# Chapter 7

# Radiation and Beam Transport in Recirculated Linacs

In this chapter we present an introduction to the electromagnetic radiation that is emitted from the electron beam in a recirculated linac. Because the electron beam is never really in equilibrium in the recirculated linac, the effects of the emitted radiation are somewhat different and more complicated than occurs in storage rings. Instead of computing equilibrium properties of the beam as in storage rings, one must track and quantify the decrease in the beam quality as the electrons traverse the recirculated linac design, essentially element by element. In the first section of this chapter, general formulas will be presented for the total radiation produced by the electrons traversing either bend magnets or radiation-producing insertion devices.

Utilizing these estimates and the semiclassical notion that energy and momentum conservation may be used to estimate the effect of the individual quantized radiation events back on the electrons, we shall evaluate the increase in emittance and energy spread for electron beams traversing either bend magnets or radiation-producing insertion devices.

The amount of degradation in beam quality depends on the linear beam optics in the accelerator and certain choices in the recirculater beam optics lead to reduced degradation. Many of the beam optical techniques used in storage ring design are beneficially employed in recirculated linac design because the quality decreases are least in recirculaters that are designed analogously to storage rings.

The chapter concludes with a discussion of some practical recirculater designs.

## 7.1 Radiation from Relavistic Electrons

Particles that are accelerated radiate electromagnetic radiation. There is a large body of theory directed to calculating the energy and radiation spectrum of the radiation emitted for various particle orbits[kim,schwinger]. In fact, many of the pricipal features of the radition have been summarized in textbooks on electromagnetism [land,Jack,Schw]. We will largely follow the standard presentation in this subject. More details are found in these references.

Begin by considering the total radiation from an electron in non-relativistic motion. It can be shown that the total energy radiated by the particle at time $t$ follows Larmor's formula

$$\frac{dE}{dt}(t) = P(t) = \frac{2e^2}{3c^3}\frac{d\boldsymbol{v}}{dt}\cdot\frac{d\boldsymbol{v}}{dt}$$

where $\boldsymbol{v}(t)$ is the particle velocity at time $t$. So the total power radiated from an electron, neglecting coherence effects, is proportional to the square of the electron acceleration. As a next step, one needs to generalize Larmor's formula to relativistic electrons. For freely propagating photons both the energy increment $dE$ and the time increment $dt$ are the 0th components of four-vectors, and their ratio must therefore be a Lorentz invariant. The only Lorentz invariant that may be constructed from the velocity and acceleration, and that reduces to the Larmor formula is

$$P = \frac{2e^2}{3c}\frac{du^\alpha}{d\tau}\frac{du_\alpha}{d\tau}$$

where $u^\alpha = (\gamma, \gamma\boldsymbol{\beta})$ is the usual relativistic velocity 4-vector and $\tau$ is the proper time. This formula, through the use of the vector relation $\boldsymbol{\beta}^2\dot{\boldsymbol{\beta}}^2 - (\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2 = \boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}^2$, becomes the Lienard result

$$P(t) = \frac{2e^2}{3c}\gamma^6(\dot{\boldsymbol{\beta}}^2 - [\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}]^2).$$

When the acceleration is separated into its component along the the velocity and into its component perpendicular to the velocity, and by use of the fact that the velocity change for longitudinal forces is very small $\dot{\boldsymbol{\beta}} = \frac{1}{\gamma^3}F_\parallel$, it is shown that the total radiation from the acceleration from the linear accelerator portions of the orbit is quite small [jack]. Therefore, one mainly needs to consider the radiation from transverse accelerations. The principal locations where substantial transverse acceleration take place are in the benders of the beam recirculation magnets and in any insertion devices installed explicitly to extract electromagnetic radiation from the electrons.

