Solutions:

2.12.a. The specified transfer matrix can be written as either

$$M_{s'+L,s'}M_{s',s} \qquad \text{or} \qquad M_{s',s}M_{s+L,s}$$
$$\therefore M_{s'+L,s'} = M_{s',s}M_{s+L,s}M_{s',s}^{-1}$$

$$M_{s'+L,s'} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} M(s)_{11} & M(s)_{12} \\ M(s)_{21} & M(s)_{22} \end{pmatrix} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix}$$
$$= \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} M(s)_{11} M_{22} - M(s)_{12} M_{21} & M(s)_{12} M_{11} - M(s)_{11} M_{12} \\ M(s)_{21} M_{22} - M(s)_{22} M_{21} & M(s)_{22} M_{11} - M(s)_{21} M_{12} \end{pmatrix}$$
$$\left(M_{s'+L,s'}\right)_{11} = M_{11}\left(M(s)_{11} M_{22} - M(s)_{12} M_{21}\right) + M_{12}\left(M(s)_{21} M_{22} - M(s)_{22} M_{21}\right)$$
$$\left(M_{s'+L,s'}\right)_{22} = M_{21}\left(M(s)_{12} M_{11} - M(s)_{11} M_{12}\right) + M_{22}\left(M(s)_{22} M_{11} - M(s)_{21} M_{12}\right)$$
$$\mathrm{Tr}\left(M_{s'+L,s'}\right) = M(s)_{11}\left(M_{11}M_{22} - M_{12}M_{21}\right) + M(s)_{22}\left(M_{11}M_{22} - M_{12}M_{21}\right) = \mathrm{Tr}\left(M_{s+L,s}\right)$$

Therefore, the phase advance of either single period unimodular matrix is the same.

2.12.b.

$$M_{s'+L,s'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos\mu_L + \begin{pmatrix} \alpha(s') & \beta(s') \\ -\gamma(s') & -\alpha(s') \end{pmatrix} \sin\mu_L =$$
$$M_{s',s}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos\mu_L + \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} \sin\mu_L\right] M_{s',s}^{-1} =$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos\mu_L + M_{s',s}\begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} M_{s',s}^{-1} \sin\mu_L$$

Subtracting the common (constant!) term from both sides of the equation and dividing by $\sin\mu_L$ gives

$$\begin{pmatrix} \alpha(s') & \beta(s') \\ -\gamma(s') & -\alpha(s') \end{pmatrix} = M_{s',s}\begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} M_{s',s}^{-1}$$

2.12.c.

$$\begin{pmatrix} \alpha(s') & \beta(s') \\ -\gamma(s') & -\alpha(s') \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix}$$

$$= \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \alpha(s)M_{22} - \beta(s)M_{21} & \beta(s)M_{11} - \alpha(s)M_{12} \\ -\gamma(s)M_{22} + \alpha(s)M_{21} & -\alpha(s)M_{11} + \gamma(s)M_{12} \end{pmatrix}$$

$$\beta(s') = M_{11}\left(\beta(s)M_{11} - \alpha(s)M_{12}\right) + M_{12}\left(-\alpha(s)M_{11} + \gamma(s)M_{12}\right)$$

$$\alpha(s') = M_{11}\left(\alpha(s)M_{22} - \beta(s)M_{21}\right) + M_{12}\left(-\gamma(s)M_{22} + \alpha(s)M_{21}\right)$$

$$\gamma(s') = -M_{21}\left(\alpha(s)M_{22} - \beta(s)M_{21}\right) - M_{22}\left(-\gamma(s)M_{22} + \alpha(s)M_{21}\right)$$

$$\beta(s') = M_{11}^2\beta(s) - 2M_{11}M_{12}\alpha(s) + M_{12}^2\gamma(s)$$

$$\alpha(s') = -M_{11}M_{21}\beta(s) + \left(M_{11}M_{22} + M_{12}M_{21}\right)\alpha(s) - M_{12}M_{22}\gamma(s)$$

$$\gamma(s') = M_{21}^2\beta(s) - 2M_{21}M_{22}\alpha(s) + M_{22}^2\gamma(s)$$

$$\beta(s+ds) \approx \beta(s) - 2ds\alpha(s) \rightarrow \frac{d\beta}{ds} = -2\alpha(s)$$

$$\alpha(s+ds) \approx K(s)ds\beta(s) + \alpha(s) - ds\gamma(s) \rightarrow \frac{d\alpha}{ds} = K(s)\beta(s) - \gamma(s)$$

$$\gamma(s+ds) \approx 2K(s)ds\alpha(s) + \gamma(s) \rightarrow \frac{d\gamma}{ds} = 2K(s)\alpha(s)$$

