Self-consistent Space Charge Distributions: Theory and Applications

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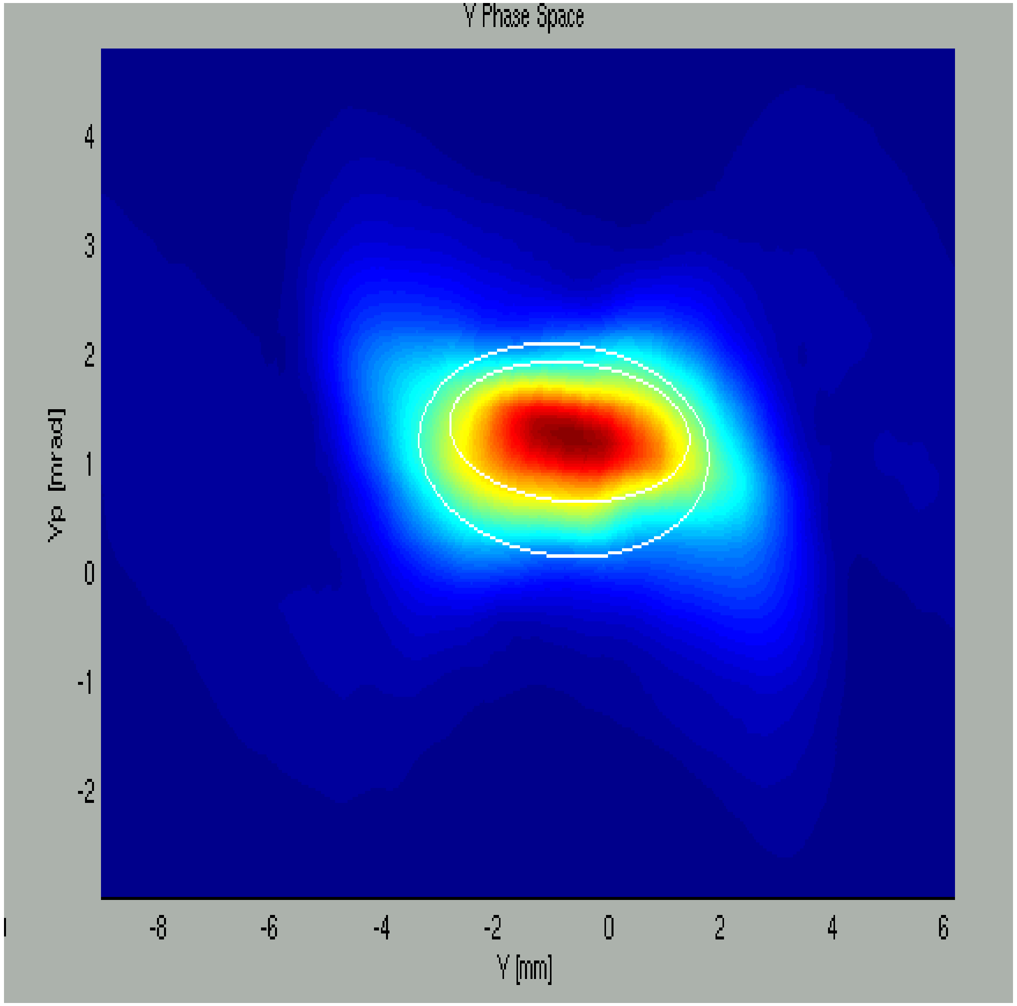
Talk Outline:

i. Problem description: size and halo growth due to space charge effects, dilution, losses.
ii. Self-consistent time dependent distributions as a solution to the problem.
iii. Present status of the problem: injection technique, losses, etc.
iv. Modification of the injection schemes: creation of self-consistent loss-free distributions
v. Acceleration of ultra-small emittance beams

Brief Problem Description

- SNS example: beam distribution after MEBT
  1) S-shape was formed;
  2) Halo strings grew up to 10 rms in y-direction.
- Reasons: tails and core have different frequencies, tails not properly populated. It causes fast dilution of the phase space. Fast dilution or/and core oscillations cause resonances; resonant particles are subject to amp. growth and losses



Self-consistent Space Charge Distributions

1) Self-consistency is a broad term:
   a) time-independent (with taking into account own space charge force);
   b) periodic;
   c) keeping same shape;
   d) under all linear transformations producing elliptical beam with uniform density;
   e) all other simplifications of general motion.

2) *b, c, d* cases relevant to this talk. If we have periodic distribution (with revolution or linac lattice period), and the shape has no (or has small) tails, the distribution produces no loss and preserves rms emittance in the course of accumulation (acceleration)

3) If we knew how to find and create them, it would be a solution to space charge problems.

How Many Analytical Solutions Found?

1) Time-independent up to 3D (Batygin, Gluckstern…).
   Their use is limited, because of the fact that the conventional focusing uses alternating gradient
2) Time-dependent with nonlinear force – none
3) Time-dependent with linear force – up to 2D (Kapchinsky-Vladimirsky distribution)

In this talk new 2D and (more important) 3D self-consistent distributions are presented. They have ellipsoidal shape that is preserved under any linear beam transport

Basic Math of Self-Consistent Distributions

$$\vec{F}(X,Y) = -\nabla U_{sc} = \int_{\Gamma} \frac{2\lambda r_0(\vec{R}-\vec{R}_i)dX_i dY_i}{\gamma_r^3 \beta_r^2((X-X_i)^2+(Y-Y_i)^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f \; dX_i' dY_i'$$

F(X,Y) **is the space charge force,** ***f*** **is the distribution function; 2D case is taken just for example.**

$$H = (p_x^2 + p_y^2)/2m + k(x^2+y^2)/2 + U_{sc}$$

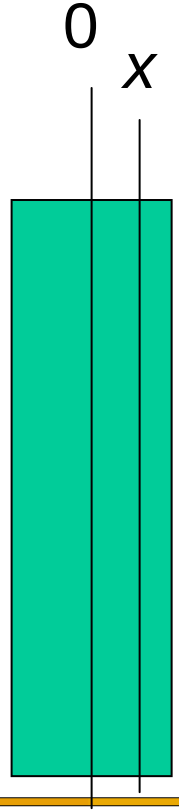
$$f = \Phi(H) \leftrightarrow any \;\; function \; \Phi \;\; of \;\; Hamiltonian$$

$$is \; the \; solution \quad to \; Vlasov \; equation$$

The first example: linear 1D case – the beam density is constant.

$$\int_{-\infty}^{\infty} f(H)\, dp_x = const = \int_{kx^2/2}^{H_b} \frac{f(H)dH}{\sqrt{H-kx^2/2}},$$

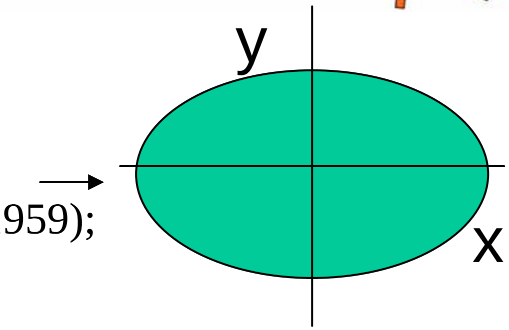
where the distribution function ***f*** **doesn't depend on phase.**
The integral equation is called Abel's Integral Equation.



