USPAS Course on Recirculated and Energy Recovered Linear Accelerators

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Lecture 5

Chapter Outline

- Particle Motion in the Linear Approximation
- Some Geometry of Ellipses
- Ellipse Dimensions in the β-function Description
- Area Theorem for Linear Transformations
- Phase Advance for a Unimodular Matrix
  - Formula for Phase Advance
  - Matrix Twiss Representation
  - Invariant Ellipses Generated by a Unimodular Linear Transformation
- Detailed Solution of Hill's Equation
  - General Formula for Phase Advance
  - Transfer Matrix in Terms of β-function
  - Periodic Solutions
- Non-periodic Solutions
  - Formulas for β-function and Phase Advance
- Beam Matching

Particle Motion in Linear Approximation

Fundamental Notion: The *Design Orbit* is a path in an Earth-fixed reference frame, i.e., a differentiable mapping from [0,1] to points within the frame. As we shall see as we go on, it generally consists of *arcs of circles* and *straight lines*.

$$\sigma : [0,1] \to \mathrm{R}^3$$

$$\sigma \;\; \to \;\; \vec{X}(\sigma) = \left(X(\sigma), Y(\sigma), Z(\sigma) \right)$$

Fundamental Notion: *Path Length*

$$ds = \sqrt{\left(\frac{dX}{d\sigma}\right)^2 + \left(\frac{dY}{d\sigma}\right)^2 + \left(\frac{dZ}{d\sigma}\right)^2} \, d\sigma$$

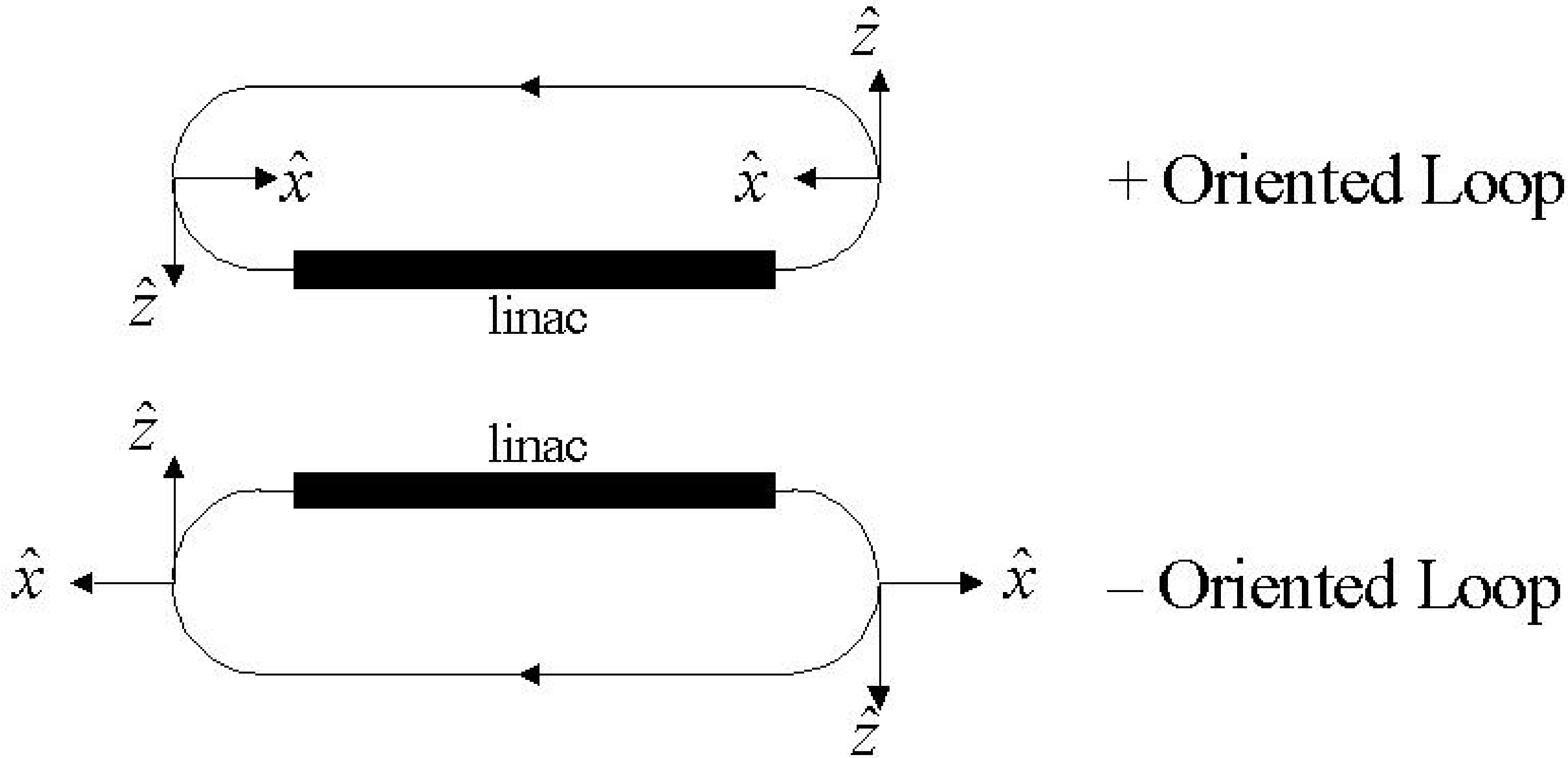
The *Design Trajectory* is the path specified in terms of the path length in the Earth-fixed reference frame. For a relativistic accelerator where the particles move at the velocity of light, $L_{tot}=ct_{tot}$.

