Physics 704/804 Electromagnetic Theory II

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For Circular Motion

$$\vec{x}(t) = \rho\left[\cos\omega t\hat{x} + \sin\omega t\hat{y}\right]$$

$$\vec{\mathrm{v}}(t) = \rho\omega\left[-\sin\omega t\hat{x} + \cos\omega t\hat{y}\right] \quad \dot{\vec{\mathrm{v}}}(t) = -\omega^2\vec{x}$$

$$\dot{\vec{\beta}}^2 = \left(\frac{\rho\omega^2}{c}\right)^2 \qquad \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2 = \left(\frac{\rho\omega}{c}\frac{\rho\omega^2}{c}\right)^2$$

$$P = \frac{q^2}{6\pi\varepsilon_0 c}\gamma^4\left(\frac{\rho\omega^2}{c}\right)^2 = \frac{q^2}{6\pi\varepsilon_0 c}\gamma^4\beta^4\frac{c^2}{\rho^2} = \frac{q^2 c}{6\pi\varepsilon_0\rho^2}\gamma^4\beta^4$$

Energy loss per turn

$$\delta E = \frac{2\pi\rho}{\beta c}\frac{q^2 c}{6\pi\varepsilon_0\rho^2}\gamma^4\beta^4 = \frac{q^2}{3\varepsilon_0\rho}\gamma^4\beta^3$$

$$\delta E(\mathrm{MeV}) = 8.85\times 0.01\frac{\left[E(\mathrm{GeV})\right]^4}{\rho(\mathrm{m})}$$

Relativistic Peaking

In far field after short acceleration

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c} \frac{\left|\hat{n}\times\left\{\left(\hat{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}\right|^2}{\left(1-\hat{n}\cdot\vec{\beta}\right)^5}$$

$$\frac{dP(t')}{d\Omega} = \frac{q^2\dot{\beta}^2}{16\pi^2\varepsilon_0 c} \frac{\sin^2\theta}{\left(1-\beta\cos\theta\right)^5}$$

$$\theta_{\max} \to \frac{1}{2\gamma}$$

For circular motions

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c} \frac{\dot{\beta}^2}{\left(1-\beta\cos\theta\right)^3}\left[1-\frac{\sin^2\theta\cos^2\varphi}{\gamma^2\left(1-\beta\cos\theta\right)^2}\right]$$

Spectrum Radiated by Motion

$$\frac{dE}{d\Omega} = \int_{-\infty}^{\infty} \frac{dP}{d\Omega} dt = \int_{-\infty}^{\infty} \vec{E} \times \vec{H} \cdot \hat{n} R^2 dt = \frac{1}{c\mu_0} \int_{-\infty}^{\infty} \left(\vec{E} \cdot \vec{E}\right) R^2 dt =$$

$$\frac{1}{c\mu_0}\left(\frac{q}{8\pi^2\varepsilon_0 c}\right)^2 \int_{-\infty}^{\infty}\int\int\int\int \left[\frac{\hat{n} \times \left\{\left(\hat{n} - \vec{\beta}\right) \times \dot{\vec{\beta}}\right\}}{\left(1 - \hat{n} \cdot \vec{\beta}\right)^2}(t')\right] \cdot \left[\frac{\hat{n} \times \left\{\left(\hat{n} - \vec{\beta}\right) \times \dot{\vec{\beta}}\right\}}{\left(1 - \hat{n} \cdot \vec{\beta}\right)^2}(t'')\right]$$

$$\times e^{i\omega\left[R\sqrt{1-2\hat{n}\cdot\vec{r}(t')/R+\left(\hat{n}\cdot\vec{r}(t')\right)^2/R^2}/c-t+t'\right]} e^{i\omega'\left[\sqrt{1-2\hat{n}\cdot\vec{r}(t'')/R+\left(\hat{n}\cdot\vec{r}(t'')\right)^2/R^2}/c-t+t''\right]} dt'\,d\omega\,dt''\,d\omega'\,dt =$$

clearing the unprimed time integral and omega prime integral with delta represntation

$$\frac{2\pi}{c\mu_0}\left(\frac{q}{8\pi^2\varepsilon_0 c}\right)^2 \int_{-\infty}^{\infty}\int\int \left[\frac{\hat{n} \times \left\{\left(\hat{n} - \vec{\beta}\right) \times \dot{\vec{\beta}}\right\}}{\left(1 - \hat{n} \cdot \vec{\beta}\right)^2}(t')\right] \cdot \left[\frac{\hat{n} \times \left\{\left(\hat{n} - \vec{\beta}\right) \times \dot{\vec{\beta}}\right\}}{\left(1 - \hat{n} \cdot \vec{\beta}\right)^2}(t'')\right]$$

$$\times e^{i\omega\left[-\hat{n}\cdot\vec{r}(t')/c-t+t'\right]} e^{-i\omega\left[-\hat{n}\cdot\vec{r}(t'')/c-t+t''\right]} dt'\,dt''\,d\omega$$

$$\frac{d^2E}{d\omega d\Omega}=\frac{q^2}{32\pi^3\varepsilon_0 c}\left|\int_{-\infty}^{\infty}\frac{\hat{n}\times\left\{\left(\hat{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}}{\left(1-\hat{n}\cdot\vec{\beta}\right)^2}e^{i\omega\left[-\hat{n}\cdot\vec{r}(t')/c-t+t'\right]}dt'\right|^2$$

$$\frac{d^2E}{d\omega d\Omega}=\frac{q^2\omega^2}{32\pi^3\varepsilon_0 c}\left|\int_{-\infty}^{\infty}\hat{n}\times\left(\hat{n}\times\vec{\beta}\right)e^{i\omega\left[t'-\hat{n}\cdot\vec{r}(t')/c\right]}dt'\right|^2$$

