UVa Course on
Physics of
Particle Accelerators

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Last Time

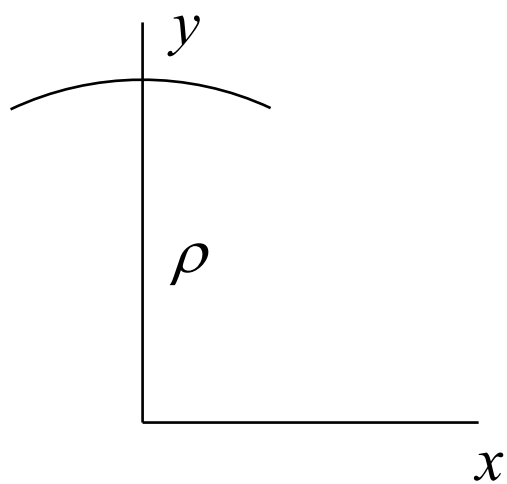
For transverse bend acceleration $\dot{\vec{\beta}} = -\frac{\beta^2 c}{\rho}\hat{r}$

$$x(t) = \rho \sin(\beta c t / \rho)$$

$$y(t) = \rho \cos(\beta c t / \rho)$$

$$\ddot{x}(t) = -\frac{\beta^2 c^2}{\rho}\sin(\beta c t / \rho)$$

$$\ddot{y}(t) = -\frac{\beta^2 c^2}{\rho}\cos(\beta c t / \rho)$$



$$\dot{\vec{\beta}} = \frac{d}{dt}\left(\beta\hat{\theta}\right) = \beta\frac{d\hat{\theta}}{dt} = -\frac{\beta^2 c}{\rho}\hat{r}$$

Fractional Energy Loss

$$\delta E = \frac{2e^2}{3\rho}\Theta\beta^3\gamma^4$$

For one turn with isomagnetic bending fields

$$\frac{\delta E}{E_{beam}} = \frac{4\pi r_e}{3\rho}\beta^3\gamma^3$$

r_e is the classical electron radius: 2.82×10^{-13} cm

Photon Number for Bends

$$P = \int_0^\infty \frac{dP}{d\omega} d\omega = \frac{\sqrt{3}}{2\pi}\frac{e^2}{\rho}\omega_c \gamma \int_0^\infty \xi \int_\xi^\infty K_{5/3}(x)\, dx\, d\xi = \frac{2e^2 c}{3\rho^2}\gamma^4$$

$$\frac{d\dot{n}}{d\omega} = \frac{1}{\hbar\omega}\frac{dP}{d\omega}$$

$$\langle \hbar\omega \rangle = \frac{\int_0^\infty \hbar\omega \frac{d\dot{n}}{d\omega} d\omega}{\int_0^\infty \frac{d\dot{n}}{d\omega} d\omega} = \frac{8}{15\sqrt{3}}\hbar\omega_c$$

$$\dot{n} = \frac{5\alpha}{2\sqrt{3}}\frac{c}{\rho}\gamma \qquad \delta n = \frac{5\alpha}{2\sqrt{3}}\Theta\gamma \qquad \alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$$

Spectral Distribution in Beam Frame

Begin with power distribution in beam frame: dipole radiation pattern

$$\frac{dP^*}{d\Omega^*} = \frac{e^2}{8\pi} k^{*4} a^2 \sin^2 \Theta^*$$

Number distribution in terms of wave number

$$\frac{dN_\gamma}{d\Omega^*} = \frac{\alpha}{4} N K^2 \frac{k_x^{*2} + k_z^{*2}}{k^{*2}}$$

Solid angle transformation

$$d\Omega^* = \frac{d\Omega}{\gamma^{*2}\left(1 - \beta^* \cos\theta\right)^2}$$

Number distribution in beam frame

$$\frac{dN_\gamma}{d\Omega} = \frac{\alpha}{4} NK^2 \frac{\sin^2\theta \sin^2\varphi + \gamma^{*2}\left(\cos\theta - \beta^*\right)^2}{\gamma^{*4}\left(1-\beta^*\cos\theta\right)^4}$$

Energy is simply

$$E(\theta) = \hbar \frac{2\pi\beta^* c}{\lambda_{ID}\left(1-\beta^*\cos\theta\right)} \qquad \hat{E}(\theta) = \frac{1}{\left(1-\beta^*\cos\theta\right)}$$

Number distribution as a function of normalized lab-frame energy

$$\frac{dN_\gamma}{d\hat{E}} = \frac{\alpha\pi}{4\gamma^{*2}\beta^{*3}} NK^2 \left[\left(\frac{\hat{E}}{\gamma^{*2}} - 1\right)^2 + \beta^{*2}\right]$$

Average Energy

Limits of integration

$$\cos\theta = 1 \qquad \hat{E} = \frac{1}{1-\beta^*} \qquad \cos\theta = -1 \qquad \hat{E} = \frac{1}{1+\beta^*}$$