$$s:[0,L_{tot}]\rightarrow \mathrm{R}^3$$

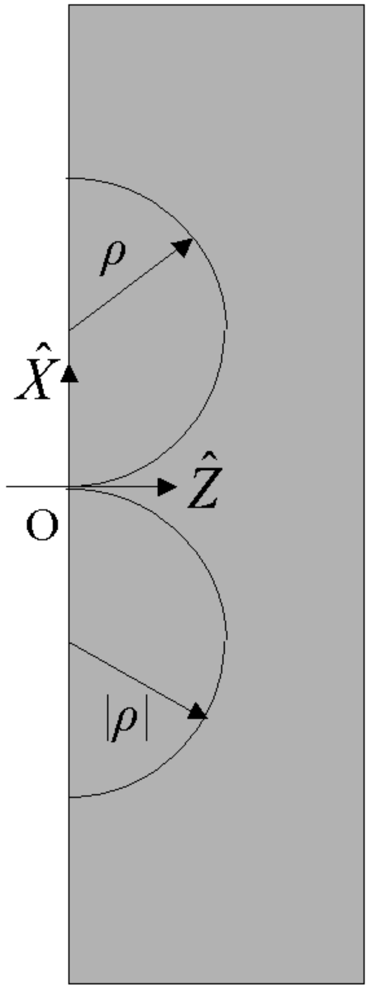
$$s \;\rightarrow\; \vec{X}(s)=\big(X(s),Y(s),Z(s)\big)$$

The first step in designing any accelerator, but in particular designing a recirculated linac, is to specify bending magnet locations that are consistent with the arc portions of the Design Trajectory.

Orientation Conventions



Bend Magnet Geometry



Positively Oriented Orbit

$\rho > 0 \qquad \Omega_c < 0$

Negatively Oriented Orbit

$\rho < 0 \qquad \Omega_c > 0$

Bend Magnet Trajectory Calculation

For a uniform magnetic field

$$\frac{d(\gamma m\vec{V})}{dt} = -e\left[\vec{E} + \vec{V}\times\vec{B}\right]$$

$$\frac{d(\gamma m V_x)}{dt} = eV_zB_y$$

$$\frac{d(\gamma m V_z)}{dt} = -eV_xB_y$$

$$\frac{d^2V_x}{dt^2} + \frac{\Omega_c^2}{\gamma^2}V_x = 0 \qquad \frac{d^2V_z}{dt^2} + \frac{\Omega_c^2}{\gamma^2}V_z = 0$$

For the solution satisfying boundary conditions: $\vec{X}(0) = 0$ $\quad \vec{V}(0) = V_{0z}\hat{z}$

$$X(t) = -\frac{p}{eB_y}\left(\cos\left(\Omega_c t/\gamma\right) - 1\right) = \rho\left(1 - \cos\left(\Omega_c t/\gamma\right)\right) \qquad \Omega_c = -eB_y/m$$

$$Z(t) = -\frac{p}{eB_y}\sin\left(\Omega_c t/\gamma\right) = -\rho\sin\left(\Omega_c t/\gamma\right)$$

Magnetic Rigidity

The magnetic rigidity is:

$$B\rho = \left|B_y \rho\right| = \frac{p}{|q|}$$

It depends only on the particle momentum and charge, and is a convenient way to characterize the magnetic field. Given magnetic rigidity and the required bend radius, the required bend field is a simple ratio. Note particles of momentum 100 MeV/*c* have a rigidity of 0.334 T m.