For an electron being accelerated by a bending magnet with bend radius $\rho$, it is easily shown that $\dot{\boldsymbol{\beta}} = \beta^2 c/\rho$ directed towards the center of curvature of the bend. Then Lienard's formula for the total radiation is

$$P(t) = \frac{2e^2c}{3\rho^2}\beta^4\gamma^4.$$

The total energy loss for a bend of angle $\Theta$ is

$$\delta E = \frac{2e^2}{3\rho}\Theta\beta^3\gamma^4.$$

Assuming that the electrons are bent by isomagnetic dipoles and that they must be bent by a full 360 degrees on each recirculation pass, the fraction of electron energy lost per recirculation pass is

$$\frac{\delta E}{E_{beam}} = \frac{4\pi}{3}\frac{r_e}{\rho}\beta^3\gamma^3$$

where $r_e$ is the classical electron radius. For a bend radius of 10 m the fractional energy loss is under 0.1% as long as $\gamma < 9460$, corresponding to a beam energy of 4.8 GeV. For smaller recirculated linacs operating at lower energies ($E_{beam} \leq$ 1 GeV), the energy loss is small enough that the beam dynamics is not highly affected by the radiation emission. The rapid increase in the power emitted as a function of the beam energy implies that, depending on the bend radius of the dipoles, the beam dynamics can be substantially affected by the radiation emission as the beam energy increases beyond 1 GeV.

In order to calculate the effects of the radiation on the electron beam, it is necessary to know the energy spectrum of the emitted photons in detail. Rather than reproducing the derivation here, electrodynamics texts may be consulted for the calculation of the energy distribution of the emitted radiation. The distribution in frequency of the power emitted is

$$\frac{dP}{d\omega} = \frac{\sqrt{3}}{2\pi}\frac{e^2}{\rho}\gamma\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{\infty} K_{5/3}(x)dx \qquad 7.1$$

where $\frac{dP}{d\omega}d\omega$ gives the power in the (angular) frequency interval $[\omega, \omega + d\omega]$, $K_{5/3}$ is the modified Bessel function of order 5/3, and $\omega_c$ is the so-called critical frequency. It is close to the frequency with maximum emitted power and is expressed as

$$\omega_c = \frac{3}{2}\gamma^3\frac{c}{\rho}.$$

A qualitative argument can be made to understand the energy scaling of the critical frequency, which relies on two facts: (1) as the energy of an electron increases, the radiation emitted from the electron tends to be confined to a cone, directed in the instantaneously forward direction, of angular width $1/\gamma$, and (2) as the energy increases, the difference between the velocity of the electron and the velocity of light goes as $1/\gamma^2$. Consider in Fig 7.1 an observer looking at the radiation emitted from a portion of the circular orbit. By (1), the observed flash of radiation must originate within that portion of the orbit within $1/\gamma$ radians of the point of tangency in the orbit. Assuming the time that the electron arrives at the point of tangency is the origin of lab time, the electron begins emitting, as seen by the observer, at the location A, which is located $\rho/\gamma$ upstream of the point tangency, and ceases emitting at B, located at $\rho/\gamma$ downstream. Now by

(2), the electron arrives at A at $t = -\rho/\gamma\beta c \approx -\rho/\gamma c - \rho/2c\gamma^3$ and at B at $t = \rho/\gamma c + \rho/2c\gamma^3$. But the light emitted at A reaches B at $t = \rho/\gamma c - \rho/2c\gamma^3$ Therefore, the total duration of the observed pulse should be of order $\rho/\gamma^3 c$, implying freqency content up to $\gamma^3 c/\rho$. The three halves in the expression for the critical frequency is a convenient choice for expressing the mathematical form of the detailed distribution.

As noted in the exercises, by utilizing a result on the integrals of the Bessel functions it is possible to verify the consistency of Eqn. 7.1 with Lienard's formula in the extreme relativistic case ($\beta \approx 1$)

$$P = \int_0^\infty \frac{dP}{d\omega} d\omega = \frac{\sqrt{3}}{2\pi}\frac{e^2}{\rho}\omega_c\gamma \int_0^\infty \xi \int_\xi^\infty K_{5/3}(x)dx d\xi = \frac{2e^2 c}{3\rho^2}\gamma^4.$$

More importantly, the average energy of the emitted photons may be obtained by noting

$$\frac{d\dot{n}}{d\omega} = \frac{1}{\hbar\omega}\frac{dP}{d\omega}d\omega$$

is the photon emission rate. It thus follows that

$$\langle\hbar\omega\rangle = \frac{\int_0^\infty \hbar\omega \frac{d\dot{n}}{d\omega} d\omega}{\int_0^\infty \frac{d\dot{n}}{d\omega} d\omega} = \frac{8}{15\sqrt{3}}\hbar\omega_c$$

is the average energy of the emitted photons.