Math for Self-Consistent Elliptical Distributions

$$1D \int f(H)dp_x = \int_{x^2/2}^{H_b} \frac{f(H)dH}{\sqrt{H - kx^2/2}} = C,\ f(H) = \frac{C}{\pi\sqrt{H_b - H}};$$

$$2D \int f(H)dp_x dp_y = C,\ f(H) = \frac{C}{\pi}\delta(H_b - H) - KV\ distribution\ (1959);$$



$$3D - no\ solutions$$

***Outstanding fact* – any linear transformation of the phase space preserves the elliptical shape. Valid for all-D cases. 1D sample drift transform proof:**

$$x^2 + p_x^2 \xrightarrow{Transf\ x_n = x + p_x} (x_n - p_x)^2 + p_x^2 \equiv x_n^{\ 2}/2 + 2(p_x - x_n/2)^2$$

$$\xrightarrow{Subst\quad x_n = X\sqrt{2}, P_x = (p_x - x_n/2)/\sqrt{2}} X^2 + P_x^2$$

After linear drift transformation there exist substitution of variables such that the density integral in new variables is exactly same. It means constant density again

$$Old - \int f(x^2 + p_x^2)dp_x = const, New - \sqrt{2}\int f(X^2 + P_x^2)dP_x = const$$

Envelope Equations

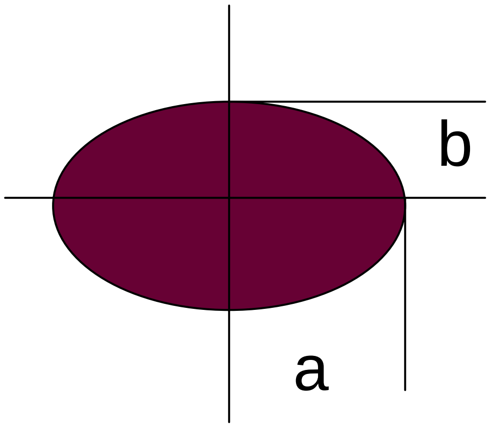
General Result – if distribution depends on quadratic form of coordinates and momenta and initially produces constant density in coordinate space, this density remains constant under all linear transformations.

Linear motion – quadratic invariant- solution to Vlasov equation for constant density as a function of this invariant- linear force-linear motion. We get closed loop of self-consistency. One final step – boundaries of the beam determine the force, force determines the particle dynamics, including dynamics of the boundary particles. Boundaries (or envelopes) must obey dynamic equations. In 2D case :

$$a'' + K_x(s)a = \varepsilon_x / a^3 + \xi /(a+b),$$

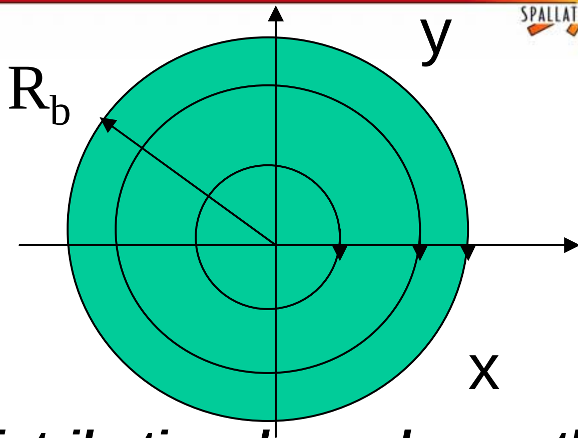
$$b'' + K_y(s)b = \varepsilon_y / b^3 + \xi /(a+b),$$

$$where \;\; \xi = 4r_0\lambda / \beta^2\gamma^3.$$



New Solutions – 2D set

Rotating disk – arrows show the velocities. In all xy, $p_x p_y$, $p_x x$, $p_y y$ projections this figure gives a disk – different topology then one of the KV distribution



The difference with previous cases- the distribution depends on other invariants, not only on Hamiltonian.

$$f = C\delta(X_0 - Y_0')\delta(Y_0 + X_0'),\ R < R_b\ (0\ otherwise)\ \{2,2\}\ case$$

Any linear transformation preserves elliptical shape. The proof: 4D boundary elliptical line remains always elliptical, the projection of elliptical line onto any plane is an ellipse, the density remains constant under any linear transformation

$$\{n,m\}\ case.\ n = \dim.\ Seek\ f\ in\ the\ form\ f \propto g(H)\delta\times\delta \ldots m\ times$$

$$f = \frac{C}{\sqrt{H_b - H}}\delta(X_0' - Y_0);\ \{2,1\}\ case,\ KV -\ \{2,0\}\ case.$$

The principle – delta-functions reduce the dimension in the density integral. Remaining eqn. for g(H) is same as in time-independent case

New Solutions – 3D set

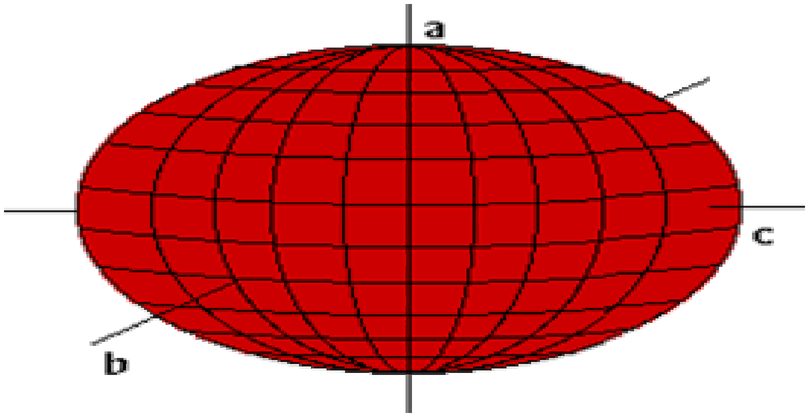
$$f = C\delta(X_0' - aX_0...)\delta(Y_0' + bX_0...)\delta(Z_0' + cX_0...),\ H < H_b (0\ otherwise)\ \{3,3\}\ case;$$

$$f = \frac{C}{\sqrt{H_b - H}}\delta(X_0' - aX_0...)\delta(Y_0' + bX_0...),\ H < H_b (0\ otherwise)\ \{3,2\}\ case;$$

$$f = C\delta(H_b - H)\delta(X_0' - aX_0...),\ H < H_b (0\ otherwise)\ \{3,1\}\ case;$$

$$no\ solution\ \ in\ \ \{3,0\}\ case.$$

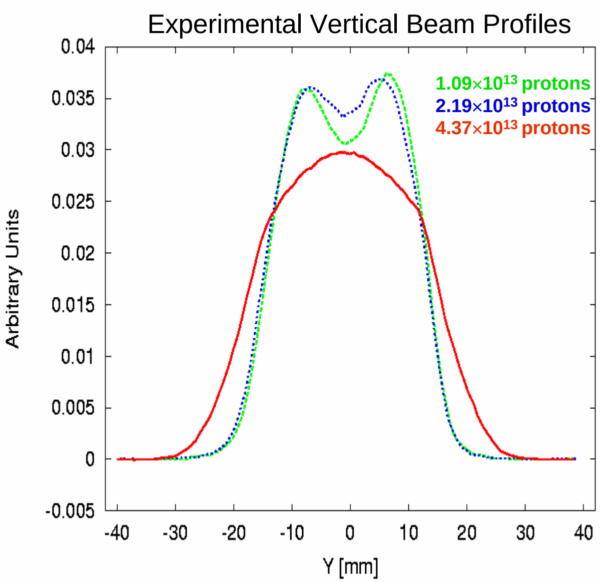
3 new 3D cases found. All have ellipsoidal shape in xyz projection. The density inside is always constant.