Long Dipole Magnet

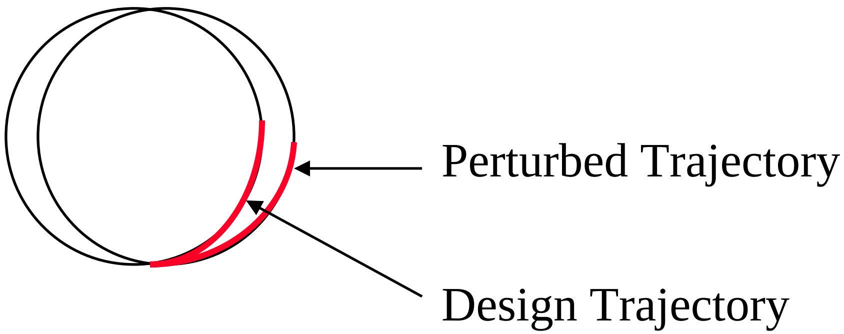
$$BL = B\rho\left(2\sin\left(\theta/2\right)\right)$$

Normal Incidence (or exit) Dipole Magnet

$$BL = B\rho\sin\left(\theta\right)$$

Operated by the Southeastern Universities Research Association for the U. S. Department of Energy

Natural Focussing Action in Bend Plane



Can show that for either a displacement perturbation or angular perturbation from the design trajectory

$$\frac{d^2x}{ds^2} = -\frac{x}{\rho_x^2(s)} \qquad \frac{d^2y}{ds^2} = -\frac{y}{\rho_y^2(s)}$$

Quadrupole Focussing

$$\vec{B}(x,y) = B'(s)(x\hat{y} - y\hat{x})$$

$$\gamma m \frac{d\mathrm{v}_{\mathrm{x}}}{ds} = -eB'(s)x \qquad \gamma m \frac{d\mathrm{v}_{\mathrm{y}}}{ds} = eB'(s)y$$

$$\frac{d^2x}{ds^2} + \frac{B'(s)}{B\rho}x = 0 \qquad \frac{d^2y}{ds^2} - \frac{B'(s)}{B\rho}y = 0$$

Combining with the previous slide

$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho_x^2(s)} + \frac{B'(s)}{B\rho}\right]x = 0 \qquad \frac{d^2y}{ds^2} + \left[\frac{1}{\rho_y^2(s)} - \frac{B'(s)}{B\rho}\right]y = 0$$

Hill's Equation

Define focussing strengths (with units of m^{-2})

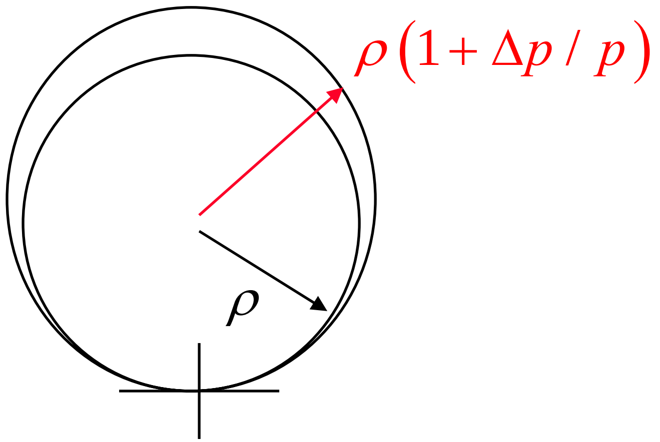
$$K_x(s) = \frac{1}{\rho_x^2(s)} + \frac{B'(s)}{B\rho} \qquad K_y = \frac{1}{\rho_y^2(s)} - \frac{B'(s)}{B\rho}$$

$$\frac{d^2x}{ds^2} + K_x(s)x = 0 \qquad \frac{d^2y}{ds^2} + K_y(s)y = 0$$

Note that this is like the harmonic oscillator, or exponential for constant *K,* but more general in that the focussing strength, and hence oscillation frequency depends on *s*

