USPAS Course on
4^{th} Generation Light Sources II
ERLs and Thomson Scattering

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Introduction to Thomson Scattering

Lecture: Introduction to Thomson Scattering

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Thomson Scattering

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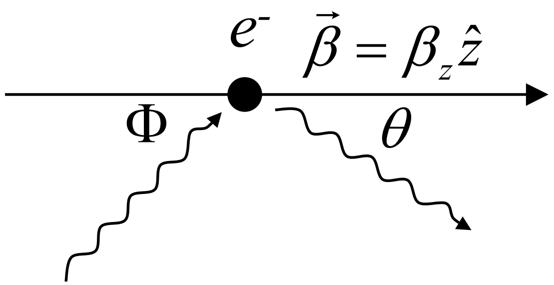
- Purely “classical” scattering of photons by electrons
- Thomson regime defined by the photon energy in the electron rest frame being small compared to the rest energy of the electron
- In this case electron radiates at the same frequency (elastic scattering!) as the incident photon for small field strengths
- Dipole radiation pattern is generated in beam frame, as for undulators
- Therefore radiation patterns can be largely copied from our previous undulator work

- Note on terminology: Some authors call any scattering of photons by free electrons Compton Scattering. Compton observed (the so-called Compton effect) frequency shifts in X-ray scattering off (resting!) electrons that depended on scattering angle. Such frequency shifts arise only when the energy of the photon in the rest frame becomes comparable with 0.511 MeV. We will reserve the words “Compton Scattering”, only for such higher energy scattering. We will talk about only one experiment in the “Compton regime”.

Simple Kinematics



Beam Frame | **Lab Frame**

$$p'_{e\mu} = \left(mc^2, 0\right)$$

$$p_{e\mu} = mc^2\left(\gamma, \gamma\beta_z \hat{z}\right)$$

$$p'_{p\mu} = \left(E'_L, \vec{E}'_L\right)$$

$$p_{p\mu} = E_L\left(1, \sin\Phi \hat{x} + \cos\Phi \hat{z}\right)$$

$$p_e \cdot p_p = mc^2 E'_L = mc^2 E_L \gamma\left(1 - \beta_z \cos\Phi\right) \qquad (3.1)$$

$$E'_L = E_L \gamma \left(1 - \beta_z \cos\Phi\right)$$

In beam frame scattered photon radiated with wave vector

$$k'_\mu = \frac{E'_L}{c}\left(1, \sin\theta' \cos\phi', \sin\theta' \sin\phi', \cos\theta'\right)$$

Back in the lab frame, the scattered photon energy E_s is

$$E_s = E'_L \, \gamma\left(1 + \beta_z \cos\theta\,'\right) = \frac{E'_L}{\gamma\left(1 - \beta_z \cos\theta\right)}$$

$$E_s = E_L \frac{\left(1 - \beta_z \cos\Phi\right)}{\left(1 - \beta_z \cos\theta\right)} \qquad (3.2)$$

Cases explored

Backscattered

$$\Phi = \pi$$

$$E_s = E_L \frac{(1+\beta_z)}{(1-\beta_z \cos\theta)} \approx 4\gamma^2 E_L \qquad \text{at} \quad \theta = 0$$

Provides highest energy photons for a given beam energy, or alternatively, the lowest beam energy to obtain a given photon wavelength. Pulse length roughly the ELECTRON bunch length

Cases explored, contd.

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Ninety degree scattering

$$\Phi = \pi / 2$$

$$E_s = E_L \frac{1}{\left(1 - \beta_z \cos\theta\right)} \approx 2\gamma^2 E_L \qquad \text{at} \quad \theta = 0$$

Provides factor of two lower energy photons for a given beam energy than the equivalent Backscattered situation. However, very useful for making short X-ray pulse lengths. Pulse length a complicated function of electron bunch length and transverse size.

Cases explored, contd.

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Small angle scattered (SATS)

$$\Phi << 1$$

$$E_s = E_L \frac{\Phi^2}{2(1-\beta_z \cos\theta)} \approx \Phi^2 \gamma^2 E_L \qquad \text{at} \quad \theta = 0$$

Provides much lower energy photons for a given beam energy than the equivalent Backscattered situation. Alternatively, need greater beam energy to obtain a given photon wavelength. Pulse length roughly the PHOTON pulse length.

Transformation of Photon Field

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Photon field for *x*-polarized plane wave traveling in the –*z* direction (i.e., for the backscattered case!)

$$A_x(t,x,y,z) = A(z+ct)e^{i(k_z z+\omega t)}$$

$$A'_x(t',x',y',z') = A(\gamma(1+\beta_z)(z'+ct'))e^{i(k'_z z'+\omega' t')}$$

because $z+ct = \gamma(1+\beta_z)(z'+ct')$

$$\omega' = \gamma(1+\beta_z)\omega \qquad (3.3)$$

$$k'_z = \gamma(1+\beta_z)k_z$$

$$E'_x = \gamma(1+\beta_z)E_x$$

$$B'_y = -E'_x = \gamma(1+\beta_z)B_y$$

Finite Pulse Effects

The main focus of the rest of the lecture today is to generalize the work done so far to cover cases with

1. High Field strength

And

2. Finite Energy spread from the pulsed photon beam itself

Roughly speaking, the conclusion is that the energy spectra of the scattered photons is increased by a width of order of $1/N$, where N is the number of oscillations the electron makes.