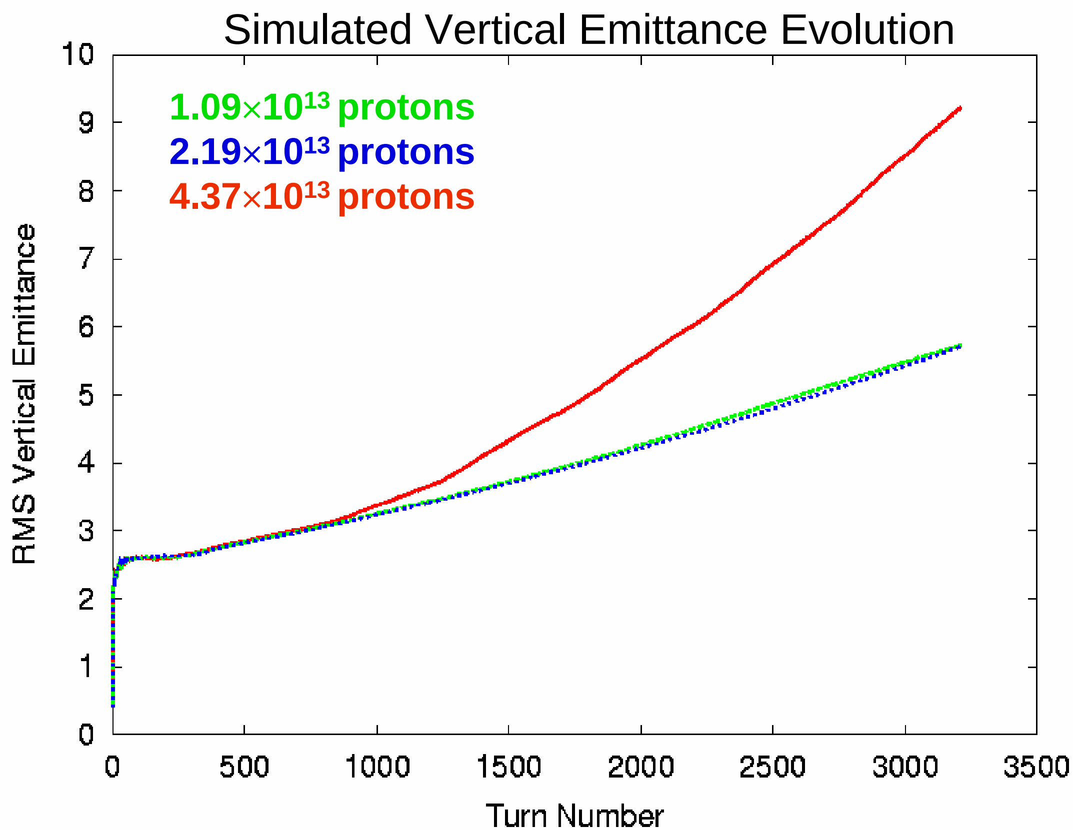


Experimental Picture: PSR case (12-16 slides - courtesy S. Cousineau)

Beam Intensity	1.09×10^{13} - 4.37×10^{13}
Accumulation Time	3214 turns (~1.16 ms)
(ν_x, ν_y)	(3.19, 2.19)

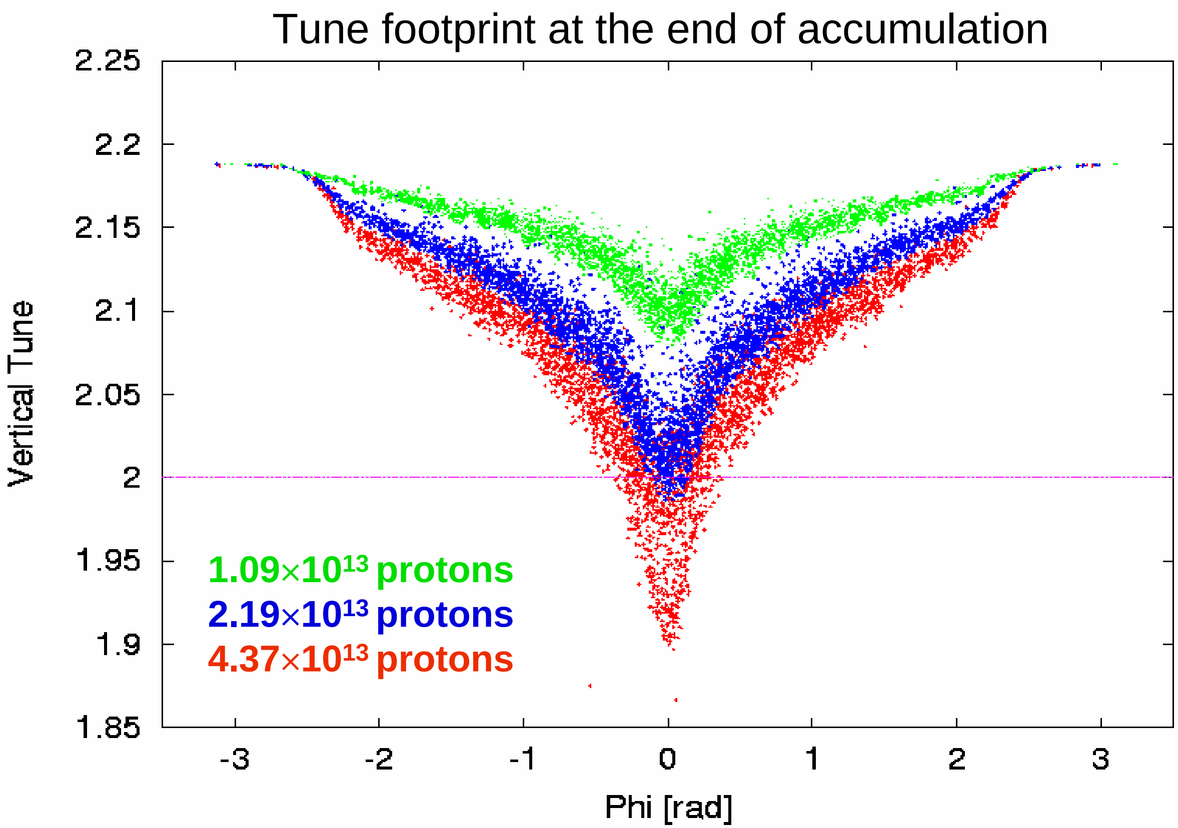
- Some emittance growth always present due to vertical injection painting.
- Space charge induced emittance growth after turn 1500 for highest intensity.



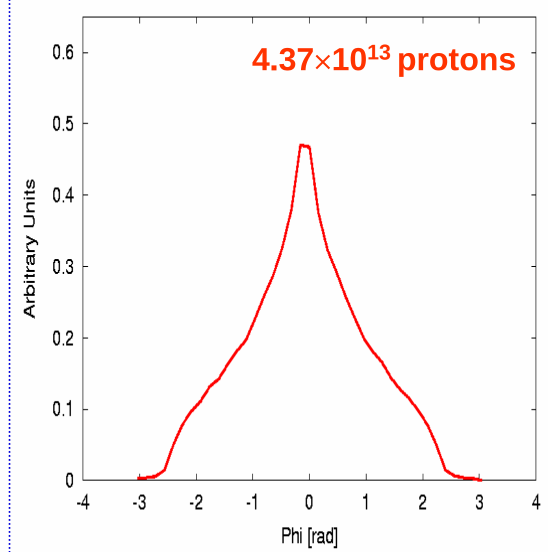


Emittance evolution of PSR beam

- Particles at the center of the beam experience the largest space charge tune depression.
- Single particle tunes reach integer values (2.0) before the onset of emittance growth.