Factor of two difference from Jackson because he combines positive frequency and negative frequency contributions in one positive frequency integral. I don't like because Parseval's formula, etc. don't work! I've written papers about performing this calculation in new regimes of high intensity pulsed lasers.

Snychrotron Radiation

Case of instantaneous circular motion

$$\xi = \frac{\omega\rho}{3c}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}$$

$$\frac{d^2E}{d\omega d\Omega} = \frac{q^2}{24\pi^3\varepsilon_0 c}\left(\frac{\omega\rho}{c}\right)^2\left(\frac{1}{\gamma^2}+\theta^2\right)^2\left[K_{2/3}^2(\xi) + \frac{\theta^2}{(1/\gamma)^2+\theta^2}K_{1/3}^2(\xi)\right]$$

$$\frac{dE}{d\omega} = \frac{\sqrt{3}q^2}{8\pi\varepsilon_0 c}\gamma\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{\infty} K_{5/3}(x)dx$$

critical frequecy

$$\omega_c = \frac{3}{2}\gamma^3\left(\frac{c}{\rho}\right)$$

Photon Interpretation

Photon Energy Distribution

$$\frac{dN}{dy} = \frac{I}{\hbar\omega_c}\frac{9\sqrt{3}}{64\pi^2\varepsilon_0}\int_y^\infty K_{5/3}(x)dx$$

Number emitted per revolution

$$N = \frac{5\pi}{\sqrt{3}}\gamma\alpha \qquad \alpha \text{ fine structure constant} = \frac{e^2}{4\pi\varepsilon_0\hbar c}$$

Average energy

$$\langle\hbar\omega\rangle = \frac{8}{15\sqrt{3}}\hbar\omega_c$$

Undulator emission

Wiggle electrons at a certain frequency with alternating pole magnets

$$K = \frac{eB\lambda}{2\pi mc}$$

emission frequency

$$\omega' = \gamma\left(1-\beta\cos\theta\right)\omega \approx \omega\left(1+\gamma^2\theta^2\right)/\,2\bar{\gamma}^2$$

$$\bar{\gamma} = \frac{\gamma}{\sqrt{1+K^2/2}}$$

By Larmor, in undulator

$$P = \frac{q^2 c \overline{\gamma}^2 K^2 k_0^2}{12\pi\varepsilon_0}$$

Number emitted per pass of undulator

$$N_\gamma = \frac{2\pi}{3} \alpha N K^2$$

By uncertainty relation, spectral width of emission is $1/N$

Thomson Scattering

Nonrelativistic Compton Scattering

$$\frac{dP}{d\Omega} = \frac{e^2}{16\pi^2\varepsilon_0 c^3}\left|\vec{\varepsilon}^{\,*}\cdot\dot{\vec{\mathrm{v}}}\right|^2$$

Incident plane wave

$$\vec{E} = \vec{\varepsilon}_0 E_0 e^{i\left(\vec{k}_0\cdot\vec{x}-\omega t\right)}$$

$$\dot{\vec{\mathrm{v}}} = \vec{\varepsilon}_0\,\frac{e}{m}E_0 e^{i\left(\vec{k}_0\cdot\vec{x}-\omega t\right)}$$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^2}{32\pi^2\varepsilon_0 c^3}\left|E_0\right|^2\left(\frac{e}{m}\right)^2\left|\vec{\varepsilon}^{\,*}\cdot\vec{\varepsilon}_0\right|^2$$

Scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{dP_{scattered} / d\Omega}{I_{incident}} = \frac{\frac{e^2}{32\pi^2\varepsilon_0 c^3}\left|E_0\right|^2\left(\frac{e}{m}\right)^2\left|\vec{\varepsilon}^* \cdot \vec{\varepsilon}_0\right|^2}{\frac{\varepsilon_0 c}{2}\left|E_0\right|^2}$$

$$= \left(\frac{e^2}{4\pi\varepsilon_0 m c^2}\right)^2 \left|\vec{\varepsilon}^* \cdot \vec{\varepsilon}_0\right|^2 = r_e^2 \left|\vec{\varepsilon}^* \cdot \vec{\varepsilon}_0\right|^2$$

r_e classical electron radius

$$\frac{d\sigma}{d\Omega} = \frac{r_e^2}{2}\left(1 + \cos^2\theta\right)$$

$$\sigma_T = \frac{8\pi}{3} r_e^2$$