Hamilton-Jacobi Method

Given single particle with Hamiltonian *H,* can obtain motion by solving the (first order) Hamilton-Jacobi partial differential equation (non-relativistic!)

$$-\frac{\partial S}{\partial t} = H\left(q_i, \frac{\partial S}{\partial q_i}\right)$$

Easy example: Free Particle Motion

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\sum_i \frac{\partial S}{\partial q_i}\cdot\frac{\partial S}{\partial q_i}$$

$$S = -\alpha t + \vec{q} \cdot \vec{k}$$

$$\alpha = \frac{1}{2m}\left|\vec{k}\right|^2$$

$$\vec{\beta} = \frac{\partial S}{\partial \vec{k}} = -\frac{\vec{k}}{m}t + \vec{q}$$ are constant

$$\vec{p} = \frac{\partial S}{\partial \vec{q}} = \vec{k}$$ ✓

$$\frac{\vec{k}}{m}\left(t-t_0\right)=\vec{q}-\vec{q}_0$$

Free particle action

$$S=-\alpha t+\vec{q}\cdot\vec{k}=\frac{m}{2}\frac{\left|\vec{q}-\vec{q}_0\right|^2}{t-t_0}$$

(by addition of an (irrelevant!) constant factor)

Homework

Write down the Hamilton-Jacobi equation for the harmonic oscillator. Assuming an action function of the form

$$S(q,\alpha,t) = W(q,\alpha) - \alpha t$$

solve the the Hamilton-Jacobi equation for the motion harmonic oscillator:

$$q(t) = \sqrt{\frac{2\alpha}{k}} \sin \omega(t+\beta)$$

Note that the constant β follows directly from the requirement

$$\beta = \frac{\partial S}{\partial \alpha}$$

Homework

To illustrate the use of the Hamilton-Jacobi method to eliminate cyclic variables, use the Hamilton-Jacobi equation for gravitational motion in the plane with Hamiltonian

$$H\left(p_r, p_\theta, r\right) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{G}{r}$$

to show the orbit is

$$\frac{1}{r(\theta)} = \frac{1}{r_0}\left(1 + \varepsilon \cos\left(\theta - \theta_0\right)\right)$$

One should assume that

$$S = W\left(r, \alpha, p_{\theta 0}\right) - \alpha t + p_{\theta 0}\theta$$

Hint: in this problem $\partial S / \partial p_{\theta 0}$ is a constant of the motion

Electron in a Plane Wave

Assume plane-polarized pulsed laser beam moving in the $-z$ direction

$$\vec{A} = A_x(z+ct)\hat{x} \equiv A_x(\xi)\hat{x}$$

Electron Langrangian (electron charge is $-e$)

$$L = -mc^2\sqrt{1-\beta^2} + e\phi - \frac{e}{c}\vec{v}\cdot\vec{A}$$

Electron canonical momentum

$$\vec{P} = \frac{\partial L}{\partial \vec{v}} = \gamma m\vec{v} - \frac{e}{c}\vec{A}$$

Electron Hamiltonian

$$H = \gamma mc^2 - e\phi$$

Hamiltonian in terms of canonical momentum

$$(H + e\phi)^2 = m^2c^4 + \left(\vec{P} + \frac{e}{c}\vec{A}\right)^2 c^2$$

Relativistic Hamilton-Jacobi Equation

$$\left(\frac{\partial S}{\partial t} + e\phi\right)^2 = m^2c^4 + \left(\frac{\partial S}{\partial \vec{r}} + \frac{e}{c}\vec{A}\right)^2 c^2 \qquad (3.4)$$

Free Particle

$$S = -k_0 ct + \vec{k} \cdot \vec{r}$$

$$k_0 = \sqrt{m^2c^2 + \left|\vec{k}\right|^2}$$

$$\vec{\beta} = \frac{\partial S}{\partial \vec{k}} = -\frac{\vec{k}c}{\sqrt{m^2c^2 + \left|\vec{k}\right|^2}}t + \vec{r}$$ are constant

$$\frac{\vec{k}c}{\sqrt{m^2c^2+\left|\vec{k}\right|^2}}\left(t-t_0\right)=\vec{r}-\vec{r}_0$$

$$\vec{k}=\frac{m\dfrac{\vec{r}-\vec{r}_0}{t-t_0}}{\sqrt{1-\left(\dfrac{\vec{r}-\vec{r}_0}{t-t_0}\right)^2/c^2}}=\gamma m\vec{v}$$

the usual formula

$$\vec{p}=\frac{\partial S}{\partial\vec{r}}=\vec{k}$$

With the Field

Assume solution of full H-J equation (3.4) of the form

$$S = -k_0 ct + \vec{k} \cdot \vec{r} + F(\xi)$$

Because the propagation vector is perpendicular to the vector potential, one obtains an ODE for *F*

$$\frac{dF}{d\xi} = \frac{1}{-k_0 - k_z}\left[\frac{m^2c^2 - k_0^2 + \left|\vec{k}\right|^2}{2} + \frac{e}{c}k_x A_x + \frac{e^2}{2c^2}A_x^2\right]$$

Obtain the energy and momentum by differentiation

$$-E=\frac{\partial S}{\partial t}=-k_0 c+\frac{-c}{k_0+k_z}\left[\frac{m^2c^2-k_0^2+\left|\vec{k}\right|^2}{2}+\frac{e}{c}k_x A_x+\frac{e^2}{2c^2}A_x^2\right]=-\gamma mc^2$$

$$P_x=\frac{\partial S}{\partial x}=k_x=\gamma m v_x-\frac{eA_x}{c}$$

$$P_y=\frac{\partial S}{\partial y}=k_y=\gamma m v_y \qquad (3.5)$$

$$P_z=\frac{\partial S}{\partial z}=k_z+\frac{1}{-k_0-k_z}\left[\frac{m^2c^2-k_0^2+\left|\vec{k}\right|^2}{2}+\frac{e}{c}k_x A_x+\frac{e^2}{2c^2}A_x^2\right]=\gamma m v_z$$