Energy Effects

$$\Delta x(s) = \frac{p}{eB_y}\frac{\Delta p}{p}\left(\cos(s/\rho)-1\right)$$



This solution is not a solution to Hill's equation directly, but *is* a solution to the inhomogeneous Hill's Equations

$$\frac{d^2x}{ds^2}+\left[\frac{1}{\rho_x^2(s)}+\frac{B'(s)}{B\rho}\right]x=-\frac{1}{\rho_x(s)}\frac{\Delta p}{p}$$

$$\frac{d^2y}{ds^2}+\left[\frac{1}{\rho_y^2(s)}-\frac{B'(s)}{B\rho}\right]y=-\frac{1}{\rho_y(s)}\frac{\Delta p}{p}$$

Dispersion

From theory of linear ordinary differential equations, the general solution to the inhomogeneous equation is the sum of **any** solution to the inhomogeneous equation, called the particular integral, plus two linearly independent solutions to the homogeneous equation, whose amplitudes may be adjusted to account for boundary conditions on the problem.

$$x(s) = x_p(s) + A_x x_1(s) + B_x x_2(s) \qquad y(s) = y_p(s) + A_y y_1(s) + B_y y_2(s)$$

Because the inhomogeneous terms are proportional to $\Delta p/p$, the particular solution can generally be written as

$$x_p(s) = D_x(s)\frac{\Delta p}{p} \qquad y_p(s) = D_y(s)\frac{\Delta p}{p}$$

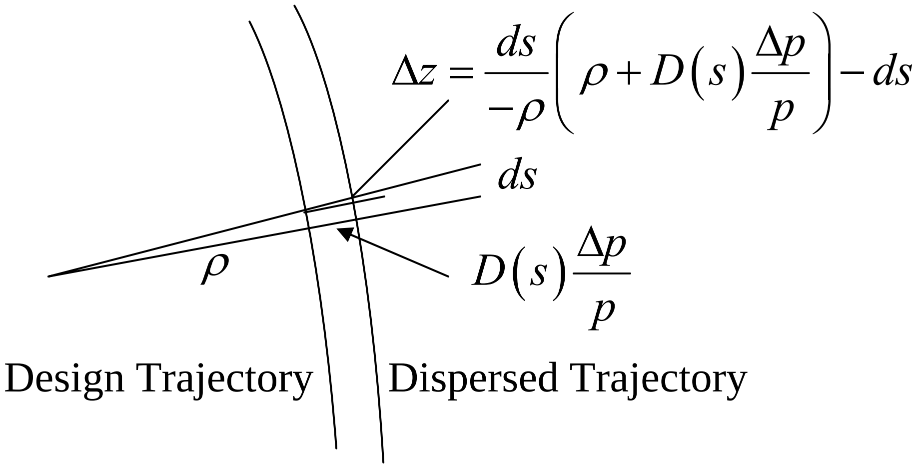
where the dispersion functions satisfy

$$\frac{d^2 D_x}{ds^2} + \left[\frac{1}{\rho_x^2(s)} + \frac{B'(s)}{B\rho}\right] D_x = -\frac{1}{\rho_x(s)} \qquad \frac{d^2 D_y}{ds^2} + \left[\frac{1}{\rho_y^2(s)} - \frac{B'(s)}{B\rho}\right] D_y = -\frac{1}{\rho_y(s)}$$

M_{56}

In addition to the transverse effects of the dispersion, there are important effects of the dispersion along the direction of motion. The primary effect is to change the time-of-arrival of the off-momentum particle compared to the on-momentum particle which traverses the design trajectory.

$$d\left(\Delta z\right) = -D\left(s\right)\frac{\Delta p}{p}\frac{ds}{\rho\left(s\right)}$$