Recall sharply peaked longitudinal density profile...

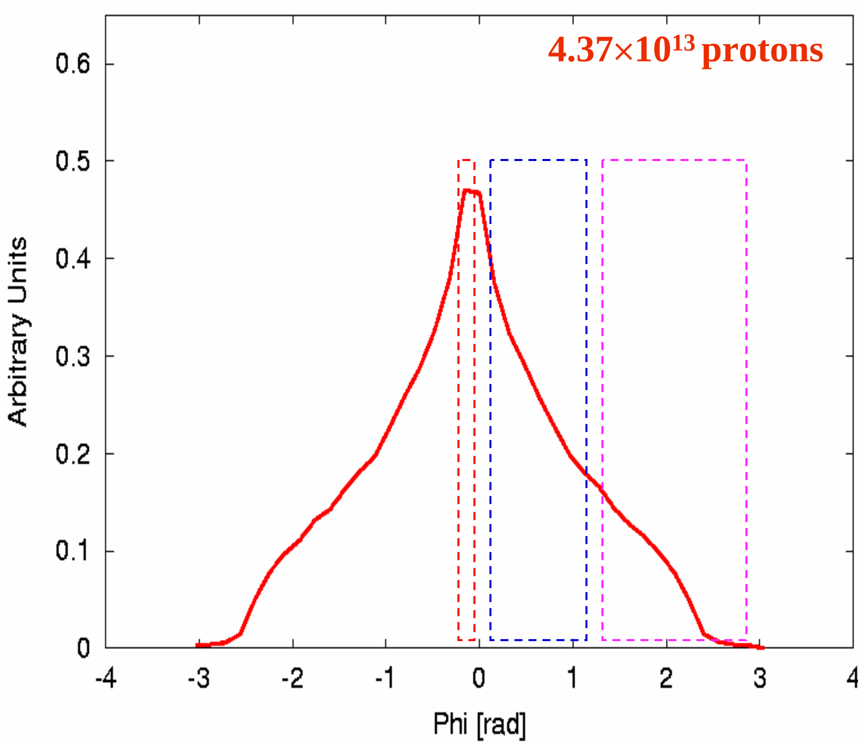


$\Rightarrow$ Need to consider coherent (envelope) motion of beam.

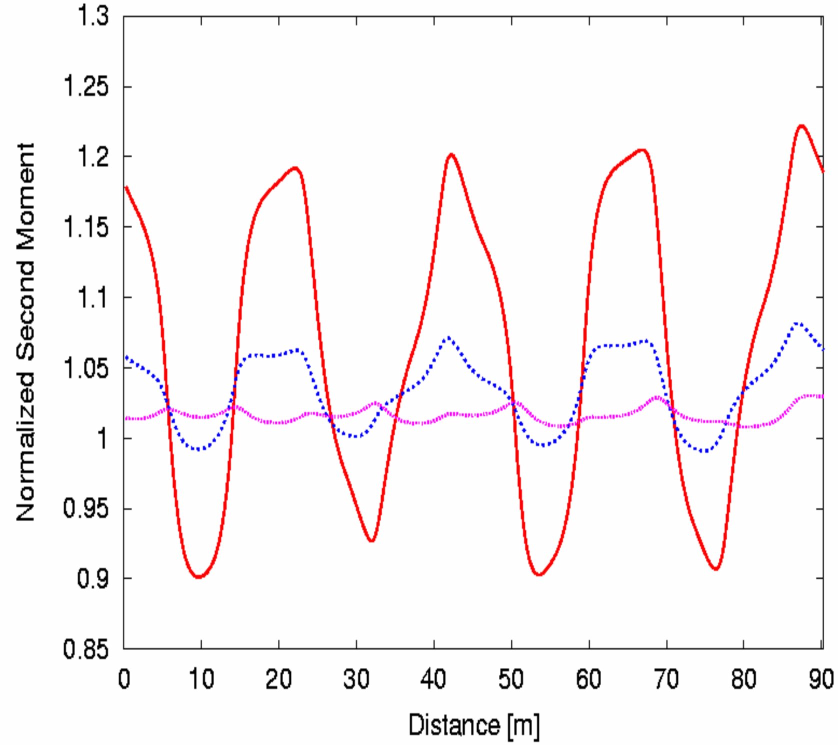
One-turn envelope motion of beam

- Envelope executes 20% oscillations about zero-space-charge envelope $((\beta\varepsilon_{rms})^{1/2})$ in center of long. distribution.
- Oscillations are nearly periodic (almost $\nu_e = 4.0$ per turn of beam).
  $\Rightarrow$ half-integer coherent resonance

Longitudinal density profile



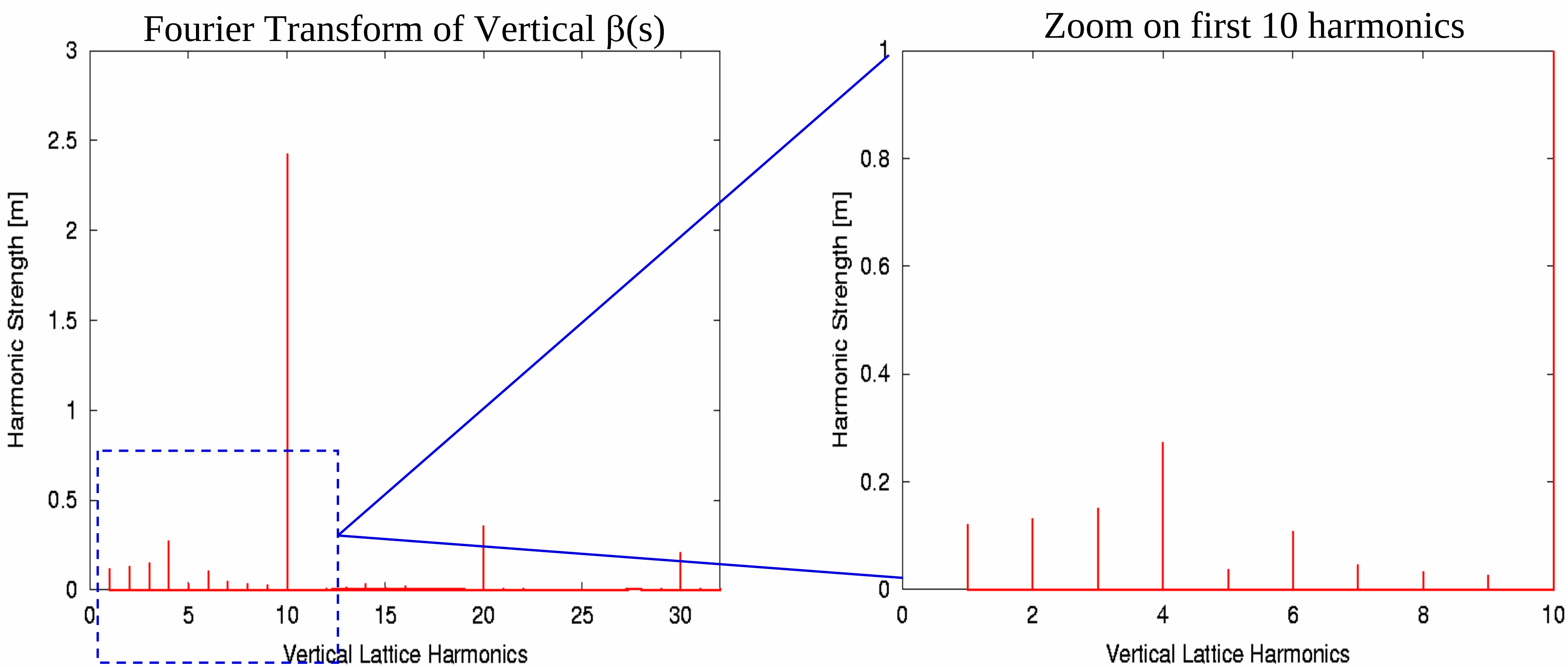
One-turn Envelope (Second-Moment) Evolution