Obtain constants of motion by differentiation w.r.t. k_i

$$\beta_0 = \frac{\partial S}{\partial k_0} = -ct + \frac{1}{(k_0 + k_z)^2} \int_{-\infty}^{\xi} \left[\frac{m^2c^2 - k_0^2 + \left|\vec{k}\right|^2}{2} + \frac{e}{c} k_x A_x + \frac{e^2}{2c^2} A_x^2 \right] d\xi' + \frac{k_0 \xi}{k_0 + k_z}$$

$$\beta_1 = \frac{\partial S}{\partial k_x} = x - \frac{1}{k_0 + k_z} \int_{-\infty}^{\xi} \frac{eA_x}{c} d\xi' - \frac{k_x \xi}{k_0 + k_z}$$

$$\beta_2 = \frac{\partial S}{\partial k_y} = y - \frac{k_y \xi}{k_0 + k_z} \tag{3.6}$$

$$\beta_3 = \frac{\partial S}{\partial k_z} = z + \frac{1}{(k_0 + k_z)^2} \int_{-\infty}^{\xi} \left[\frac{m^2c^2 - k_0^2 + \left|\vec{k}\right|^2}{2} + \frac{e}{c} k_x A_x + \frac{e^2}{2c^2} A_x^2 \right] d\xi' - \frac{k_z \xi}{k_0 + k_z}$$

Boundary Conditions for Beam Frame

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$$\dot{x}' = \dot{y}' = \dot{z}' = 0 \qquad \text{as} \qquad t \to -\infty$$

$$x' = y' = z' = 0 \qquad \text{as} \qquad t \to -\infty$$

$$k'_x = 0 \quad k'_y = 0 \quad k'_z = 0 \quad k'_0 = mc$$

$$\gamma' = 1 + \frac{e^2 A'^2_x}{2m^2c^4}$$

$$\frac{v'_x}{c} = \frac{eA'_x}{\gamma' mc^2} \tag{3.7}$$

$$\frac{v'_z}{c} = \frac{-\dfrac{e^2 A'^2_x}{2m^2c^4}}{\gamma'} = \frac{1-\gamma'}{\gamma'}$$

Ponderomotive Force along *-z*

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Simplification Using the Proper Time

As in the discussion of undulators, the orbit in the beam frame is expressed simply in terms of the proper time by differentiating the time equation

$$c\frac{dt'}{d\xi'} = 1 + \frac{e^2 A'^2_x}{2m^2c^4} = \gamma' \qquad \therefore\ \xi' = \tau + \text{constant}$$

Direct differentiation of Eqn. (3.6b,d) or ratios using Eqn. (3.7) yields

$$\frac{dx'}{d\xi'} = \frac{eA'_x}{mc^2}$$

$$\frac{dz'}{d\xi'} = -\frac{e^2 A'^2_x}{2m^2c^4}$$

The electron moves on a modulated sinusoid in proper time with angular frequency Ω'_0 in x and $2\Omega'_0$ in z where

$$\Omega'_0 = \gamma(1+\beta_z)\omega_0$$

$$\rho'(x',y',z',t') = -e\delta(x'-x'(\xi'(t')))\delta(y')\delta(z'-z'(\xi'(t')))$$

$$\begin{aligned}&\vec{J}'(x',y',z',t') = \\ &-e(dx'/d\xi')(d\xi'/dt')\hat{x}\delta(x'-x'(\xi'(t')))\delta(y')\delta(z'-z'(\xi'(t'))) \\ &-e(dz'/d\xi')(d\xi'/dt')\hat{z}\delta(x'-x'(\xi'(t')))\delta(y')\delta(z'-z'(\xi'(t')))\end{aligned}$$

Radiation Potentials

$$\Phi'(\vec{r}\,',t')=\int\frac{1}{R'}\rho'\left(\vec{r}\,'',t'-\frac{R'}{c}\right)dx''dy''dz''=-\frac{e}{2\pi}\int\frac{e^{i\omega'(t''-t'+R'/c)}}{R'}dt''d\omega'$$

$$=-\frac{e}{2\pi c}\int\left[1+\frac{e^2A'^2_x(\xi')}{2m^2c^4}\right]\frac{e^{i\omega'\left(\frac{\xi'}{c}+\frac{1}{c}\int_{-\infty}^{\xi'}\frac{e^2A'^2_x(\xi'')}{2m^2c^4}d\xi''-t'+R'/c\right)}}{R'}d\xi'd\omega'$$

$$A'_x(\vec{r}\,',t')=\int\frac{1}{R'c}J'_x\left(\vec{r}\,'',t'-\frac{R'}{c}\right)dx''dy''dz''=-\frac{e}{2\pi}\int\frac{v'_x(t'')e^{i\omega'(t''-t'+R'/c)}}{R'c}dt''d\omega'$$

$$=-\frac{e}{2\pi c}\int\frac{eA'_x(\xi')}{mc^2}\frac{e^{i\omega'\left(\frac{\xi'}{c}+\frac{1}{c}\int_{-\infty}^{\xi'}\frac{e^2A'^2_x(\xi'')}{2m^2c^4}d\xi''-t'+R'/c\right)}}{R'}d\xi'd\omega'$$

$$A'_z(\vec{r}\,',t')=\int\frac{1}{R'c}J'_z\left(\vec{r}\,'',t'-\frac{R'}{c}\right)dx''dy''dz''=-\frac{e}{2\pi}\int\frac{v'_z(t'')e^{i\omega'(t''-t'+R'/c)}}{R'c}dt''d\omega'$$

$$=\frac{e}{2\pi c}\int\frac{e^2A'^2_x(\xi')}{2m^2c^4}\frac{e^{i\omega'\left(\frac{\xi'}{c}+\frac{1}{c}\int_{-\infty}^{\xi'}\frac{e^2A'^2_x(\xi'')}{2m^2c^4}d\xi''-t'+R'/c\right)}}{R'}d\xi'd\omega'$$