Given the total power radiated and the energy of a typical photon one can estimate the total expected number of emission events. The average number of photons emitted per unit time is

$$\dot{n} = \frac{5\alpha}{2\sqrt{3}}\frac{c}{\rho}\gamma$$

where $\alpha$ is the fine structure constant. The mean number of photons emitted for a bend of angle $\Theta$ is consequently

$$\delta n = \frac{5\alpha}{2\sqrt{3}}\Theta\gamma.$$

Note that this result is independent of bend radius and depends only on the beam energy. For 1 GeV electrons there are almost 130 photons emitted per turn. For moderately relativistic beams, this circumstance allows one to calculate the radiation effects back on the electrons statistically. It should be noted that for a uniform field bending dipole, the photons are emitted with equal probability throughout the bend.

As in storage rings, the other main transversely accelerating radiation producing devices in recirculated linacs are insertion devices placed in the beam path to generate electromagnetic radiation. In an insertion device the electron is made to oscillate transversely in a sinusoidal way by special purpose magnets whose polarity alternates along the beam path. In Fig. 7.3 a photograph is shown of a permanent magnet undulater designed for the Advanced Photon Source at Argonne National Laboratory. The device consists of SmCo material arranged physically so that the magnetic field on axis is very close to sinusoidal of a single frequency.

By proper insertion device design and parameter choices, it is possible to arrange for most of the power emitted to emerge within a relatively narrow range of frequencies. For the purposes of this discussion it will be assumed that all of the emission will happen within the lowest frequency fundamental band of the insertion device, and leave some discussion of the corrections to this approximation to the exercises and references. Also, we shall assume that the effects of the insertion device can be modelled as one dimensional magnets. Assume

$$B(z) \approx B_0 \cos(2\pi z/\lambda)$$

where $B_0$ is the peak field on axis and $\lambda$ is the period of the insertion device. The number of transverse oscillations is given by $N = L/\lambda$ where $L$ is the total length of the wiggler. In the one dimensional approximation this magnetic field can be represented by a single transverse vector potential component

$$A(z) \approx \frac{B_0 \lambda}{2\pi} \sin(2\pi z/\lambda).$$

Using conservation of transverse canonical momentum it is straightforward to show that the electron orbit within the alternating magnetic field has

$$v_x(z) = \frac{eA(z)}{\gamma mc} = \frac{K}{\gamma} \sin(2\pi z/\lambda)$$

and

$$x(z) = \int \frac{v_x}{v_z} dz \approx -\frac{1}{\langle\beta_z\rangle} \frac{K}{\gamma} \frac{\lambda}{2\pi} \cos(2\pi z/\lambda)$$

where $\langle\beta_z\rangle$ is the average longitudinal velocity divided by the speed of light and the field strength parameter, $K$, is defined to be

$$K = \frac{eB_0\lambda}{2\pi mc^2}.$$

The orbit makes a maximum angular excursion of $K/\gamma$ away from the insertion device axis. From this result, and the fact that the angular emission tends to be confined to a cone of order $1/\gamma$, one concludes that interference and coherent superposition of the emission from the different parts of the orbit is possible when $K << 1$, but for $K >> 1$ the emission will consist of the incoherent sum of the emission from the $N$ separate bends. Thus in principal the emission at the fundamental, because of the coherent superposition of the emission from various parts of the orbit, can be much greater than in a bend source with similar magnetic field strength, at least for for certain frequencies harmonically related to the fundamental. The drawback is, consistent with the Larmor/Lienard theorem on the total emission from the orbit, that the radiation must be emitted into a smaller bandwidth. For a finite perfectly sinusoidal oscillation with $N$ oscillations, the power emitted per unit frequency increases roughly a factor $N$ and the bandwidth of the emission is $1/N$.

Because $\gamma$ is conserved in an interaction between an electron and a static magnetic field, the average longitudinal velocity of the electron inside the insertion device may be calculated from

$$\beta_z(z) = \sqrt{1 - \frac{1}{\gamma^2} - \beta_x^2(z)},$$

with the result

$$\beta^{*2} \equiv \langle\beta_z\rangle^2 = 1 - \frac{1}{\gamma^2} - \frac{K^2}{2\gamma^2}.$$