Driving term for the envelope resonance

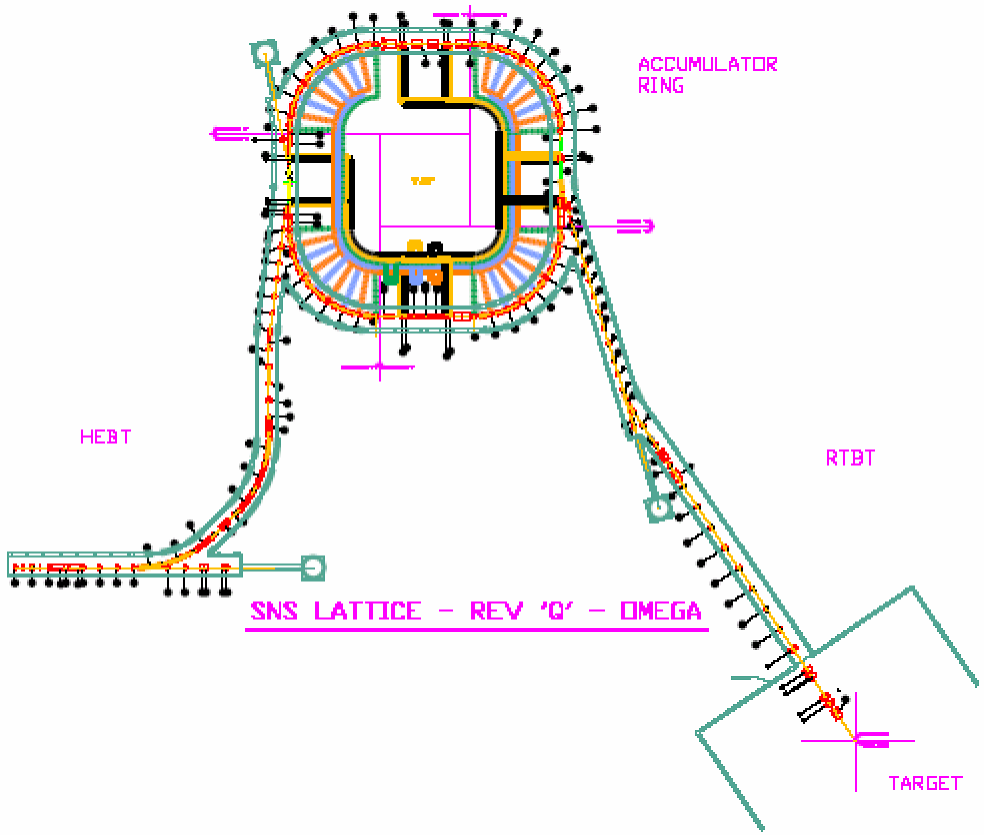
- The coherent resonance ($\nu_e \approx 4.0$) is driven by an n=4 harmonic term.
  - *Where does the driving term come from?*

Try the lattice!



- Besides the structure harmonics of the PSR ring (10, 20, 30...), the n=4 harmonic is the strongest harmonic in the ring!

SNS Ring Example



• The ring has 4-fold symmetry – 12th beta-function structure harmonic is large. The tunes are close to 6, therefore the envelope oscillation is a problem when the depressed tunes approach the integer resonance

Ring SC Distributions – Difference from Linac Case

1) Vacuum chamber shielding – longitudinal force reduced
2) The SC force depends on relative longitudinal coordinates of particles
3) Dispersion changes the transverse size
4) Chromaticity plus energy spread yield betatron frequency spread

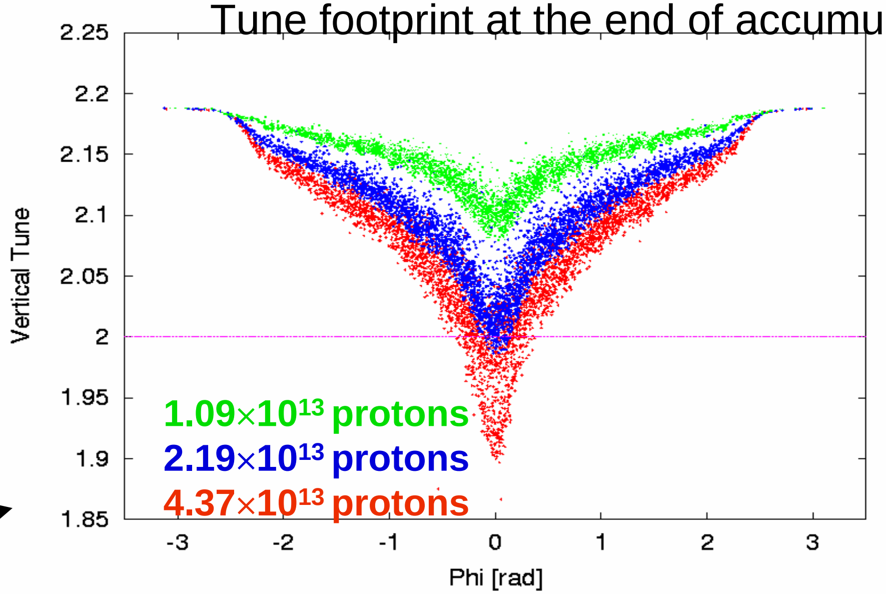
1) $$\Delta E = -\frac{2e^2\Pi}{\gamma^2}\ln(b/a)\lambda'$$

2) $$x_{tot} = x + \eta\frac{\Delta p}{p}$$

3) Chromatic spread can be larger than the SC tuneshift

4)