Magnetic Field

Space differentiate the potentials to obtain the magnetic field

$$\vec{B}' \approx \frac{e}{2\pi c^2 r'} \sin\Theta' \hat{\Phi}' \int \frac{eA'_x(\xi')}{mc^2} i\omega' e^{i\omega'\left(\frac{\xi'}{c}+\frac{1}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi'' - t' + R'/c\right)} d\xi' d\omega'$$

$$-\frac{e}{2\pi c^2 r'} \sin\theta' \hat{\phi}' \int \frac{e^2 A'^2_x(\xi')}{2m^2c^4} i\omega' e^{i\omega'\left(\frac{\xi'}{c}+\frac{1}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi'' - t' + R'/c\right)} d\xi' d\omega'$$

Far field

$$R' = \sqrt{\left(x' - x'(\xi')\right)^2 + y'^2 + \left(z' - z'(\xi')\right)^2}$$

$$\approx r'\left(1 - \frac{x'}{r'^2}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi'' + \frac{z'}{r'^2}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi''\right)$$

Fourier Transformed

$$\vec{B}'(\vec{r}',t') \approx \frac{e}{2\pi c^2 r'} \sin\Theta' \hat{\Phi}' \int \frac{eA'_x(\xi')}{mc^2} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi'' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi'' - t' + r'/c\right)} d\xi' d\omega'$$

$$-\frac{e}{2\pi c^2 r'} \sin\theta' \hat{\phi}' \int \frac{e^2 A'^2_x(\xi')}{2m^2c^4} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi'' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi'' - t' + r'/c\right)} d\xi' d\omega'$$

In Thomson limit, spectral content of the emitted radiation *depends "simply" on emission angle in the beam frame*

$$\tilde{\vec{B}}'(\vec{r}',\omega') \approx \frac{e}{c^2 r'} \sin\Theta' \hat{\Phi}' \int \frac{eA'_x(\xi')}{mc^2} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi''\right)} d\xi'$$

$$-\frac{e}{c^2 r'} \sin\theta' \hat{\phi}' \int \frac{e^2 A'^2_x(\xi')}{2m^2c^4} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi'' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi''\right)} d\xi'$$

Equivalent Dipole Moment Spectra

So the emission is calculated (exactly, in the far-field limit) using an equivalent *x*-dipole spectra

$$D'_x(\omega') = \int \frac{eA'_x(\xi')}{mc^2} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi'' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi''\right)} d\xi'$$

and an equivalent *z*-dipole spectra

$$D'_z(\omega') = \int \frac{e^2 A'^2_x(\xi')}{2m^2c^4} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi'' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi''\right)} d\xi'$$

Given the photon beam vector potential, one obtains the spectral distribution of the radiation by performing these “simple” integrals

Polarization Energy Distributions

Compare with Eqns. (2.14) and (2.16)

$$\frac{dE'_{perp}}{d\omega' d\Omega'} = \frac{e^2}{8\pi^2 c^3} \left| D'_x(\omega') \right|^2 \sin^2 \phi'$$

$$\frac{dE'_{par}}{d\omega' d\Omega'} = \frac{e^2}{8\pi^2 c^3} \left| D'_x(\omega') \cos\theta' \cos\phi' + D'_z(\omega') \sin\theta' \right|^2$$

Results "exact" for electrons in a plane-wave and in the far-field limit. They can be used to reproduce what we previously have obtained!

Case I: Low Field Limit

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$$\frac{eA'_x(\xi')}{mc^2} << 1$$

$$D'_x(\omega') = \int \frac{eA'_x(\xi')}{mc^2} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi'' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi''\right)} d\xi' \approx -c\int \frac{eE'_x(\xi')}{mc^2} e^{i\omega'\frac{\xi'}{c}} d\xi'$$

$$D'_z(\omega') << D'_x(\omega')$$

Therefore, for low fields the spectral content of the emitted radiation *is independent of the emission angle in the beam frame.* The electron re-radiates at the same frequency as it moves.

Energy Spectral Distributions

$$\frac{dE'_{perp}}{d\omega' d\Omega'} = \frac{e^2}{8\pi^2 c} \frac{e^2 \left| \tilde{E}'_x (\omega'/c) \right|^2}{m^2 c^4} \sin^2 \phi'$$

$$\frac{dE'_{par}}{d\omega' d\Omega'} = \frac{e^2}{8\pi^2 c} \frac{e^2 \left| \tilde{E}'_x (\omega'/c) \right|^2}{m^2 c^4} \cos^2 \theta' \cos^2 \phi'$$

Or in the lab frame

$$\frac{dE_{perp}}{d\omega d\Omega} = \frac{e^2}{8\pi^2 c} \frac{e^2 \left| \tilde{E}_x \left(\omega (1 - \beta_z \cos\theta) / (c(1+\beta_z))\right) \right|^2}{m^2 c^4 \gamma^2 (1 - \beta_z \cos\theta)^2} \sin^2 \phi$$

$$(3.8)$$

$$\frac{dE_{par}}{d\omega d\Omega} = \frac{e^2}{8\pi^2 c} \frac{e^2 \left| \tilde{E}_x \left(\omega (1 - \beta_z \cos\theta) / (c(1+\beta_z))\right) \right|^2}{m^2 c^4 \gamma^2 (1 - \beta_z \cos\theta)^2} \left(\frac{\cos\theta - \beta_z}{1 - \beta_z \cos\theta} \right)^2 \cos^2 \phi$$

