Physics 704/804 Electromagnetic Theory II

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Radiation From Relativistic Electrons

From discussion early in the course, in the Lorenz gauge the equation for the potentials is

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi=-\frac{\rho}{\varepsilon_0}$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A}=-\mu_0\vec{J}$$

The solution, using the retarded Green Function is

$$\phi\left(\vec{x},t\right)=\frac{1}{4\pi\varepsilon_0}\int d^3x'dt'\,\frac{\rho\left(\vec{x}',t'\right)}{\left|\vec{x}-\vec{x}'\right|}\delta\left(\left|\vec{x}-\vec{x}'\right|/c-t+t'\right)$$

$$\vec{A}\left(\vec{x},t\right)=\frac{\mu_0}{4\pi}\int d^3x'dt'\,\frac{\vec{J}\left(\vec{x}',t'\right)}{\left|\vec{x}-\vec{x}'\right|}\delta\left(\left|\vec{x}-\vec{x}'\right|/c-t+t'\right)$$

Delta Function Representation

$$\phi(\vec{x},t)=\frac{1}{8\pi^2\varepsilon_0}\int d^3x'dt'd\omega\,\frac{\rho(\vec{x}',t')}{|\vec{x}-\vec{x}'|}e^{i\omega[|\vec{x}-\vec{x}'|/c-(t-t')]}$$

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{8\pi^2}\int d^3x'dt'd\omega\,\frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|}e^{i\omega[|\vec{x}-\vec{x}'|/c-(t-t')]}$$

$$\rho(\vec{x},t)=q\delta^3(\vec{x}-\vec{r}(t)) \qquad \vec{J}(\vec{x},t)=q\vec{\mathrm{v}}(t)\delta^3(\vec{x}-\vec{r}(t))$$

$$\phi(\vec{x},t)=\frac{q}{8\pi^2\varepsilon_0}\int dt'd\omega\,\frac{1}{|\vec{x}-\vec{r}(t')|}e^{i\omega[|\vec{x}-\vec{r}(t')|/c-(t-t')]}$$

$$\vec{A}(\vec{x},t)=\frac{q\mu_0}{8\pi^2}\int dt'd\omega\,\frac{\vec{\mathrm{v}}(t')}{|\vec{x}-\vec{r}(t')|}e^{i\omega[|\vec{x}-\vec{r}(t')|/c-(t-t')]}$$

Lienard-Weichert Potentials

$$\phi(\vec{x},t) = \frac{q}{4\pi\varepsilon_0}\int dt' \frac{\delta\left(\left|\vec{x}-\vec{r}(t')\right|/c-(t-t')\right)}{\left|\vec{x}-\vec{r}(t')\right|}$$

$$\vec{A}(\vec{x},t) = \frac{q\mu_0}{4\pi}\int dt' \frac{\vec{\mathrm{v}}(t')\,\delta\left(\left|\vec{x}-\vec{r}(t')\right|/c-(t-t')\right)}{\left|\vec{x}-\vec{r}(t')\right|}$$

$$\phi(\vec{x},t) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\left|\vec{x}-\vec{r}(t')\right|\left(1-\hat{n}\cdot\vec{\beta}(t')\right)}\right)_{ret}$$

$$\vec{A}(\vec{x},t) = \frac{q\mu_0}{4\pi}\left(\frac{\vec{\mathrm{v}}(t')}{\left|\vec{x}-\vec{r}(t')\right|\left(1-\hat{n}\cdot\vec{\beta}(t')\right)}\right)_{ret}$$