Tune footprint at the end of accumulation



SNS Injection – How the Distribution is Formed

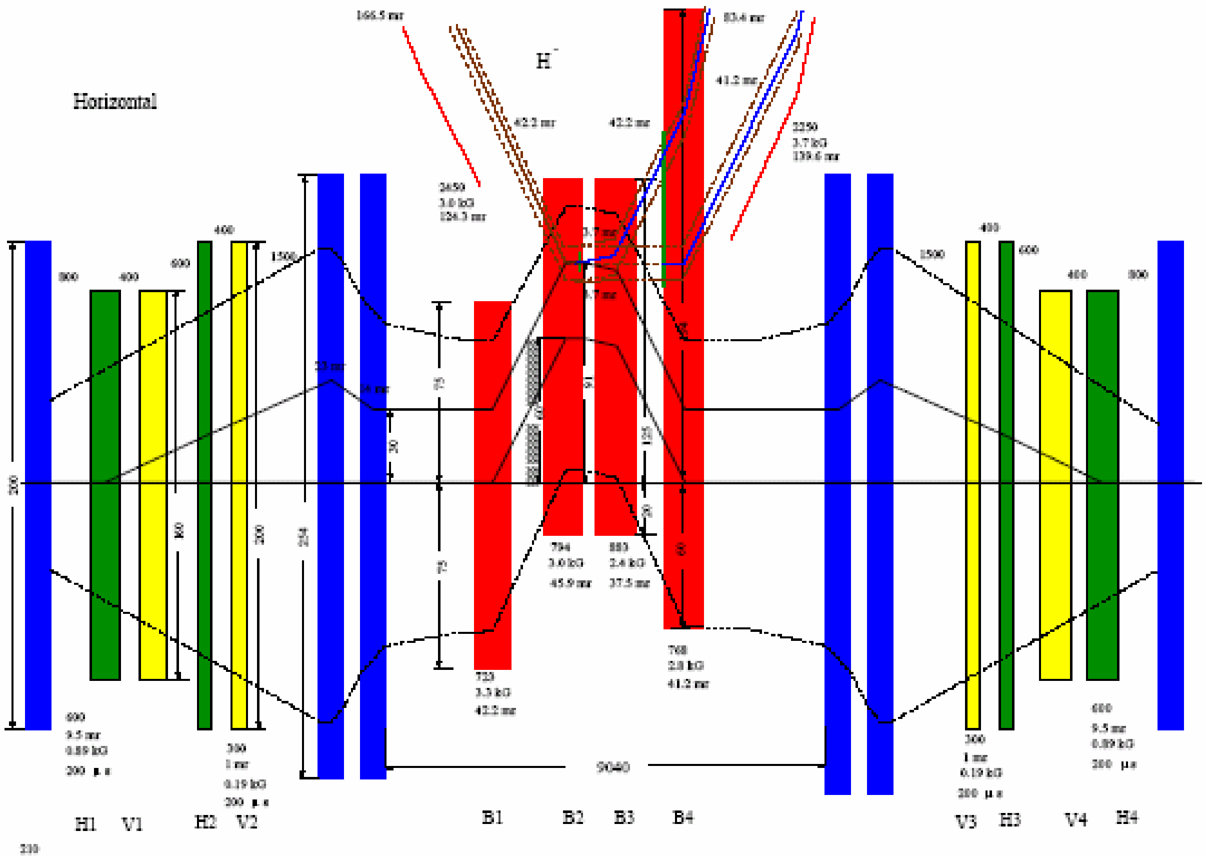
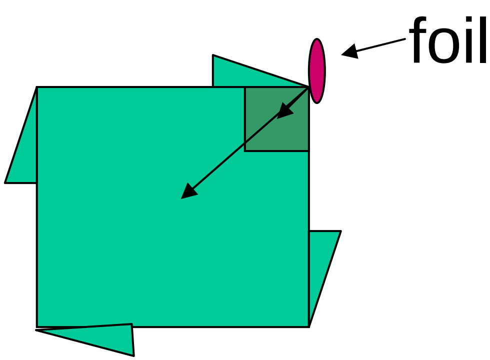


Figure 12: Schematic layout of the beam injection region of the proposed hybrid lattice ring.

Sf-Consistency - no-Chromaticity Low-Dispersion Case

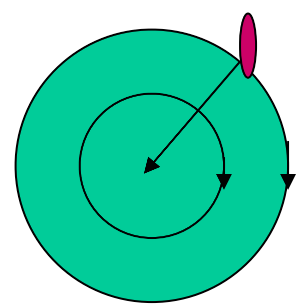
- **SNS case – chromaticity can be eliminated be sextupoles, dispersion exist only in the arcs (no dispersion in the drifts, etc.), but there is strong double harmonic - 2D case is enough (neglect longitudinal space charge force.**

1) Present injection – moving the closed orbit from the foil, no injected particle angles



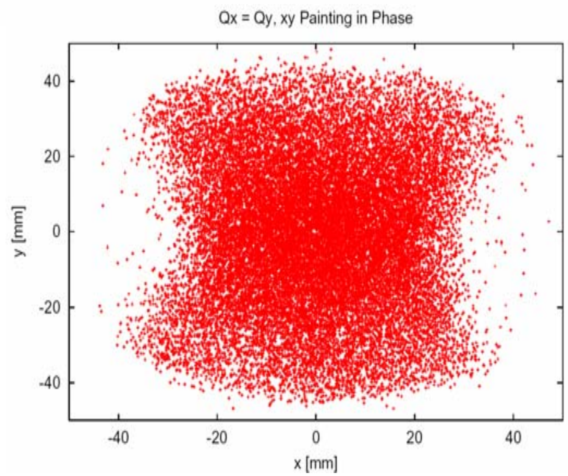
Growing tails because of space charge- is a typical dilution effect

2) Modified injection – needs particle angle on the foil, round betatron modes (e.g. equal betatron tunes).

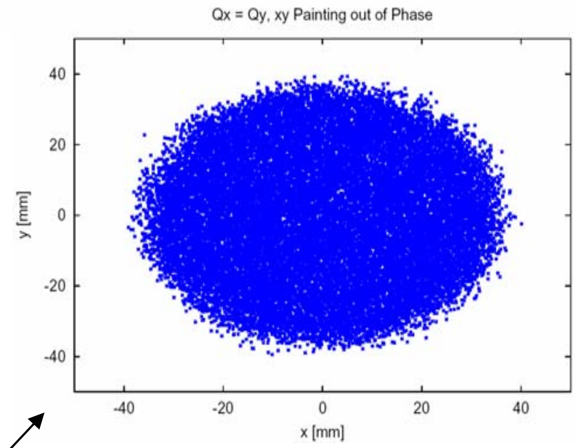


This {2,2} distribution has outstanding property – it retains its self-consistent shape at any moment of injection

No-Chromaticity Low-Dispersion Case cont.



Painting without angle

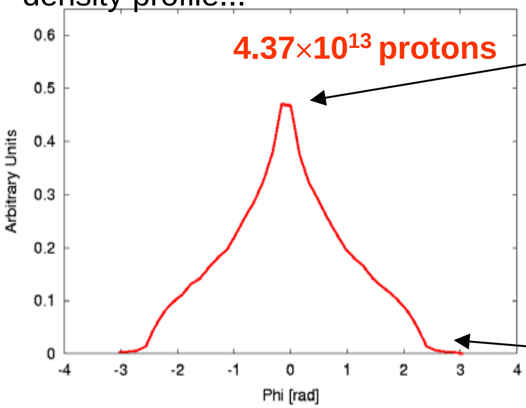


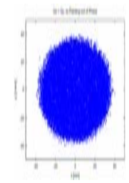
Painting with angle

Tr. size: center

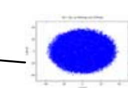
Finally, self consistent distribution is one having transverse elliptical shape, and its size should correspond to envelope parameters along longitudinal coordinate – the last condition can be met when we introduce injected beam beta function variation along the beam. It needs an additional fast kicker or RF quadrupole.

Recall sharply peaked longitudinal density profile...