Average energy is also analytically calculable

$$\langle E \rangle = \frac{\int_0^\infty E \frac{dN_\gamma}{d\hat{E}} d\hat{E}}{\int_0^\infty \frac{dN_\gamma}{d\hat{E}} d\hat{E}} = \gamma^{*2} \hbar 2\pi \beta^* c / \lambda_{ID} \approx \frac{E_{\max}}{2}$$

Effects back on particle beam

Classically; radiation causes damping of the transverse and longitudinal (energy or synchrotron) oscillations in the particle beam. For ring accelerators the time scale for the damping of the oscillations is easy to estimate

$$\tau_{damping} \equiv \frac{1}{f_{rev}} \frac{E}{\delta E_{turn}}$$

In more detail, the oscillations will follow a damped sinusoid

$$\begin{pmatrix} x \\ y \\ \Delta\phi \end{pmatrix}(t) = \begin{pmatrix} x_0 \exp\left(-t/\tau_x\right)\cos\left(\omega_x t - \delta_x\right) \\ y_0 \exp\left(-t/\tau_y\right)\cos\left(\omega_y t - \delta_y\right) \\ \Delta\phi_0 \exp\left(-t/\tau_{\Delta\phi}\right)\cos\left(\omega_{\Delta\phi} t - \delta_{\Delta\phi}\right) \end{pmatrix}$$

Damping rates in detail

For vertical (out of bend plane) damping

$$\tau_y \equiv 2\tau_{damping}$$

For horizontal (in bend plane) damping

$$\tau_x \equiv 2\tau_{damping} / \left(1 - \vartheta\right)$$

For vertical (out of bend plane) damping

$$\tau_{\Delta\phi} \equiv 2\tau_{damping} / \left(2 + \vartheta\right)$$

Radiation Partition

The damping is partitioned into the various degrees of freedom by

$$\vartheta = \frac{\oint \left(D / \rho^2 \right) \left(1 + 2\rho^2 K \right) ds}{\oint ds / \rho^2}$$

By suitable design (i.e., choice of D, ρ, K) one can shift greater or less damping between the horizontal and longitudinal degrees of freedom.

Robinson's Theorem

No matter what choice is made for the radiation partition

$$\frac{1}{\tau_x}+\frac{1}{\tau_y}+\frac{1}{\tau_{\Delta\phi}}=\frac{1}{2\tau_{damping}}\left(1-\vartheta+1+2+\vartheta\right)=\frac{2}{\tau_{damping}}$$

Equilibrium Energy Spread

Quantized photon emission events act to stimulate motion in all three degrees of freedom. Thus, the oscillations do not damp to zero.

$$\left\langle \Delta A^2 \right\rangle = \left\langle A_1^2 - A_0^2 \right\rangle = e^2$$

$$\left\langle \frac{dA^2}{dt} \right\rangle = \int_0^\infty e^2 \dot{n}(e)\, de = \dot{N}_{ph} \left\langle e^2 \right\rangle$$

In equilibrium.

$$\left\langle A^2 \right\rangle = \frac{\tau_{\Delta\phi}}{2} \dot{N}_{ph} \left\langle e^2 \right\rangle$$

$$\frac{\sigma_E^2}{E^2} = \frac{55}{32\sqrt{3}} \frac{\hbar c}{mc^2} \frac{\gamma^2}{2+\vartheta} \frac{\left\langle 1/\rho^3 \right\rangle}{\left\langle 1/\rho^2 \right\rangle}$$

Equilibrium Emittance

$$\left\langle \delta a^2 \right\rangle = \frac{e^2}{E_o^2} H(s)$$

$$H(s) = \beta D'^2 + 2\alpha D D' + \gamma D^2$$

In equilibrium, averaged over the ring.

$$\left\langle A^2 \right\rangle = \frac{\tau_{\Delta\phi}}{2} \dot{N}_{ph} \left\langle e^2 \right\rangle$$

$$\frac{\sigma_{x,y}^2}{\beta_{x,y}} = \frac{55}{32\sqrt{3}} \frac{\hbar c}{mc^2} \frac{\gamma^2}{J_{x,y}} \frac{\left\langle H/\rho^3 \right\rangle}{\left\langle 1/\rho^2 \right\rangle}$$

Emittance and energy spread increases

For recirculated linacs, there is no equilibrium and these estimates are used to compute emittance and energy spread increases

$$\Delta\varepsilon_{x,y} = \frac{1}{2cE_0^2}\int \dot{N}_{ph}\left(e^2\right)H\left(s\right)ds = \frac{55C_\gamma\hbar c\left(mc^2\right)^2}{64\pi\sqrt{3}}\gamma^5\int\frac{H}{\rho^3}ds$$

$$\Delta\frac{\sigma_E^2}{E^2} = \frac{55}{32\sqrt{3}}\frac{\hbar c}{mc^2}\frac{\gamma^2}{2+\vartheta}\frac{\left\langle 1/\rho^3\right\rangle}{\left\langle 1/\rho^2\right\rangle}$$