Comparison to Weak Field Undulator

This result is identical to the weak field undulator result (2.71) with the replacement of the magnetic field Fourier transform by the electric field Fourier transform

	Undulator	Thomson Backscatter
Driving Field	$\tilde{B}_y\left(\omega\left(1-\beta_z\cos\theta\right)/c\beta_z\right)$	$\tilde{E}_x\left(\omega\left(1-\beta_z\cos\theta\right)/\left(c\left(1+\beta_z\right)\right)\right)$
Forward Frequency	$\lambda \approx \dfrac{\lambda_0}{2\gamma^2}$	$\lambda \approx \dfrac{\lambda_0}{4\gamma^2}$
	Lorentz contract + Doppler	Double Doppler

The discussion following the weak-field undulator result applies with trivial modification

Case II: Longitudinally Flat Photon Pulse

Suppose (unrealistically!), that the photon pulse is hard-edge and flat with N_p periods

$$A_x(\xi) = A_0 \cos\left(\frac{2\pi}{\lambda_0}\xi\right)\left[\Theta(\xi) - \Theta\left(\xi - N_p\lambda_0\right)\right] \qquad \text{define } a \equiv \frac{eA_0}{mc^2}$$

$$D'_x(\omega') = \int \frac{eA'_x(\xi')}{mc^2} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'}\frac{eA'_x(\xi'')}{mc^2}d\xi'' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'}\frac{e^2A'^2_x(\xi'')}{2m^2c^4}d\xi''\right)} d\xi' =$$

$$i\omega' a \int \cos(k'_z\xi') e^{i\omega'\left(\frac{\xi'}{c} - \frac{a\sin\theta'\cos\phi'}{k'_z c}\sin(k'_z\xi') + \frac{a^2(1+\cos\theta')}{4c}\left(\xi' + \frac{\sin(2k'_z\xi')}{2k'_z}\right)\right)} d\xi'$$

$$D'_z(\omega') = \int \frac{e^2A'^2_x(\xi')}{2m^2c^4} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'}\frac{eA'_x(\xi'')}{mc^2}d\xi' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'}\frac{e^2A'^2_x(\xi'')}{2m^2c^4}d\xi''\right)} d\xi' =$$

$$\frac{i\omega' a^2}{2} \int \cos^2(k'_z\xi') e^{i\omega'\left(\frac{\xi'}{c} - \frac{a\sin\theta'\cos\phi'}{k'_z c}\sin(k'_z\xi') + \frac{a^2(1+\cos\theta')}{4c}\left(\xi' + \frac{\sin(2k'_z\xi')}{2k'_z}\right)\right)} d\xi'$$

Utilizing Bessel Function expansion and identities that we've seen before

$$D'_x = ic\sum_{n=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\frac{n+2k'}{\sin\theta'\cos\phi'}J_{n+2k'}\left(\frac{a\omega'}{k'_z c}\sin\theta'\cos\phi'\right)J_{k'}\left(\frac{a^2}{8}\frac{\omega'}{k'_z c}(1+\cos\theta')\right)$$
$$\times\sigma_n\left(\frac{\omega'}{k'_z c}\left(1+\left(a^2/4\right)\left(1+\cos\theta'\right)\right)\right)$$

$$D'_z = ic\sum_{n=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\left(\frac{a^2}{4}\frac{\omega'}{k'_z c}+\frac{2k'}{1+\cos\theta'}\right)J_{n+2k'}\left(\frac{a\omega'}{k'_z c}\sin\theta'\cos\phi'\right)J_{k'}\left(\frac{a^2}{8}\frac{\omega'}{k'_z c}(1+\cos\theta')\right)$$
$$\times\sigma_n\left(\frac{\omega'}{k'_z c}\left(1+\left(a^2/4\right)\left(1+\cos\theta'\right)\right)\right)$$

$$\sigma_n(A) = e^{iA\pi N_p}\frac{\sin\left(A\pi N_p\right)}{\sin\left(A\pi/n\right)}\frac{\sin\left(A\pi/n\right)}{A-n}$$

$$\frac{dE'_{perp}}{d\omega' d\Omega'} = \frac{e^2 \left| D'_x(\omega') \right|^2}{8\pi^2 c^3} \sin^2 \phi'$$

$$\frac{dE'_{par}}{d\omega' d\Omega'} = \frac{e^2}{8\pi^2 c^3} \left| D'_x(\omega') \cos\theta' \cos\phi' + D'_z(\omega') \sin\theta' \right|^2$$

Energy Spectral Distributions: Beam Frame

$$\frac{dE'_{perp,n}}{d\omega' d\Omega'} = \frac{e^2 n^2}{2\pi^2 c} \frac{1}{\sin^2\theta' \cos^2\phi'} \left[S_{1n} + S_{2n}/n\right]^2 \sin^2\phi' \left|\sigma_n(A)\right|^2$$

$$\frac{dE'_{par,n}}{d\omega' d\Omega'} = \frac{e^2 n^2}{2\pi^2 c} \left[\begin{matrix} S_{1n}\left(\dfrac{\cos\theta'}{\sin\theta'} + \dfrac{\omega'}{nk'_z c}\dfrac{a^2}{4}\sin\theta'\right) \\ \dfrac{S_{2n}}{n}\left(\dfrac{\cos\theta'}{\sin\theta'} + \dfrac{\sin\theta'}{(1+\cos\theta')}\right) \end{matrix} \right]^2 \left|\sigma_n(A)\right|^2$$

$$S_{1n} \equiv \sum_{k'=-\infty}^{\infty} J_{n+2k'}\left(\frac{a\omega'}{k'_z c}\sin\theta'\cos\phi'\right) J_{k'}\left(\frac{a^2}{8}\frac{\omega'}{k'_z c}(1+\cos\theta')\right)$$

$$S_{2n} \equiv \sum_{k'=-\infty}^{\infty} 2k' J_{n+2k'}\left(\frac{a\omega'}{k'_z c}\sin\theta'\cos\phi'\right) J_{k'}\left(\frac{a^2}{8}\frac{\omega'}{k'_z c}(1+\cos\theta')\right)$$