2.13.a. $M_{s'',s'}M_{s',s} =$

$$\begin{pmatrix} \sqrt{\dfrac{\beta(s'')}{\beta(s')}}\left[\cos\Delta\mu_{s'',s'} + \alpha(s')\sin\Delta\mu_{s'',s'}\right] & \sqrt{\beta(s'')\beta(s')}\sin\Delta\mu_{s'',s'} \\ -\dfrac{1}{\sqrt{\beta(s'')\beta(s')}}\begin{Bmatrix} \left[1+\alpha(s'')\alpha(s')\right]\sin\Delta\mu_{s'',s'} \\ \left[\alpha(s'')-\alpha(s')\right]\cos\Delta\mu_{s'',s'} \end{Bmatrix} & \sqrt{\dfrac{\beta(s')}{\beta(s'')}}\left[\cos\Delta\mu_{s'',s'} - \alpha(s'')\sin\Delta\mu_{s'',s'}\right] \end{pmatrix}$$

$$\times \begin{pmatrix} \sqrt{\dfrac{\beta(s')}{\beta(s)}}\left[\cos\Delta\mu_{s',s} + \alpha(s)\sin\Delta\mu_{s',s}\right] & \sqrt{\beta(s')\beta(s)}\sin\Delta\mu_{s',s} \\ -\dfrac{1}{\sqrt{\beta(s')\beta(s)}}\begin{Bmatrix} \left[1+\alpha(s')\alpha(s)\right]\sin\Delta\mu_{s',s} \\ \left[\alpha(s')-\alpha(s)\right]\cos\Delta\mu_{s',s} \end{Bmatrix} & \sqrt{\dfrac{\beta(s)}{\beta(s')}}\left[\cos\Delta\mu_{s',s} - \alpha(s')\sin\Delta\mu_{s',s}\right] \end{pmatrix} =$$

$$
\begin{pmatrix}
\sqrt{\dfrac{\beta(s'')}{\beta(s)}}\left[\begin{array}{l}
\cos\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s} \\
+\alpha(s')\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s} \\
+\alpha(s)\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
+\alpha(s')\alpha(s)\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
-\left[1+\alpha(s')\alpha(s)\right]\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
-\left[\alpha(s')-\alpha(s)\right]\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}
\end{array}\right] &
\sqrt{\beta(s'')\beta(s)}\left[\begin{array}{l}
\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
+\alpha(s')\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
+\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s} \\
-\alpha(s')\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}
\end{array}\right] \\
-\dfrac{1}{\sqrt{\beta(s'')\beta(s)}}\left[\begin{array}{l}
\left[1+\alpha(s'')\alpha(s')\right]\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s} \\
+\left[\alpha(s'')-\alpha(s')\right]\cos\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s} \\
+\alpha(s)\left[1+\alpha(s'')\alpha(s')\right]\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
+\alpha(s)\left[\alpha(s'')-\alpha(s')\right]\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
+\left[1+\alpha(s')\alpha(s)\right]\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
+\left[\alpha(s')-\alpha(s)\right]\cos\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s} \\
-\alpha(s'')\left[1+\alpha(s')\alpha(s)\right]\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
-\alpha(s'')\left[\alpha(s')-\alpha(s)\right]\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}
\end{array}\right] &
\sqrt{\dfrac{\beta(s)}{\beta(s'')}}\left[\begin{array}{l}
-\left[1+\alpha(s'')\alpha(s')\right]\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
-\left[\alpha(s'')-\alpha(s')\right]\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
\cos\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s} \\
-\alpha(s'')\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s} \\
-\alpha(s')\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
+\alpha(s'')\alpha(s')\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}
\end{array}\right]
\end{pmatrix} =
$$