EM Field Radiated

$$\phi(\vec{x},t) = \frac{q}{8\pi^2\varepsilon_0}\int dt'd\omega \frac{1}{\left|\vec{x}-\vec{r}(t')\right|} e^{i\omega\left[\left|\vec{x}-\vec{r}(t')\right|/c-(t-t')\right]}$$

$$\vec{A}(\vec{x},t) = \frac{q\mu_0}{8\pi^2}\int dt'd\omega \frac{\vec{\mathrm{v}}(t')}{\left|\vec{x}-\vec{r}(t')\right|} e^{i\omega\left[\left|\vec{x}-\vec{r}(t')\right|/c-(t-t')\right]}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{n}-\vec{\beta}}{\gamma^2\left(1-\hat{n}\cdot\vec{\beta}\right)^3 R^2}\right]_{ret} + \frac{q}{4\pi\varepsilon_0 c}\left[\frac{\hat{n}\times\left\{\left(\hat{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}}{\left(1-\hat{n}\cdot\vec{\beta}\right)^3 R}\right]_{ret}$$

$$\vec{B} = \hat{n}\times\vec{E}\,/\,c$$

$$\vec{\nabla}\frac{1}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}=-\frac{\hat{n}}{\left|\vec{x}-\vec{r}\left(t'\right)\right|^2} \qquad \vec{\nabla}\left|\vec{x}-\vec{r}\left(t'\right)\right|=\hat{n}$$

$$\frac{d}{dt'}\frac{1}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}=\frac{\hat{n}\cdot\vec{\beta}c}{\left|\vec{x}-\vec{r}\left(t'\right)\right|^2} \qquad \frac{d\hat{n}}{dt'}=\frac{-d\vec{r}\,/\,dt'+\hat{n}\left(\hat{n}\cdot d\vec{r}\,/\,dt'\right)}{\left|\vec{x}-\vec{r}\left(t'\right)\right|} \qquad \cdots$$

$$-\nabla\phi\left(\vec{x},t\right)=\frac{q}{8\pi^2\varepsilon_0}\int dt'd\omega\,\frac{\hat{n}\left(1-i\omega\left|\vec{x}-\vec{r}\left(t'\right)\right|/c\right)}{\left|\vec{x}-\vec{r}\left(t'\right)\right|^2}e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}$$

$$-\frac{\partial}{\partial t}\vec{A}\left(\vec{x},t\right)=\frac{q\mu_0}{8\pi^2}\int dt'd\omega\,\frac{\vec{\mathrm{v}}\left(t'\right)i\omega}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}$$

$$\frac{d}{dt'}e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}=i\omega\left(1-\vec{\beta}\left(t'\right)\cdot\hat{n}\left(t'\right)\right)e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}$$

$$\vec{E}(\vec{x},t)=\frac{q}{8\pi^2\varepsilon_0}\int dt'd\omega e^{i\omega\left[\left|\vec{x}-\vec{r}(t')\right|/c-(t-t')\right]}\left[\frac{\hat{n}}{\left|\vec{x}-\vec{r}(t')\right|^2}+\frac{i\omega\left(\vec{\beta}-\hat{n}\right)}{c\left|\vec{x}-\vec{r}(t')\right|}\right]$$

integrate by parts to get final result

$$\vec{E}(\vec{x},t)_{vel}=\frac{q}{8\pi^2\varepsilon_0}\int dt'd\omega\frac{e^{i\omega\left[\left|\vec{x}-\vec{r}(t')\right|/c-(t-t')\right]}}{\left(1-\vec{\beta}\cdot\hat{n}\right)^2\left|\vec{x}-\vec{r}(t')\right|^2}$$

$$\times\left[\begin{array}{l}\hat{n}\left(\begin{array}{l}1-2\vec{\beta}\cdot\hat{n}+\left(\vec{\beta}\cdot\hat{n}\right)^2+\vec{\beta}\cdot\hat{n}-\left(\vec{\beta}\cdot\hat{n}\right)^2+\vec{\beta}\cdot\hat{n}-\left(\vec{\beta}\cdot\hat{n}\right)^2\\ -\beta^2+\left(\vec{\beta}\cdot\hat{n}\right)^2\end{array}\right)\\ -\vec{\beta}\left(1-\vec{\beta}\cdot\hat{n}+\vec{\beta}\cdot\hat{n}-\left(\vec{\beta}\cdot\hat{n}\right)^2-\beta^2+\left(\vec{\beta}\cdot\hat{n}\right)^2\right)\end{array}\right]$$