Now, basic kinematic arguments from relativity theory may be used to obtain the frequency of the emitted photons. In the average rest frame of the electron, the insertion device is Lorentz contracted so its wavelength is $\lambda^* = \lambda/\beta^*\gamma^*$, where

$$\gamma^{*2} = \frac{1}{1-\beta^{*2}} = \gamma^2/(1+K^2/2)$$

The sinusoidal wiggle in the beam frame emits at angular frequency $2\pi c/\lambda^*$. The relativistic Doppler shift for the photon is obtained from the Lorentz transformation formulas for the wave vector

$$k_x^* = k_x = k\sin\theta\cos\phi$$

$$k_y^* = k_y = k\sin\theta\sin\phi$$

$$k_z^* = \gamma^* k(\cos\theta - \beta^*)$$

$$k^* = \gamma^* k(1 - \beta^*\cos\theta),$$

which implies $\cos\theta^* = (\cos\theta - \beta^*)/(1 - \beta^*\cos\theta)$.

The wave vector in the beam frame has

$$k = \frac{k^*}{\gamma^*(1-\beta^*\cos\theta)} = \frac{2\pi\beta^* c}{\lambda(1-\beta^*\cos\theta)},$$

yielding the condition

$$\lambda_e \approx \frac{\lambda}{2\gamma^2}(1+K^2/2),$$

where $\lambda_e$ is the wavelength emitted in the forward direction with $\cos\theta = 1$. This resonance condition is highly important for calculating the frequency emitted from insertion device magnets and Free Electron Lasers. As one observes off of the insertion device axis, the energy of the emitted photons decreases rapidly.

Now one can make calculations similar to those done for bend radiation. First, the Larmor/Lienard theorem is used to estimate the total power radiated by the electron when it is in the insertion device

$$\langle P\rangle = \frac{2e^2}{3}\gamma^4\beta_z^{*2}c\left(\frac{K}{\gamma}\right)^2\left(\frac{2\pi}{\lambda}\right)^2\frac{1}{2}.$$

The total energy radiated after one passage of the insertion device is

$$\delta E = 2\pi^2 \frac{2e^2}{3\lambda}\gamma^2\beta_z^* N K^2.$$

As in the case of synchrotron radiation, it is necessary to obtain the photon emission spectrum over all frequencies in order to properly evaluate the effect on the beam in the next section. To perform this calculation in detail we follow the spirit of Jackson's presentation on undulaters, but generalize it to arbitrarily large scattering angles, approaching $\pi$ yielding the lowest emitted photon energy of $\hbar\omega_u$.

Begin by noting that in the average rest frame the photon distribution is straightforward to calculate. When $K << 1$, the electron executes an $N$-period sinusoidal motion along the $x$-axis in this frame, of nearly a single frequency. The Larmor/Lienard formula, applied in the average rest frame, implies the total power emitted during the oscillation is

$$\langle P^* \rangle = \frac{2e^2}{3} c K^2 \left(\frac{2\pi}{\lambda^*}\right)^2 \frac{1}{2}.$$

But the energy of each photon is nearly $\hbar 2\pi c/\lambda^*$. This means that the total number of photons produced is

$$N_\gamma = \frac{2\pi}{3}\alpha N K^2. \qquad 7.2$$

For a typical insertion device with 50-100 periods and operated with a field strength around 1, one concludes that about 1 photon is emitted, on average, for every passage of an electron through the device. Now the number of photons emitted is a Lorentz invariant quantity, and can provide a check on our final answer for the energy distribution of the photons.

The distribution of photons in energy may be obtained from the distribution of photons in solid angle and through relativistic arguments. Begin with the distribution in power of dipole radiation from an electron executing sinusoidal motion in the $x$-direction. The dipole power radiated into the solid angle $d\Omega^*$ is

$$\frac{dP^*}{d\Omega^*} = \frac{e^2 c}{8\pi} k^{*4} a^2 \sin^2\Theta^*$$

where $\Theta^*$ is the angle between the propagation vector and the $x$-axis and $a$ is the size of the oscillation. In the low field strength limit with $K << 1$, one can replace $\gamma$ by $\gamma^*$ with the result $k^* a = K$; the more general case with $K \approx 1$ is discussed in the exercises and in the references. Converting Eqn. 7.2 into a number density and integrating for $N$ oscillations yields

$$\frac{dN_\gamma}{d\Omega^*} = \frac{\alpha}{4} N K^2 (k_y^{*2} + k_z^{*2})/k^{*2}$$

when expressed in terms of the photon propagation vector in the beam frame. To get the photon number distribution in the lab frame apply the Lorentz transformation formulas for the wave vector and the solid angle transformation formula $d\Omega^* = d\Omega/[\gamma^{*2}(1-\beta^*\cos\theta)^2]$ to obtain

$$\frac{dN_\gamma}{d\Omega} = \frac{\alpha}{4} N K^2 \frac{\sin^2\theta\sin^2\phi + \gamma^{*2}(\cos\theta - \beta^*)^2}{\gamma^{*4}(1-\beta^*\cos\theta)^4} \qquad 7.3$$