$$
\begin{pmatrix}
\sqrt{\dfrac{\beta(s'')}{\beta(s)}}\left[\begin{array}{l}
\cos\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s} \\
+\alpha(s)\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
-\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
+\alpha(s)\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}
\end{array}\right] &
\sqrt{\beta(s'')\beta(s)}\left[\begin{array}{l}
\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
+\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}
\end{array}\right] \\
-\dfrac{1}{\sqrt{\beta(s'')\beta(s)}}\left[\begin{array}{l}
\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s} \\
+\alpha(s'')\cos\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s} \\
+\alpha(s)\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
+\alpha(s)\alpha(s'')\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
+\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
-\alpha(s)\cos\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s} \\
-\alpha(s'')\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
-\alpha(s'')\alpha(s)\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}
\end{array}\right] &
\sqrt{\dfrac{\beta(s)}{\beta(s'')}}\left[\begin{array}{l}
-\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
-\alpha(s'')\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s} \\
\cos\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s} \\
-\alpha(s'')\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}
\end{array}\right]
\end{pmatrix} =
$$

$$
\begin{pmatrix}
\sqrt{\dfrac{\beta(s'')}{\beta(s)}}\left[\begin{array}{l}
\cos\left(\Delta\mu_{s'',s'}+\Delta\mu_{s',s}\right) \\
+\alpha(s)\sin\left(\Delta\mu_{s'',s'}+\Delta\mu_{s',s}\right)
\end{array}\right] &
\sqrt{\beta(s'')\beta(s)}\sin\left(\Delta\mu_{s'',s'}+\Delta\mu_{s',s}\right) \\
-\dfrac{1}{\sqrt{\beta(s'')\beta(s)}}\left\{\begin{array}{l}
\left[1+\alpha(s'')\alpha(s)\right]\sin\left(\Delta\mu_{s'',s'}+\Delta\mu_{s',s}\right) \\
+\left[\alpha(s'')-\alpha(s)\right]\cos\left(\Delta\mu_{s'',s'}+\Delta\mu_{s',s}\right)
\end{array}\right\} &
\sqrt{\dfrac{\beta(s)}{\beta(s'')}}\left[\begin{array}{l}
\cos\left(\Delta\mu_{s'',s'}+\Delta\mu_{s',s}\right) \\
-\alpha(s'')\sin\left(\Delta\mu_{s'',s'}+\Delta\mu_{s',s}\right)
\end{array}\right]
\end{pmatrix} =
$$

$$\begin{pmatrix} \sqrt{\dfrac{\beta(s")}{\beta(s)}}\left[\cos\Delta\mu_{s",s}+\alpha(s)\sin\Delta\mu_{s",s}\right] & \sqrt{\beta(s")\beta(s)}\sin\Delta\mu_{s",s} \\ -\dfrac{1}{\sqrt{\beta(s")\beta(s)}}\begin{Bmatrix}\left[1+\alpha(s")\alpha(s)\right]\sin\Delta\mu_{s",s} \\ +\left[\alpha(s")-\alpha(s)\right]\cos\Delta\mu_{s",s}\end{Bmatrix} & \sqrt{\dfrac{\beta(s)}{\beta(s")}}\left[\cos\Delta\mu_{s",s}-\alpha(s")\sin\Delta\mu_{s",s}\right] \end{pmatrix}=M_{s",s}$$

2.13.b.

$$\left(M_{s',s}\right)_{11}=\sqrt{\frac{\beta(s')}{\beta(s)}}\left[\cos\Delta\mu_{s',s}+\alpha(s)\sin\Delta\mu_{s',s}\right]$$

$$\left(M_{s',s}\right)_{12}=\sqrt{\beta(s')\beta(s)}\sin\Delta\mu_{s',s}$$

$$\beta(s)\left(M_{s',s}\right)_{11}-\alpha(s)\left(M_{s',s}\right)_{12}=\sqrt{\beta(s')\beta(s)}\cos\Delta\mu_{s',s}$$

$$\therefore \tan\Delta\mu_{s',s}=\frac{\left(M_{s',s}\right)_{12}}{\beta(s)\left(M_{s',s}\right)_{11}-\alpha(s)\left(M_{s',s}\right)_{12}}$$

2.13.c. Let M_1 be the transfer matrix from s to s' and M_2 be the transfer matrix from s' to $s"$

$$\tan\Delta\mu_{s',s}=\frac{(M_1)_{12}}{\beta(s)(M_1)_{11}-\alpha(s)(M_1)_{12}}$$

$$\tan\Delta\mu_{s",s'}=\frac{(M_2)_{12}}{\beta(s')(M_2)_{11}-\alpha(s')(M_2)_{12}}$$

and the phase advance from s to $s"$ must satisfy

$$\tan\Delta\mu_{s",s}=\frac{(M_2M_1)_{12}}{\beta(s)(M_2M_1)_{11}-\alpha(s)(M_2M_1)_{12}}=$$

$$\frac{(M_2)_{11}(M_1)_{12}+(M_2)_{12}(M_1)_{22}}{\beta(s)\left((M_2)_{11}(M_1)_{11}+(M_2)_{12}(M_1)_{21}\right)-\alpha(s)\left((M_2)_{11}(M_1)_{12}+(M_2)_{12}(M_1)_{22}\right)}.$$