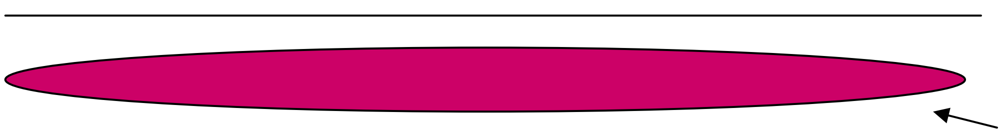
tail



Ring High-Dispersion Case

$$\overline{D}\frac{\delta p}{p} \approx \sigma_h \rightarrow \sigma_{tot}\ \textit{significantly greater than}\ \sigma_h$$

- Dispersion effect is stabilizing
- Not every energy distribution provides linearity of the transverse force
- The condition for self-consistency – any transformation **x -> x + D δp/p** should preserve the linearity of the force
- Newly found 3D distributions satisfy the conditions because the transformation **x -> x + D δp/p** is linear in 6D phase space.
- The longitudinal force, which is proportional to the derivative of the linear density is also linear!!!



fully self-consistent ring distribution is a uniform ellipsoid in projection to 3D real space

Chromaticity is equal to zero. Chromatic cases not investigated

Summary on self-consistent distributions

• New 2D and 3D SC self-consistent time dependent distributions found (V. Danilov *et al*, PRST-AB **6**, 094202 (2003))

•They directly applicable to linacs

• They applicable to rings with large energy spread and dispersion with no tune and beta-function chromaticity

• The forming of the distribution (and as a result – no loss accumulation and transport) requires relatively small modifications in the injection painting

Problems:

• Inclusion of all chromatic effects – chromaticity of the tunes and beta functions

•Nonlinear force distributions

One outstanding application of 2D self consistent distributions – acceleration of small emittance beams (follows)

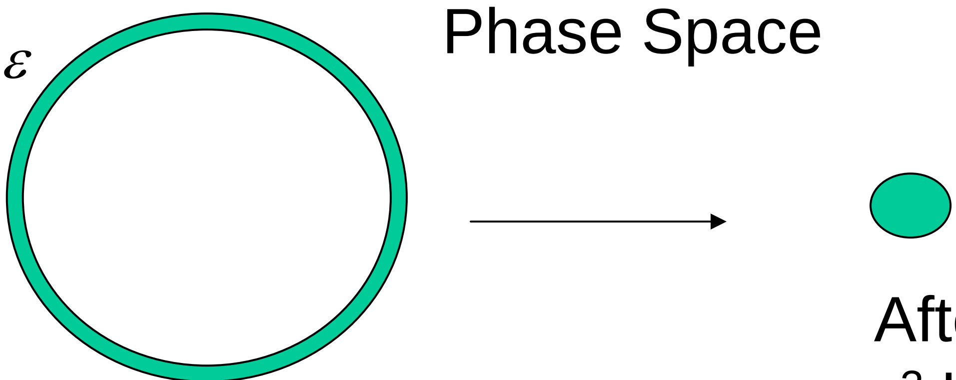
Acceleration of Small Emittance Beams

• By using special distributions, we can form the beam with ultra small emittance, but large size to avoid Space Charge blow-up, then accelerate beams and return the beam to normal distribution (Ya.S. Derbenev, NIM A **441**, (2000) p.223-233)

$$\Delta\nu_b \approx -\frac{\lambda r_0 R^2}{\nu_b \sigma_t^2 \beta^2 \gamma^3};|\Delta\nu_b| \le 0.1 \to \varepsilon_{sc} \ge \frac{10\lambda r_0 R^2}{\nu_b \beta_f \beta^2 \gamma^3}(\sigma_t^2 = \beta_f \varepsilon)$$

Phase Space

$\sigma_t^2 \ne \beta_f \varepsilon$



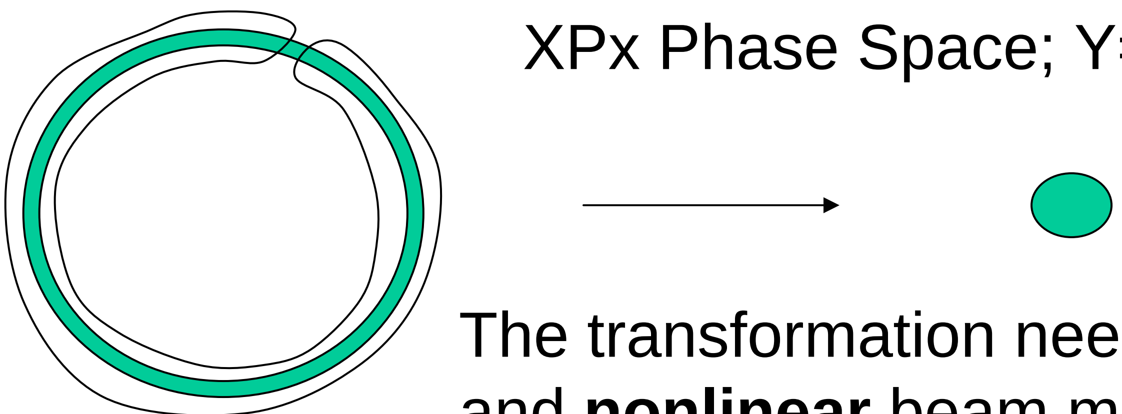
The size is large, but the occupied phase space is small

After acceleration factor γ^3 becomes dominant and Space Charge effects vanish

Acceleration of Small Emittance Beams (cont.)

- In 2D case the looking alike candidate is {2,0} KV distribution

$$f(4D) = C\delta(H_0 - H)$$



The transformation needs nonlinear separatrix and **nonlinear** beam manipulations

The linac emittance has to be much smaller than one in the ring, which is determined by space charge.
SNS Example: linac ε=0.3 mm·mrad, ring ε=100 mm·mrad

The question: can we find distributions with large size, which can be converted to small-emittance beam by linear transformation?

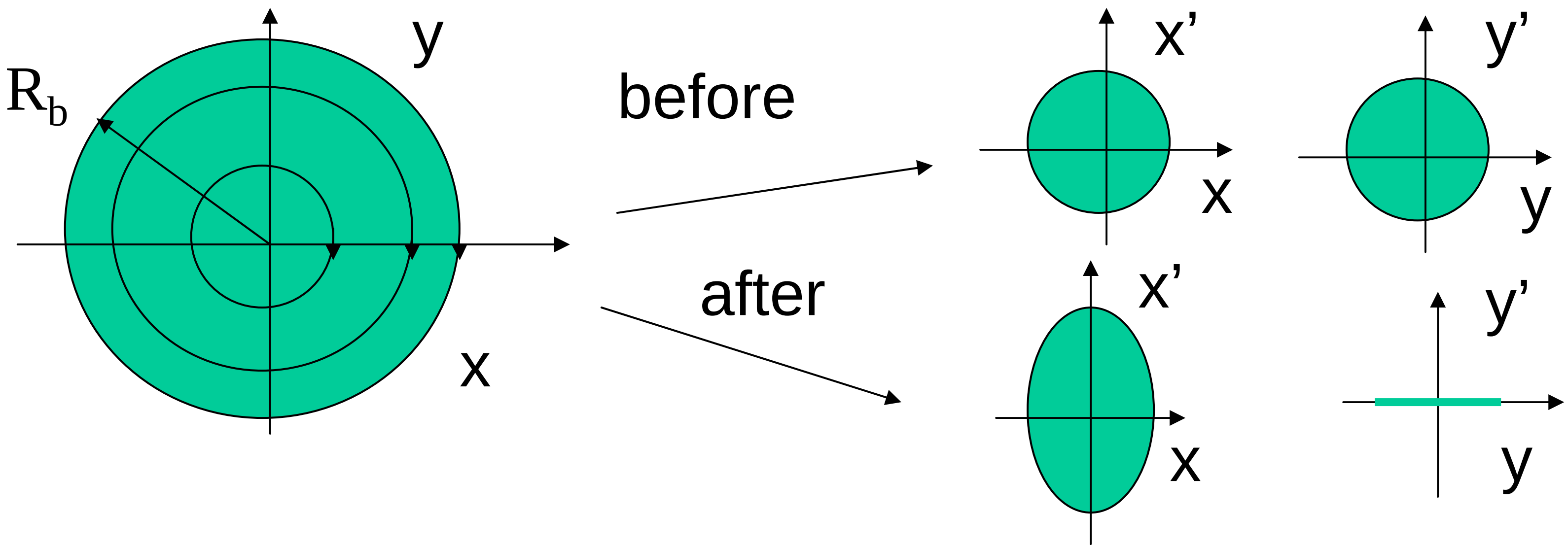
{2,2} rotating disk case

$$X^2+Y^2=R^2,\quad X'=Y/\beta_f,\ Y'=-X/\beta_f$$

$$\xrightarrow{\text{skew quad}}$$

$$\Delta X'=Y/\beta_f,\ \Delta Y'=X/\beta_f$$

$$X^2+Y^2=R^2,\quad X'=2Y/\beta_f,\ Y'=0$$



The conclusion: the rotating disk self-consistent distribution can be transformed by skew quad into distribution with one (e.g. vertical) zero emittance. This is the ideal choice for collider with flat beams

Amazing detail: regularly looking beam (in x and y phase spaces) transformed into zero emittance beam. Reason – xy correlations.

