \dot{\vec{\beta}} \right)^2 \right] = \frac{2}{3} \frac{e^2}{c} \gamma^4 \left[\left(\dot{\vec{\beta}} \right)^2 + \gamma^2 \left(\vec{\beta} \cdot \dot{\vec{\beta}} \right)^2 \right]$$

Barut's Version

$$\frac{dE}{d\tau} = \frac{2}{3} \frac{e^2}{c^3} \frac{dt}{d\tau} \frac{d^2 x^\mu}{d\tau^2} \frac{d^2 x_\mu}{d\tau^2}$$

$$E = \frac{2e^2}{3} \int_{-\infty}^{\infty} \left[\gamma^2 (1 - \beta \cos \Phi) \left(\frac{df}{d\xi} \right)^2 + \frac{f^2}{2} \left(\frac{df}{d\xi} \right)^2 \right] d\xi$$

Usual Larmor term

From ponderomotive
dipole



Some Cases

Total radiation from electron initially at rest

$$E = \frac{2e^2}{3} \int_{-\infty}^{\infty} \left[\left(\frac{df}{d\xi} \right)^2 + \frac{f^2}{2} \left(\frac{df}{d\xi} \right)^2 \right] d\xi$$

For a flat pulse exactly (Sarachik and Schappert)

$$\frac{dE}{dt} = \frac{1}{3} \frac{e^2 \omega^2}{c} a^2 \left(1 + a^2 / 8 \right)$$



Total radiation from electron in the co-moving rest frame for flat laser pulse (Sarachik and Schappert)

$$E' = \frac{2e^2}{3} \int_{-\infty}^{\infty} \left[\left(1 + \frac{\langle f \rangle^2}{2} \right) \left(\frac{df}{d\xi'} \right)^2 + \frac{f^2}{2} \left(\frac{df}{d\xi'} \right)^2 \right] d\xi'$$

$$\frac{dE'}{dt'} = \frac{1}{3} \frac{e^2 \omega'^2}{c} a^2 \left(1 + 3a^2 / 8 \right)$$



Other Flat Pulse Cases

Backscatter

$$\frac{dE}{dt} = \frac{1 + \beta}{3} \frac{e^2 \omega^2}{c} \gamma^2 a^2 \left(1 + a^2 / \gamma^2 8(1 + \beta)\right)$$

90 Degree Scattering

$$\frac{dE}{dt} = \frac{1}{3} \frac{e^2 \omega^2}{c} \gamma^2 a^2 \left(1 + a^2 / \gamma^2 8\right)$$



Undulator

Exact formula for the “1-D” undulator, $f = -eA_x / mc^2$

$$E = \frac{2}{3} e^2 \int_{-\infty}^{\infty} \sqrt{\beta^2 - \frac{f^2}{\gamma^2}} \left[\gamma^2 \left(\frac{df}{dz} \right)^2 + \frac{f^2 (df/dz)^2}{\beta^2 - \frac{f^2}{\gamma^2}} \right] dz$$

For any practical undulator, with $K \ll \gamma$

$$\frac{dE}{dt} \approx \frac{\beta}{3} \frac{e^2 \omega^2}{c} \gamma^2 K^2 \left(1 + K^2 / \beta^2 \gamma^2 8 \right)$$



For Circular Polarization

$$\vec{A}_{inc}(\xi) = A(\xi) \left\{ \cos(2\pi\xi / \lambda) \hat{x} \pm \sin(2\pi\xi / \lambda) [-\cos\Phi \hat{y} + \sin\Phi \hat{z}] \right\}$$

$$E = \frac{2e^2}{3} \int_{-\infty}^{\infty} \left[\gamma(n_{inc\mu} u^\mu(-\infty)) - f_{\pm} \beta \gamma \sin\Phi + \frac{\hat{A}^2}{2} \right]$$

$$\hat{A} = -eA / mc^2 \quad \times \left[\left(\frac{d\hat{A}}{d\xi} \right)^2 + \left(\frac{2\pi}{\lambda} \right)^2 \hat{A}^2 \right] d\xi$$

Only specific case I can find in literature completely calculated has $\sin\Phi = 0$ and flat pulses ($dA/d\xi = 0$). The orbits are then pure circles



For zero average velocity in middle of pulse

$$\gamma \vec{\beta}(-\infty) = -\frac{\vec{n}_{inc}}{n_{incv} u^v(-\infty)} \frac{c \hat{A}^2}{2} \rightarrow \gamma^2 n_{incv} u^v(-\infty) / c = 1 + \frac{\hat{A}^2}{2}$$

Sokolov and Ternov, in *Radiation from Relativistic Electrons*, give

$$\frac{dE'}{dt'} = \frac{2}{3} \frac{e^2 \omega'^2}{c} a^2 (1 + a^2)$$

and the general formula checks out



Conclusions

- An introduction to Thomson Scatter source radiation calculations and a general formula for obtaining the spectral angular energy distribution has been given
- I've shown how dipole solutions to the Maxwell Equations can be used to obtain and understand very general expressions for the spectral angular energy distributions for weak field Insertion Devices and general weak field Thomson Scattering photon sources
- A “new” calculation scheme for high intensity pulsed laser Thomson Scattering has been developed. This same scheme can be applied to calculate spectral properties of “short”, high- K wigglers.
- Due to ponderomotive broadening, it is simply wrong to use single-frequency estimates of flux and brilliance in situations where the square of the field strength parameter becomes comparable to or exceeds the $(1/N)$ spectral width of the induced electron wobble
- The new theory is especially useful when considering Thomson scattering of Table Top TeraWatt lasers, which have exceedingly high field and short pulses. Any calculation that does not include ponderomotive broadening is incorrect.



Conclusions

- Because the laser beam in a Thomson scatter source can interact with the electron beam non-collinearly with the beam motion (a piece of physics that cannot happen in an undulator), ponderomotively driven transverse dipole motion is now possible
- This motion can generate radiation at the second harmonic of the up-shifted incident frequency. The dipole direction is in the direction of laser incidence.
- Because of Doppler shifts generated by the ponderomotive displacement velocity induced in the electron by the intense laser, the frequency of the emitted radiation has an angular asymmetry.
- Sum rules for the total energy radiated, which generalize the usual Larmor/Lenard sum rule, have been obtained.