But the tangent addition formula is

$$\tan\left(\Delta\mu_{s",s'}+\Delta\mu_{s',s}\right)=\frac{\tan\Delta\mu_{s",s'}+\tan\Delta\mu_{s',s}}{1-\tan\Delta\mu_{s",s'}\tan\Delta\mu_{s',s}}=$$

$$\frac{\dfrac{(M_1)_{12}}{\beta(s)(M_1)_{11}-\alpha(s)(M_1)_{12}}+\dfrac{(M_2)_{12}}{\beta(s')(M_2)_{11}-\alpha(s')(M_2)_{12}}}{1-\dfrac{(M_1)_{12}}{\beta(s)(M_1)_{11}-\alpha(s)(M_1)_{12}}\dfrac{(M_2)_{12}}{\beta(s')(M_2)_{11}-\alpha(s')(M_2)_{12}}}=$$

$$\frac{\beta(s)}{\beta(s)}\frac{(M_1)_{12}\left(\beta(s')(M_2)_{11}-\alpha(s')(M_2)_{12}\right)+(M_2)_{12}\left(\beta(s)(M_1)_{11}-\alpha(s)(M_1)_{12}\right)}{\left(\beta(s')(M_2)_{11}-\alpha(s')(M_2)_{12}\right)\left(\beta(s)(M_1)_{11}-\alpha(s)(M_1)_{12}\right)-(M_1)_{12}(M_2)_{12}}.$$

Because

$$\text{Numerator} = (M_1)_{12}\,\beta(s)\left[\beta(s')(M_2)_{11}-\alpha(s')(M_2)_{12}\right]+(M_2)_{12}\,\beta(s)\left[\beta(s)(M_1)_{11}-\alpha(s)(M_1)_{12}\right]$$

$$= (M_1)_{12}(M_2)_{11}\left[(M_1)_{12}^2\left(1+\alpha^2(s)\right)-2(M_1)_{12}(M_1)_{11}\,\alpha(s)\beta(s)+(M_1)_{11}^2\,\beta^2(s)\right]$$

$$-(M_1)_{12}(M_2)_{12}\left[-(M_1)_{22}(M_1)_{12}\left(1+\alpha^2(s)\right)+\left((M_1)_{11}(M_1)_{22}+(M_1)_{12}(M_1)_{21}\right)\alpha(s)\beta(s)-(M_1)_{21}(M_1)_{11}\,\beta^2(s)\right]$$

$$+(M_2)_{12}\,\beta^2(s)(M_1)_{11}\left[(M_1)_{11}(M_1)_{22}-(M_1)_{12}(M_1)_{21}\right]$$

$$-(M_2)_{12}\,\beta(s)\alpha(s)(M_1)_{12}\left[(M_1)_{11}(M_1)_{22}-(M_1)_{12}(M_1)_{21}\right]$$

$$=(M_1)_{12}(M_2)_{11}\left[(M_1)_{12}^2+\left(\beta(s)(M_1)_{11}-\alpha(s)(M_1)_{12}\right)^2\right]$$

$$+(M_1)_{22}(M_2)_{12}(M_1)_{12}^2+(M_1)_{22}(M_2)_{12}(M_1)_{12}^2\,\alpha^2(s)$$

$$-2(M_2)_{12}(M_1)_{12}(M_1)_{11}(M_1)_{22}\,\beta(s)\alpha(s)$$

$$+(M_2)_{12}(M_1)_{11}(M_1)_{11}(M_1)_{22}\,\beta^2(s)$$

$$=\left[(M_2)_{11}(M_1)_{12}+(M_2)_{12}(M_1)_{22}\right]\left[(M_1)_{12}^2+\left(\beta(s)(M_1)_{11}-\alpha(s)(M_1)_{12}\right)^2\right]$$