{2,2} pulsating disk case

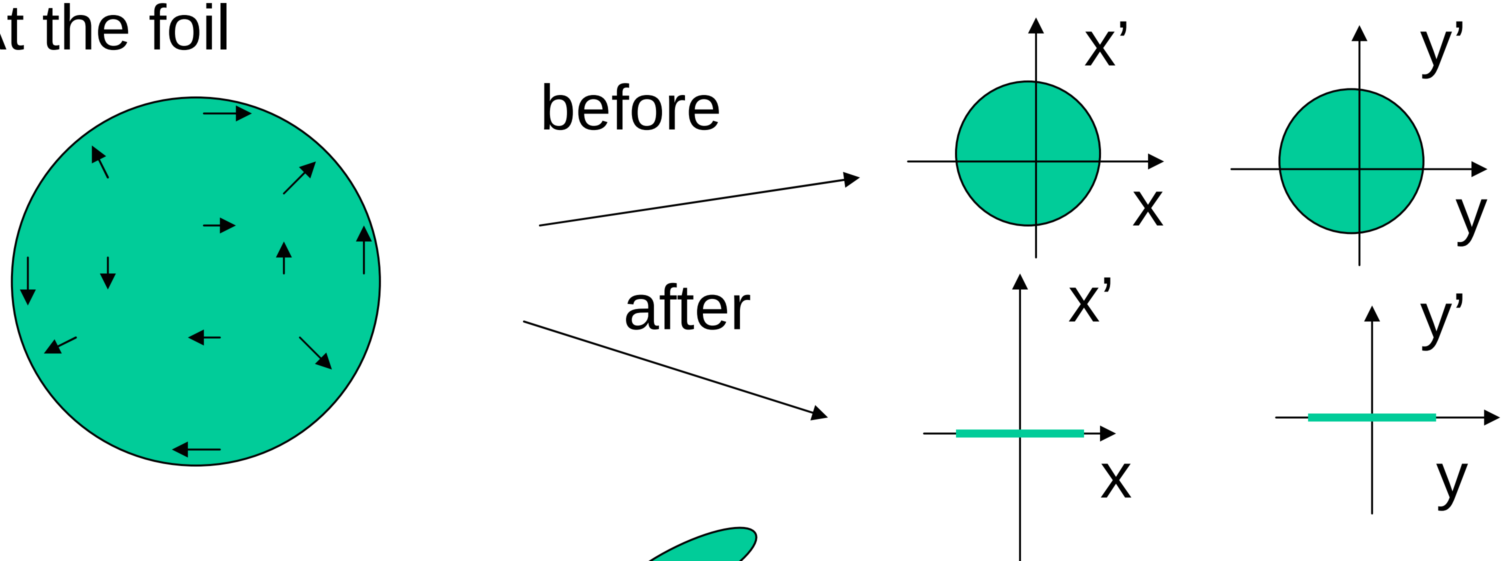
$$X^2 + Y^2 = R^2,$$
$$X' = Y / \beta_f, Y' = X / \beta_f$$

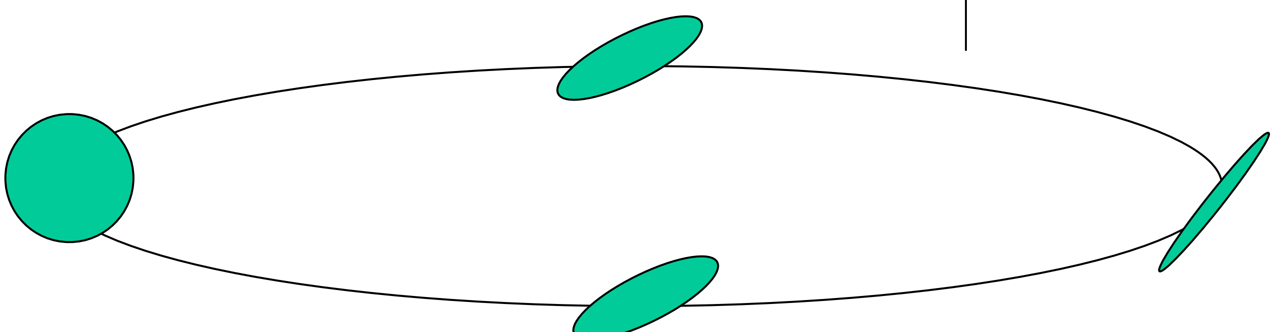
$\xrightarrow{\text{skew quad}}$

$$\Delta X' = -Y / \beta_f, \Delta Y' = -X / \beta_f$$

$$X^2 + Y^2 = R^2,$$
$$X' = 0, Y' = 0$$

At the foil

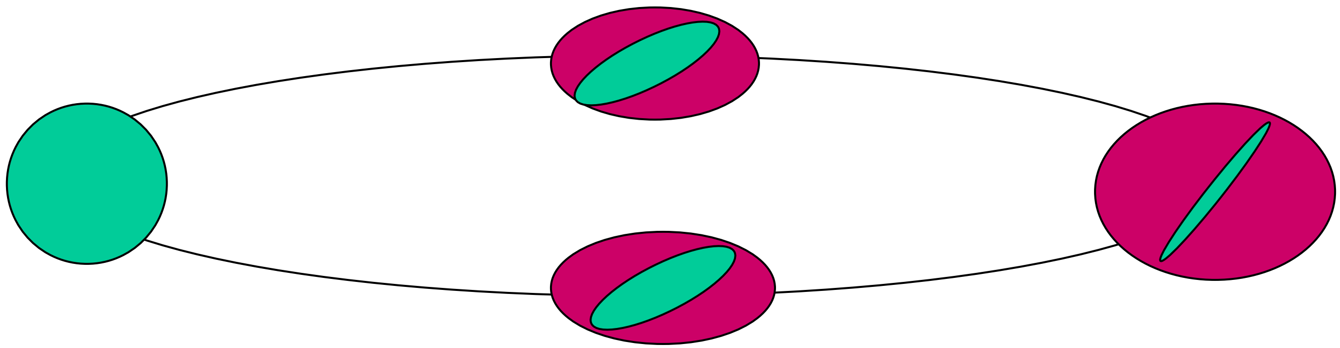




changing shape around the ring

{2,2} pulsating disk case (cont.)

Total transverse size (red) modified by dispersion.
It prevents its from shrinking



If $D^2\left(\frac{\Delta p}{p}\right)^2 + \sigma_x^2 = const$, the size will remain large around the ring

The accelerated bunch can be converted to one with both zero emittances.

The conclusion:

some SC {2,2} distributions in xy projection have large size, but in xp_x and yp_y (one or both) the emittance vanish. The reason: 4D volume of the figure is zero and it has special xy correlations