Energy Spectral Distributions: Lab Frame

For the general distribution at high *a,* due to the non-trivial transformation properties of the polarization vectors, it is easier and makes sense to first transform the electromagnetic field into the lab frame, and then perform the polarization dot product. In general,

$$\vec{E}'(\vec{r}',t') = \frac{e}{2\pi c^2}\hat{n}'\int\left[1+\frac{e^2A'^2_x(\xi')}{2m^2c^4}\right]\frac{i\omega'}{R'}dH'$$

$$-\frac{e}{2\pi c^2}\hat{x}\int\frac{eA'_x(\xi')}{mc^2}\frac{i\omega'}{R'}dH' + \frac{e}{2\pi c^2}\hat{z}\int\frac{e^2A'^2_x(\xi')}{2m^2c^4}\frac{i\omega'}{R'}dH'$$

where

$$dH' = e^{i\frac{\omega'}{c}\left[\xi'+\int_{-\infty}^{\xi'}\frac{e^2A'^2_x(\xi'')}{2m^2c^4}d\xi''-t'c+R'\right]}d\xi'\,d\omega'$$

$$\vec{B}'(\vec{r}\,',t')=\frac{e}{2\pi c^2}\sin\theta'\sin\phi'\,\hat{x}\int\frac{e^2A'^2_x(\xi')}{2m^2c^4}\frac{i\omega'}{R'}dH'$$

$$-\frac{e}{2\pi c^2}\cos\theta'\,\hat{y}\int\frac{eA'_x(\xi')}{mc^2}\frac{i\omega'}{R'}dH'+\frac{e}{2\pi c^2}\sin\theta'\cos\phi'\,\hat{y}\int\frac{e^2A'^2_x(\xi')}{2m^2c^4}\frac{i\omega'}{R'}dH'$$

$$\frac{e}{2\pi c^2}\sin\theta'\sin\phi'\,\hat{x}\int\frac{eA'_x(\xi')}{mc^2}\frac{i\omega'}{R'}dH'$$

By Lorentz Transformation Formulas

$$E_x=\gamma\left(E'_x+\beta_z B'_y\right)\qquad B_x=\gamma\left(B'_x-\beta_z E'_y\right)$$

$$E_y=\gamma\left(E'_y-\beta_z B'_x\right)\qquad B_y=\gamma\left(B'_y+\beta_z E'_x\right)$$

$$E_z=E'_z\qquad B_z=B'_z$$

$$\vec{E}(\vec{r},t)=\frac{e}{2\pi c^2}\hat{n}\int\left[\frac{\gamma\beta_z}{\gamma(1+\beta_z)}+\frac{1}{\gamma^2(1+\beta_z)^2}\frac{e^2A_x^2(\xi)}{2m^2c^4}\right]\frac{i\omega}{R}dH$$

$$-\frac{e}{2\pi c^2}\hat{x}\int\frac{1}{\gamma(1+\beta_z)}\frac{eA_x(\xi)}{mc^2}\frac{i\omega}{R}dH+\frac{e}{2\pi c^2}\hat{z}\int\left[\frac{\gamma}{\gamma(1+\beta_z)}+\frac{1}{\gamma^2(1+\beta_z)^2}\frac{e^2A_x^2(\xi)}{2m^2c^4}\right]\frac{i\omega}{R}dH$$

$$\vec{B}(\vec{r},t)=-\frac{e}{2\pi c^2}\sin\theta\sin\phi\hat{x}\int\left[\frac{\gamma\beta_z}{\gamma(1+\beta_z)}-\frac{1}{\gamma^2(1+\beta_z)^2}\frac{e^2A_x^2(\xi)}{2m^2c^4}\right]\frac{i\omega}{R}dH$$

$$-\frac{e\cos\theta}{2\pi c^2}\hat{y}\int\frac{1}{\gamma(1+\beta_z)}\frac{eA_x(\xi)}{mc^2}\frac{i\omega}{R}dH+\frac{e\sin\theta\cos\phi}{2\pi c^2}\hat{y}\int\left[\frac{\gamma\beta_z}{\gamma(1+\beta_z)}-\frac{1}{\gamma^2(1+\beta_z)^2}\frac{e^2A_x^2(\xi)}{2m^2c^4}\right]\frac{i\omega}{R}dH$$

$$\frac{e}{2\pi c^2}\sin\theta\sin\phi\hat{z}\int\frac{1}{\gamma(1+\beta_z)}\frac{eA_x(\xi)}{mc^2}\frac{i\omega}{R}dH$$

$$dH=e^{i\frac{\omega}{c}\left[\xi+\int_{-\infty}^{\xi'}\frac{e^2A_x^2(\xi')}{2m^2c^4}d\xi'-tc+R\right]}d\xi d\omega$$