$$\text{Denominator} = \beta(s)\left[\beta(s')(M_2)_{11}-\alpha(s')(M_2)_{12}\right]\left[\beta(s)(M_1)_{11}-\alpha(s)(M_1)_{12}\right]-\beta(s)(M_1)_{12}(M_2)_{12}$$

$$= (M_2)_{11}\left[(M_1)_{12}^2+\left(\beta(s)(M_1)_{11}-\alpha(s)(M_1)_{12}\right)^2\right]\left[\beta(s)(M_1)_{11}-\alpha(s)(M_1)_{12}\right]$$

$$-(M_2)_{12}\left[-(M_1)_{22}(M_1)_{12}\left(1+\alpha^2(s)\right)+\left((M_1)_{11}(M_1)_{22}+(M_1)_{12}(M_1)_{21}\right)\alpha(s)\beta(s)-(M_1)_{21}(M_1)_{11}\,\beta^2(s)\right]$$

$$\times\left[\beta(s)(M_1)_{11}-\alpha(s)(M_1)_{12}\right]$$

$$-\beta(s)(M_1)_{12}(M_2)_{12}\left[(M_1)_{11}(M_1)_{22}-(M_1)_{12}(M_1)_{21}\right]$$

$$=\left[(M_1)_{12}^2+\left(\beta(s)(M_1)_{11}-\alpha(s)(M_1)_{12}\right)^2\right]\left[\beta(s)(M_2)_{11}(M_1)_{11}-\alpha(s)(M_2)_{11}(M_1)_{12}\right]$$

$$-(M_2)_{12}(M_1)_{22}(M_1)_{12}(M_1)_{12}\,\alpha^3(s)$$

$$+\left[2(M_2)_{12}(M_1)_{22}(M_1)_{12}(M_1)_{11}+(M_2)_{12}(M_1)_{12}(M_1)_{21}(M_1)_{12}\right]\alpha^2(s)\beta(s)$$

$$-\left[(M_2)_{12}(M_1)_{11}(M_1)_{22}(M_1)_{11}+2(M_2)_{12}(M_1)_{21}(M_1)_{12}(M_1)_{11}\right]\alpha(s)\beta^2(s)$$

$$+(M_2)_{12}(M_1)_{21}(M_1)_{11}(M_1)_{11}\,\beta^3(s)$$

$$+\beta(s)(M_1)_{12}(M_2)_{12}(M_1)_{12}(M_1)_{21}-\alpha(s)(M_2)_{12}(M_1)_{22}(M_1)_{12}(M_1)_{12}$$

$$=\left[(M_1)_{12}^2+\left(\beta(s)(M_1)_{11}-\alpha(s)(M_1)_{12}\right)^2\right]\left[\beta(s)(M_2)_{11}(M_1)_{11}-\alpha(s)(M_2)_{11}(M_1)_{12}\right]$$

$$+\left[(M_1)_{12}^2+\left(\beta(s)(M_1)_{11}-\alpha(s)(M_1)_{12}\right)^2\right]\left[\beta(s)(M_2)_{12}(M_1)_{21}-\alpha(s)(M_2)_{12}(M_1)_{22}\right]$$

$$=\left[(M_1)_{12}^2+\left(\beta(s)(M_1)_{11}-\alpha(s)(M_1)_{12}\right)^2\right]$$

$$\times\left[\beta(s)\left((M_2)_{11}(M_1)_{11}+(M_2)_{12}(M_1)_{21}\right)-\alpha(s)\left((M_2)_{11}(M_1)_{12}+(M_2)_{12}(M_1)_{22}\right)\right]$$

Therefore

$$\tan\left(\Delta\mu_{s'',s'}+\Delta\mu_{s',s}\right)=\frac{(M_2)_{11}(M_1)_{12}+(M_2)_{12}(M_1)_{22}}{\beta(s)\left((M_2)_{11}(M_1)_{11}+(M_2)_{12}(M_1)_{21}\right)-\alpha(s)\left((M_2)_{11}(M_1)_{12}+(M_2)_{12}(M_1)_{22}\right)}$$
$$=\tan\Delta\mu_{s'',s}$$

Now because we know both the sign of the numerator and denominator, the inverse tangent function gives a unique solution in the principal branch. So

$$\Delta\mu_{s'',s'}+\Delta\mu_{s',s}=\Delta\mu_{s'',s}+2\pi k \quad \text{for some } k$$

Requiring the phase advance be a continuous function of s throughout its domain ensures that k must be zero.