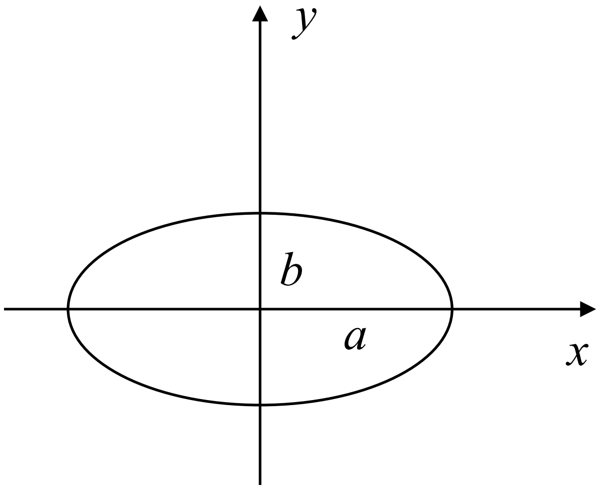


$$M_{56} = \int_{s_1}^{s_2}\left\{\frac{D_x\left(s\right)}{\rho_x\left(s\right)} + \frac{D_y\left(s\right)}{\rho_y\left(s\right)}\right\}ds$$

Some Geometry of Ellipses

Equation for an upright ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



In beam optics, the equations for ellipses are normalized (by multiplication of the ellipse equation by *ab*) so that the area of the ellipse divided by π appears on the RHS of the defining equation. For a general ellipse

$$Ax^2 + 2Bxy + Cy^2 = D$$

The area is easily computed to be

$$\frac{\text{Area}}{\pi} \equiv \varepsilon = \frac{D}{\sqrt{AC - B^2}} \qquad \text{Eqn. (1)}$$

So the equation is equivalently

$$\gamma x^2 + 2\alpha xy + \beta y^2 = \varepsilon$$

$$\gamma = \frac{A}{\sqrt{AC - B^2}}, \quad \alpha = \frac{B}{\sqrt{AC - B^2}}, \quad \text{and} \quad \beta = \frac{C}{\sqrt{AC - B^2}}$$

When normalized in this manner, the equation coefficients clearly satisfy

$$\beta\gamma - \alpha^2 = 1$$

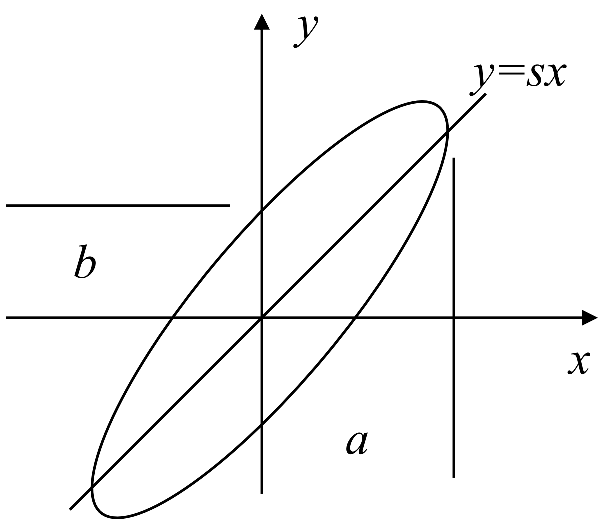
For example, the defining equation for the upright ellipse may be rewritten in following suggestive way

$$\frac{b}{a}x^2 + \frac{a}{b}y^2 = ab = \varepsilon$$

$\beta = a/b$ and $\gamma = b/a$, note $x_{\max} = a = \sqrt{\beta\varepsilon}$, $y_{\max} = b = \sqrt{\gamma\varepsilon}$

General Tilted Ellipse

Needs 3 parameters for a complete description. One way



$$\frac{b}{a}x^2 + \frac{a}{b}(y - sx)^2 = ab = \varepsilon$$

where s is a slope parameter, a is the maximum extent in the x-direction, and the y-intercept occurs at $\pm b$, and again ε is the area of the ellipse divided by π

$$\frac{b}{a}\left(1 + s^2\frac{a^2}{b^2}\right)x^2 - 2s\frac{a}{b}xy + \frac{a}{b}y^2 = ab = \varepsilon$$