General Expressions

$$\frac{dE_{perp}}{d\omega d\Omega} = \frac{e^2 \left| D_x(\omega) \right|^2}{8\pi^2 c^3} \sin^2\phi$$

$$\frac{dE_{par}}{d\omega d\Omega} = \frac{e^2}{8\pi^2 c^3} \left| D_x(\omega)\cos\theta\cos\phi - D_z(\omega)\sin\theta \right|^2$$

$$D_x(\omega) = \int \frac{1}{\gamma(1+\beta_z)} \frac{eA_x(\xi)}{mc^2} i\omega e^{i\omega\left(\frac{\xi(1-\beta_z\cos\theta)}{c(1+\beta_z)} - \frac{\sin\theta\cos\phi}{c\gamma(1+\beta_z)}\int_{-\infty}^{\xi}\frac{eA_x(\xi')}{mc^2}d\xi' + \frac{(1+\cos\theta)}{c\gamma^2(1+\beta_z)^2}\int_{-\infty}^{\xi}\frac{e^2A_x^2(\xi')}{2m^2c^4}d\xi'\right)} d\xi$$

$$D_z(\omega) = \int \left[\frac{\gamma\beta_z}{\gamma(1+\beta_z)} - \frac{1}{\gamma^2(1+\beta_z)^2} \frac{e^2A_x^2(\xi)}{2m^2c^4} \right]$$

$$\times i\omega e^{i\omega\left(\frac{\xi(1-\beta_z\cos\theta)}{c(1+\beta_z)} - \frac{\sin\theta\cos\phi}{c\gamma(1+\beta_z)}\int_{-\infty}^{\xi}\frac{eA_x(\xi')}{mc^2}d\xi' + \frac{(1+\cos\theta)}{c\gamma^2(1+\beta_z)^2}\int_{-\infty}^{\xi}\frac{e^2A_x^2(\xi')}{2m^2c^4}d\xi'\right)} d\xi$$

For Flat Pulse

An integration by parts gives $$D_z(\omega)=\frac{\beta_z\sin\theta\cos\phi}{(1-\beta_z\cos\theta)}D_x(\omega)$$

$$-\frac{1+\beta_z}{(1-\beta_z\cos\theta)}\int\frac{1}{\gamma^2(1+\beta_z)^2}\frac{e^2A_x^2(\xi)}{2m^2c^4}i\omega e^{i\omega\left(\frac{\xi(1-\beta_z\cos\theta)}{c(1+\beta_z)}-\frac{\sin\theta\cos\phi}{c\gamma(1+\beta_z)}\int_{-\infty}^{\xi}\frac{eA_x(\xi')}{mc^2}d\xi'+\frac{(1+\cos\theta)}{c\gamma^2(1+\beta_z)^2}\int_{-\infty}^{\xi}\frac{e^2A_x^2(\xi')}{2m^2c^4}d\xi'\right)}d\xi,$$

$$D_x=ic\sum_{n=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\frac{n+2k'}{\sin\theta\cos\phi}J_{n+2k'}\left(\frac{a\omega}{k_zc}\frac{\sin\theta\cos\phi}{\gamma(1+\beta_z)}\right)J_{k'}\left(\frac{a^2}{8}\frac{\omega}{k_zc}\frac{(1+\cos\theta)}{\gamma^2(1+\beta_z)^2}\right)$$
$$\times\sigma_n\left(\frac{\omega}{k_zc}\frac{\left(1-\beta_z\cos\theta+(a^2/4)(1-\beta_z)(1+\cos\theta)\right)}{1+\beta_z}\right)$$

$$D_z=\frac{\beta_z\sin\theta\cos\phi}{(1-\beta_z\cos\theta)}D_x(\omega)+\frac{ic(1+\beta_z)}{(1-\beta_z\cos\theta)}\sum_{n=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\left(\frac{1}{\gamma^2(1+\beta_z)^2}\frac{a^2}{4}\frac{\omega}{k_zc}+\frac{2k'}{(1+\cos\theta)}\right)$$
$$\times J_{n+2k'}\left(\left(\frac{a\omega}{k_zc}\frac{\sin\theta\cos\phi}{\gamma(1+\beta_z)}\right)\right)J_{k'}\left(\frac{a^2}{8}\frac{\omega}{k_zc}\frac{(1+\cos\theta)}{\gamma^2(1+\beta_z)^2}\right)\sigma_n\left(\frac{\omega}{k_zc}\frac{\left(1-\beta_z\cos\theta+(a^2/4)(1-\beta_z)(1+\cos\theta)\right)}{1+\beta_z}\right)$$

Energy Spectral Distributions: Lab Frame

$$\frac{dE_{perp,n}}{d\omega d\Omega}=\frac{e^2}{2c}\frac{\sin^2\phi}{\sin^2\theta\cos^2\phi}\left[S_{1n}+S_{2n}/n\right]^2 f_{nN}^2\left(\omega;n\omega(\theta)\right)$$

$$\frac{dE_{par,n}}{d\omega d\Omega}=\frac{e^2}{2c}\left[S_{1n}\left(\frac{\cos\theta-\beta_z}{\left(1-\beta_z\cos\theta\right)\sin\theta}+\frac{1}{\gamma^2\left(1+\beta_z\right)}\frac{a^2}{4}\frac{\omega}{nk_zc}\frac{\sin\theta}{\left(1-\beta_z\cos\theta\right)}\right)+\frac{S_{2n}}{n}\left(\frac{1}{\sin\theta}\right)\right]^2 f_{nN}^2 \tag{3.9}$$

$$S_{1n}\equiv\sum_{k'=-\infty}^{\infty}J_{n+2k'}\left(\frac{a\omega}{k_zc}\frac{\sin\theta\cos\phi}{\gamma\left(1+\beta_z\right)}\right)J_{k'}\left(\frac{a^2}{8}\frac{\omega}{k_zc}\frac{\left(1+\cos\theta\right)}{\gamma^2\left(1+\beta_z\right)^2}\right)$$

$$S_{2n}\equiv\sum_{k'=-\infty}^{\infty}2k'J_{n+2k'}\left(\frac{a\omega}{k_zc}\frac{\sin\theta\cos\phi}{\gamma\left(1+\beta_z\right)}\right)J_{k'}\left(\frac{a^2}{8}\frac{\omega}{k_zc}\frac{\left(1+\cos\theta\right)}{\gamma^2\left(1+\beta_z\right)^2}\right)$$

$$\omega(\theta)=\frac{\left(1+\beta_z\right)\left(2\pi c/\lambda_0\right)}{1-\beta_z\cos\theta+\left(a^2/4\right)\left(1-\beta_z\right)\left(1+\cos\theta\right)}$$

