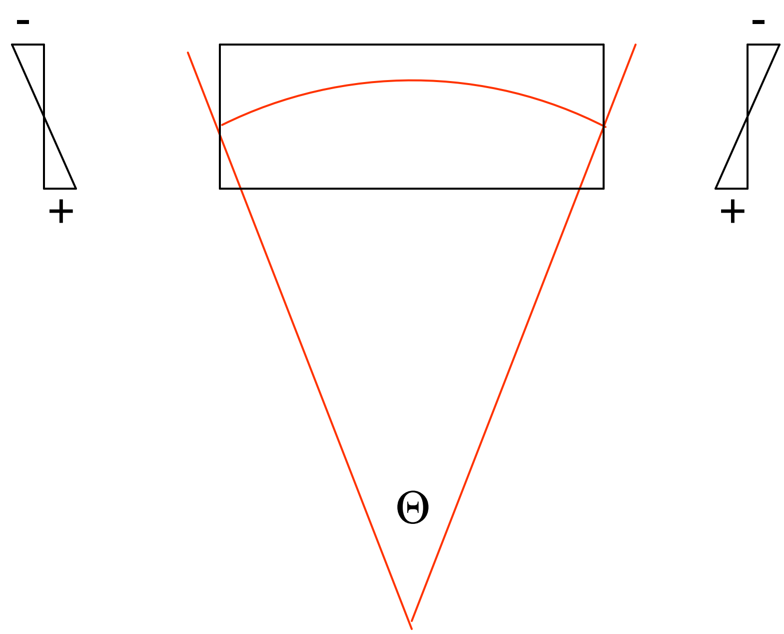
Solutions:

1. The 3×3 transfer matrix for a horizontal bend with bend angle Θ is

$$M_{\text{sector}} = \begin{pmatrix} \cos\Theta & \rho\sin\Theta & \rho(1-\cos\Theta) \\ -\sin\Theta/\rho & \cos\Theta & \sin\Theta \\ 0 & 0 & 1 \end{pmatrix}$$

The figure shows the geometry, + means magnetic field must be added to the sector magnet to get the rectangular magnet and – means magnetic field must be subtracted.



The transfer matrices for the wedges must have (12), (13), and (23) elements zero because, for small Θ the wedges are “thin”. The kick angle, by the wedge has

$$\left.\frac{dx}{ds}\right|_{after} = \left.\frac{dx}{ds}\right|_{before} + \frac{dL}{\rho} \approx \left.\frac{dx}{ds}\right|_{before} + \frac{x}{\rho}\tan\frac{\Theta}{2}$$

where dL is the extra path length through the wedge. The sign follows from the fact that positive x leads to less bending than the design orbit, meaning a positive dx/ds is generated. The wedge matrix is

$$M_{\text{wedge}} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan\frac{\Theta}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiplying out yields

$$M_{\text{tot}} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan\frac{\Theta}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\Theta & \rho\sin\Theta & \rho(1-\cos\Theta) \\ -\sin\Theta/\rho & \cos\Theta & \sin\Theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan\frac{\Theta}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan\frac{\Theta}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\Theta + \sin\Theta\tan\Theta/2 & \rho\sin\Theta & \rho(1-\cos\Theta) \\ \frac{1}{\rho}\left(-\sin\Theta + \cos\Theta\tan\frac{\Theta}{2}\right) & \cos\Theta & \sin\Theta \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \rho\sin\Theta & \rho(1-\cos\Theta) \\ 0 & 1 & 2\tan\frac{\Theta}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

The solution for the 2×2 portion of this matrix is discussed in detail on page 145 and 146 in Wiedemann.

2.a. From the diagram the length of OAB is $r_1\theta$-(r_1-r_2)sin θ. The length of OC is $r_2\theta$. Therefore the path length difference is

$$(r_1 - r_2)(\theta - \sin\theta).$$

2.b For an order p polytron, $\theta = 2\pi/p$. Between the linacs in the polytron need the path length difference to be (2 is from fact that there are two such bends, spreading and recombining, between each linac)

$$n\lambda_{RF} = 2(r_1 - r_2)(\theta - \sin\theta),$$

where n is an integer. But

$$(r_1 - r_2) = \frac{\Delta\gamma c}{2\pi f_c},$$

and so

$$\frac{2\pi f_c n\lambda_{RF}}{c} = 2\Delta\gamma\frac{2\pi}{p}\left(1 - (p/2\pi)\sin(2\pi/p)\right),$$

or

$$\Delta\gamma = n\frac{p}{2}\frac{f_c}{f_{RF}}\frac{1}{\left(1 - (p/2\pi)\sin(2\pi/p)\right)}.$$

But n is the increment of the path length between linacs and p/2 is the number of linacs. So the total per turn increment in the path length is $v = np/2$ RF wavelengths.

3. The formula for the synchrotron oscillation frequency is

$$M_{s'+L,s'} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} M(s)_{11} & M(s)_{12} \\ M(s)_{21} & M(s)_{22} \end{pmatrix} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix}$$

$$= \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} M(s)_{11} M_{22} - M(s)_{12} M_{21} & M(s)_{12} M_{11} - M(s)_{11} M_{12} \\ M(s)_{21} M_{22} - M(s)_{22} M_{21} & M(s)_{22} M_{11} - M(s)_{21} M_{12} \end{pmatrix}$$

$$\left(M_{s'+L,s'}\right)_{11} = M_{11}\left(M(s)_{11} M_{22} - M(s)_{12} M_{21}\right) + M_{12}\left(M(s)_{21} M_{22} - M(s)_{22} M_{21}\right)$$

$$\left(M_{s'+L,s'}\right)_{22} = M_{21}\left(M(s)_{12} M_{11} - M(s)_{11} M_{12}\right) + M_{22}\left(M(s)_{22} M_{11} - M(s)_{21} M_{12}\right)$$

$$\mathrm{Tr}\left(M_{s'+L,s'}\right) = M(s)_{11}\left(M_{11}M_{22} - M_{12}M_{21}\right) + M(s)_{22}\left(M_{11}M_{22} - M_{12}M_{21}\right) = \mathrm{Tr}\left(M_{s+L,s}\right)$$