Identify

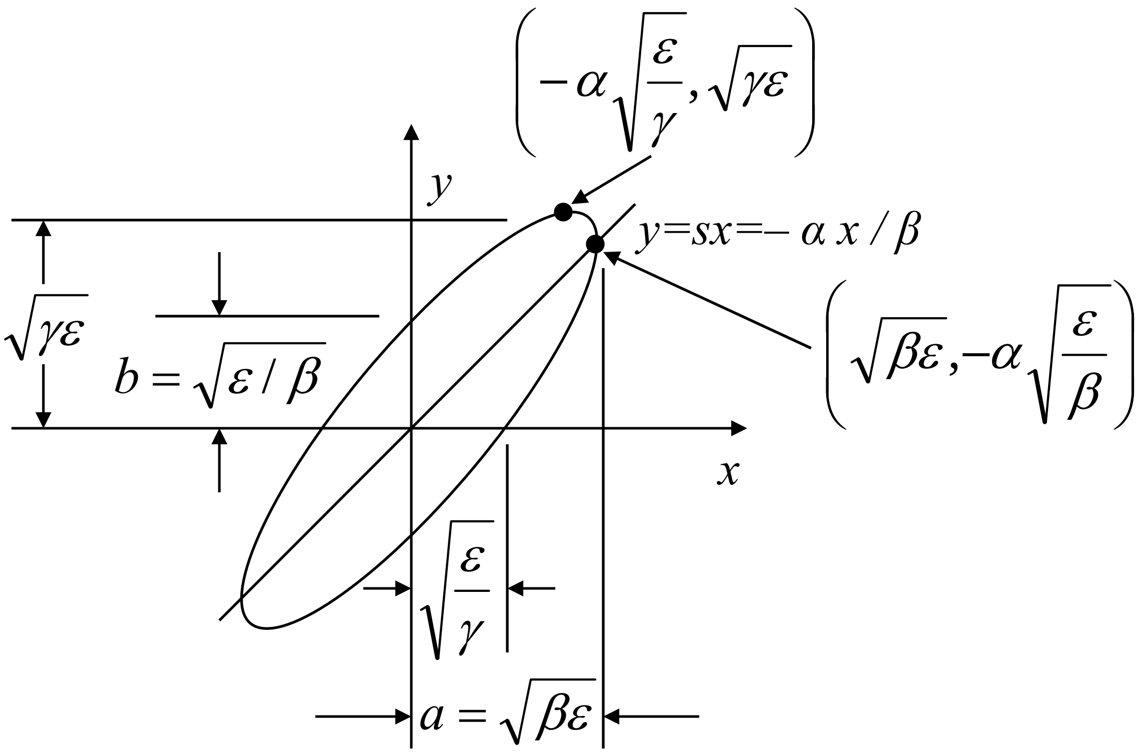
$$\gamma = \frac{b}{a}\left(1 + s^2 \frac{a^2}{b^2}\right), \quad \alpha = -\frac{a}{b} s, \quad \beta = \frac{a}{b}$$

Note that $\beta\gamma - \alpha^2 = 1$ automatically, and that the equation for ellipse becomes

$$x^2 + \left(\beta y + \alpha x\right)^2 = \beta\varepsilon$$

by eliminating the (redundant!) parameter γ

Ellipse Dimensions in the β-function Description



As for the upright ellipse $\quad x_{\max}=\sqrt{\beta\varepsilon}, \quad y_{\max}=\sqrt{\gamma\varepsilon}$

Area Theorem for Linear Optics

Under a general linear transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

an ellipse is transformed into another ellipse. Furthermore, if det (M) = 1, the area of the ellipse after the transformation is the same as that before the transformation.

Pf: Let the initial ellipse, normalized as above, be

$$\gamma_0 x^2 + 2\alpha_0 xy + \beta_0 y^2 = \varepsilon_0$$

Because

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \left(M^{-1}\right)_{11} & \left(M^{-1}\right)_{12} \\ \left(M^{-1}\right)_{21} & \left(M^{-1}\right)_{22} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

The transformed ellipse is

$$\gamma x^2 + 2\alpha xy + \beta y^2 = \varepsilon_0$$

$$\gamma = \left(M^{-1}\right)_{11}^2 \gamma_0 + 2\left(M^{-1}\right)_{11}\left(M^{-1}\right)_{21}\alpha_0 + \left(M^{-1}\right)_{21}^2 \beta_0$$

$$\alpha = \left(M^{-1}\right)_{11}\left(M^{-1}\right)_{12}\gamma_0 + \left(\left(M^{-1}\right)_{11}\left(M^{-1}\right)_{22} + \left(M^{-1}\right)_{12}\left(M^{-1}\right)_{21}\right)\alpha_0 + \left(M^{-1}\right)_{21}\left(M^{-1}\right)_{22}\beta_0$$

$$\beta = \left(M^{-1}\right)_{12}^2 \gamma_0 + 2\left(M^{-1}\right)_{12}\left(M^{-1}\right)_{22}\alpha_0 + \left(M^{-1}\right)_{22}^2 \beta_0$$