Comparison to High K Undulators

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The main results are very similar to those from undulators with the following correspondences

	Undulator	Thomson Backscatter
Field Strength	K	a
Forward Frequency	$\lambda \approx \frac{\lambda_0}{2\gamma^2}\left(1+\frac{K^2}{2}\right)$	$\lambda \approx \frac{\lambda_0}{4\gamma^2}\left(1+\frac{a^2}{2}\right)$
Transverse Pattern	$\beta^*_z + \cos\theta'$	$1+\cos\theta'$

NB, be careful with the radiation pattern, it is the same at small angles, but quite a bit different at large angles

Case III: Realistic Pulse Distribution at High a

Utilize two-timing approximation (i.e., the laser pulse is a slowly varying sinusoid with amplitude $a(\xi)$), and the fundamental expressions for D_x and D_z, allows one to write the energy distribution at any angle in terms of the Bessel function expansions and a ξ integral over the modulation amplitude. In the forward direction

$$\frac{dE_{perp,n}}{d\omega d\Omega} = \frac{e^2}{2c}\gamma^2 \langle F_n(a(\xi))\rangle \sin^2\phi f_{nN}^2(\omega; n\omega(\theta=0))$$

$$\frac{dE_{par,n}}{d\omega d\Omega} = \frac{e^2}{2c}\gamma^2 \langle F_n(a(\xi))\rangle \cos^2\phi f_{nN}^2(\omega; n\omega(\theta=0))$$

The "average" F_n is

$$\left\langle F_n\left(a(\xi)\right)\right\rangle = \frac{\int_{-\infty}^{\infty} F_n\left(a(\xi)\right)d\xi}{\left(\int_{-\infty}^{\infty} a(\xi)d\xi\right)^2}$$

and

$$\omega(\theta=0) = \frac{\gamma^2\left(1+\beta_z\right)^2}{1+a^2/2} \approx \frac{4\gamma^2}{1+a^2/2}$$

Brilliance

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The angular source size

$$\sigma_{r'} = \frac{1}{2\gamma}\sqrt{\frac{\left(1+K^2/2\right)}{nN}} = \sqrt{\frac{\lambda_n}{2L}} \qquad \lambda_n = c/n\omega(\theta=0)$$

is now much bigger than for undulators because the L_{eff} is much smaller. Solidly in “transition region” for most TS sources of X-rays!

$$B(0,0;0) = \frac{F}{(2\pi)^2\sigma_{Tx}\sigma_{Tx'}\sigma_{Ty}\sigma_{Ty}}$$

$$\sigma_{Tx}^2 = \sigma_r^2 + \sigma_x^2 + a^2 + \tfrac{1}{12}\sigma_{x'}^2 L^2 + \tfrac{1}{36}\varphi^2 L^2 \qquad \sigma_{Tx'}^2 = \sigma_{r'}^2 + \sigma_{x'}^2$$

$$\sigma_{Ty}^2 = \sigma_r^2 + \sigma_y^2 + \tfrac{1}{12}\sigma_{y'}^2 L^2 + \tfrac{1}{36}\psi^2 L^2 \qquad \sigma_{Ty'}^2 = \sigma_{r'}^2 + \sigma_{y'}^2$$

Homework

Solve the Hamilton-Jacobi Equation with the lab-frame boundary conditions

$$\dot{x} = \dot{y} = 0, \qquad \dot{z} = \beta_z c \qquad \text{as} \qquad t \to -\infty$$

$$x = y = 0, \qquad z = \beta_z ct \qquad \text{as} \qquad t \to -\infty$$

Verify the Lorentz Transformation Formulas hold between this solution and the beam-frame solution used in the lectures. With this solution, verify Eqns. 3.9 by a lab-frame computation with the standard formula from Jackson

$$\frac{d^2 E}{d\omega d\Omega} = \frac{e^2\omega^2}{8\pi^2 c}\left|\int_{-\infty}^{\infty} \vec{n} \times \left(\vec{n} \times \vec{\beta}\right) e^{i\omega\left(t - \vec{n}\cdot\vec{r}(t)/c\right)} dt\right|^2$$

Homework

Utilizing the fact that the delta-function in the retarded Green function of the wave equation picks out

$$t - t(\xi) = R/c = \sqrt{(x - x(\xi))^2 + (y - y(\xi))^2 + (z - z(\xi))^2}\,/c,$$

the Lorentz transformation formula for the coordinates, and the beam-frame and lab-frame solutions to the Hamilton-Jacobi Equation, demonstrate

$$R' = R\gamma(1 - \beta_z \cos\theta)$$

To Lorentz transform the field, this formula is needed

Conclusions

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- We've discussed how dipole solutions to the Maxwell Equations can be used to obtain very general expressions for the spectral angular energy distributions for Thomson Scattering photon sources
- We've given an introduction to Thomson Scatter source radiation calculations and a general formula for obtaining the spectral brilliance
- By application of yesterday's material, we've given how brilliance scales with various beam parameters
- The Thomson scatter resonance condition is

$$\lambda = \frac{\lambda_0}{4\gamma^2}\left(1+\frac{a^2}{2}\right)$$