It is an exercise to show that this distribution, when converted into a distribution in energy becomes

$$\frac{dN_\gamma}{d\hat{E}} = \frac{\alpha\pi}{4\gamma^{*2}\beta^{*3}} N K^2 \left[\left(\frac{\hat{E}}{\gamma^{*2}} - 1\right)^2 + \beta^{*2}\right] \qquad 7.4$$

where $\hat{E}$ is the energy normalized by $\hbar 2\pi\beta^* c/\lambda$. The minimum angular frequency is $2\pi\beta^* c/(1+\beta^*)\lambda \approx \pi c/\lambda$ emitted in the upstream direction along the insertion device axis and the maximum angular frequency is $2\pi\beta^* c/(1-\beta^*)\lambda = (1+\beta^*)\gamma^{*2} 2\pi\beta^* c/\lambda$ emitted in the downstream direction along the insertion device axis. Therefore, the integration limits on $\hat{E}$ extend from $1/(1+\beta^*)$ to $1/(1-\beta^*)$. Integrating Eqn. 7.4 over the full energy range reproduces Eqn. 7.2, as it should by the Lorentz invariance of the photon number.

As before, it is possible to calculate the average energy of the emitted photons analytically with the result

$$\langle E\rangle = \frac{\int_0^\infty E \frac{dN_\gamma}{d\hat{E}} d\hat{E}}{\int_0^\infty \frac{dN_\gamma}{d\hat{E}} d\hat{E}} = \gamma^{*2}\hbar 2\pi\beta^* c/\lambda.$$

The angle of emission of the photons possessing the average energy is $\cos\theta = \beta^*$ and they have $\cos\theta^* = 0$ in the beam frame. Thus the photons having the average energy are those emitted without any longitudinal velocity in the beam frame. By the symmetry of the dipole radiation pattern in the beam frame, one indeed expects one-half of the photons to have energies greater than the average and one-half of the photons to have energies less than the average.

As shown in Fig.7.3, the energy distribution of the emitted photons is clearly symmetrical about the average energy of $\hbar 2\pi\beta * c/\gamma *^2 \lambda$, which is close to one-half the maximum energy emitted along the insertion device axis.

Because the emission is highly aligned with the beam motion, there is a highly useful model describing synchrotron emission events, that we shall follow in making estimates of the degradation of beam properties going through bends and insertion devices. In this model it is assumed that the emission events change the electron energy and change the electron angle by the projection of the momentum on the transverse direction. We further assume that the particle position is unchanged affter an emission event.

cAt the electron, in the

$$\frac{du^\alpha}{d\tau} \cdot \frac{du_\alpha}{d\tau}$$

### 7.1.1 Quantum Fluctuations and Particle Diffusion

### 7.1.2 Aberations and Higher Order Transfer Maps

### 7.1.3 Practical Designs for Recirculaters

### 7.1.4 Ion Accumulation Effects

## 7.2 Single Bunch Instabilities

Verify Eqn. 7.1 utilizing the Bessel function integral

$$\int_0^\infty x^2 K_{5/3}(x)dx = \frac{16\pi}{9\sqrt{3}}.$$

Calculate the mean energy of the emitted photons utilizing a and

$$\int_0^\infty x K_{5/3}(x)dx = \frac{5\pi}{3}.$$

Verify the solid angle transformation rule

Verify Eqn. 7.3 leads exactly to Eqn. 7.4

Integrate Eqn. 7.4 verifying the distribution is normalized to give the correct number of photons in the lab frame.

Verify the average energy of the photons.

Although the derivation of the emission spectrum in the chapter assumes $K << 1$, many of the characteristics of the spectrum at $K$ of order one follow can be obtained with very similar arguments.

$K \approx 1$ max flux