$$\vec{E}(\vec{x},t)_{acc} = \frac{q}{8\pi^2\varepsilon_0 c}\int dt'd\omega \frac{e^{i\omega\left[\left|\vec{x}-\vec{r}(t')\right|/c-(t-t')\right]}}{\left(1-\vec{\beta}\cdot\hat{n}\right)^2\left|\vec{x}-\vec{r}(t')\right|}$$

$$\times\begin{bmatrix} \hat{n}\left(\dot{\vec{\beta}}\cdot\hat{n}\right) \\ -\dot{\vec{\beta}}\left(1-\vec{\beta}\cdot\hat{n}\right)-\vec{\beta}\left(\dot{\vec{\beta}}\cdot\hat{n}\right) \end{bmatrix}$$

$$= \frac{q}{8\pi^2\varepsilon_0 c}\int dt'd\omega \frac{e^{i\omega\left[\left|\vec{x}-\vec{r}(t')\right|/c-(t-t')\right]}}{\left(1-\vec{\beta}\cdot\hat{n}\right)^2\left|\vec{x}-\vec{r}(t')\right|}\left[\hat{n}\times\left\{\left(\hat{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}\right]$$

Larmor's Formula

For small velocities can neglect retardation

$$\vec{E}\left(\vec{x},t\right)_{acc} = \frac{q}{4\pi\varepsilon_0 c}\left[\hat{n}\times\left\{\hat{n}\times\dot{\vec{\beta}}\right\}\right]/R$$

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0\mu_0 c^3}\left|\left[\hat{n}\times\left\{\hat{n}\times\dot{\vec{\beta}}\right\}\right]\right|^2$$

$$= \frac{q^2}{16\pi^2\varepsilon_0 c^3}\left|\dot{\vec{\mathrm{v}}}\right|^2 \sin^2\theta$$

$$P = \frac{q^2}{6\pi\varepsilon_0 c^3}\left|\dot{\vec{\mathrm{v}}}\right|^2$$

Lienard's Relativistic Generalization

For photon emission, the emitted energy increment dE and duration increment dt are time components of (different) 4-vectors. Therefore their ratio, the power emitted

$$P = \frac{dE}{dt}$$

is a Lorentz invariant! Must be of form

$$P = -\frac{q^2}{6\pi\varepsilon_0 m^2 c^3}\left(\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau}\right)$$

$$E = \gamma mc^2 \qquad \vec{p} = \gamma m\vec{v} \qquad d\tau = dt/\gamma$$

$$P = -\frac{q^2}{6\pi\varepsilon_0 m^2 c^3}\left(\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau}\right) = \frac{q^2}{6\pi\varepsilon_0 c}\gamma^6\left(\left(\dot{\vec{\beta}}\right)^2 - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right)$$

Very strong energy dependence!

For Circular Motion

$$\vec{x}(t) = \rho\left[\cos\omega t\hat{x} + \sin\omega t\hat{y}\right]$$

$$\vec{\mathrm{v}}(t) = \rho\omega\left[-\sin\omega t\hat{x} + \cos\omega t\hat{y}\right] \quad \dot{\vec{\mathrm{v}}}(t) = -\omega^2\vec{x}$$

$$\dot{\vec{\beta}}^2 = \left(\frac{\rho\omega^2}{c}\right)^2 \qquad \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2 = \left(\frac{\rho\omega}{c}\frac{\rho\omega^2}{c}\right)^2$$

$$P = \frac{q^2}{6\pi\varepsilon_0 c}\gamma^4\left(\frac{\rho\omega^2}{c}\right)^2 = \frac{q^2}{6\pi\varepsilon_0 c}\gamma^4\beta^4\frac{c^2}{\rho^2} = \frac{q^2 c}{6\pi\varepsilon_0\rho^2}\gamma^4\beta^4$$

Energy loss per turn

$$\delta E = \frac{2\pi\rho}{\beta c}\frac{q^2 c}{6\pi\varepsilon_0\rho^2}\gamma^4\beta^4 = \frac{q^2}{3\varepsilon_0\rho}\gamma^4\beta^3$$

$$\delta E(\mathrm{MeV}) = 8.85\times 0.01\frac{\left[E(\mathrm{GeV})\right]^4}{\rho(\mathrm{m})}$$