Because (verify!)

$$\beta\gamma-\alpha^2=\left(\beta_0\gamma_0-\alpha_0^2\right)$$
$$\times\left(\left(M^{-1}\right)_{21}^2\left(M^{-1}\right)_{12}^2+\left(M^{-1}\right)_{11}^2\left(M^{-1}\right)_{22}^2-2\left(M^{-1}\right)_{11}\left(M^{-1}\right)_{22}\left(M^{-1}\right)_{12}\left(M^{-1}\right)_{21}\right)$$
$$=\left(\beta_0\gamma_0-\alpha_0^2\right)\left(\det M^{-1}\right)^2$$

the area of the transformed ellipse (divided by π) is, by Eqn. (1)

$$\frac{\text{Area}}{\pi}=\varepsilon=\frac{\varepsilon_0}{\sqrt{\beta_0\gamma_0-\alpha_0^2}\left|\det M^{-1}\right|}=\varepsilon_0\,|\det M\,|$$

Example: Tilted ellipse from the upright ellipse

In the tilted ellipse the *y*-coordinate is raised by the slope with respect to the un-tilted ellipse

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\gamma_0 = \frac{b}{a}, \qquad \alpha_0 = 0, \qquad \beta_0 = \frac{a}{b}, \qquad \left(M^{-1}\right)_{21} = -s$$

$$\therefore \quad \gamma = \frac{b}{a} + \frac{a}{b} s^2, \qquad \alpha = -\frac{a}{b} s, \qquad \beta = \frac{a}{b}$$

Because det (*M*)=1, the tilted ellipse has the same area as the upright ellipse, i.e., $\varepsilon = \varepsilon_0$.

Phase Advance for a Unimodular Matrix

Any two-by-two unimodular (Det (M) = 1) matrix with $|\mathrm{Tr}\ M| < 2$ can be written in the form

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu)$$

The *phase advance* of the matrix, μ, gives the eigenvalues of the matrix $\lambda = e^{\pm i\mu}$, and $\cos \mu = (\mathrm{Tr}\ M)/2$

Pf: The equation for the eigenvalues of M is

$$\lambda^2 - \left(M_{11} + M_{22}\right)\lambda + 1 = 0$$

Because M is real, both λ and λ^* are solutions of the quadratic. Because

$$\lambda = \frac{\mathrm{Tr}(M)}{2} \pm i\sqrt{1-(\mathrm{Tr}(M)/2)^2}$$

For $|\mathrm{Tr}\ M| < 2$, $\lambda\,\lambda^* = 1$ and so $\lambda_{1,2} = e^{\pm i\mu}$. Consequently $\cos\mu = (\mathrm{Tr}\ M)/2$. Now the following matrix is trace-free.

$$M - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(\mu) = \begin{pmatrix} \dfrac{M_{11}-M_{22}}{2} & M_{12} \\ M_{21} & \dfrac{M_{22}-M_{11}}{2} \end{pmatrix}$$

Simply choose

$$\alpha = \frac{M_{11} - M_{22}}{2\sin\mu}, \quad \beta = \frac{M_{12}}{\sin\mu}, \quad \gamma = -\frac{M_{21}}{\sin\mu}$$

and the sign of μ to properly match the individual matrix elements with $\beta > 0$. It is easily verified that $\beta\gamma - \alpha^2 = 1$. Now

$$M^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(2\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(2\mu)$$

and more generally

$$M^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(n\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(n\mu)$$

Therefore, because sin and cos are both bounded functions, the matrix elements of any power of M remain bounded as long as $|\mathrm{Tr}\,(M)| < 2$.

NB, in some beam dynamics literature it is (incorrectly!) stated that the less stringent $|\mathrm{Tr}\,(M)| \leq 2$ ensures boundedness and/or stability. That equality cannot be allowed can be immediately demonstrated by counterexample. The upper triangular or lower triangular subgroups of the two-by-two unimodular matrices, i.e., matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

clearly have unbounded powers if $|x|$ is not equal to 0.