Physics 704/804 Electromagnetic Theory II

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Covariance of Electrodynamics

Electric charge a Lorentz invariant

Non-relativistic Lorentz Force

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)$$

$$p^{\mu} = m\left(U^{0}, \vec{U}\right) = m\left(\gamma c, \gamma \vec{\mathrm{v}}\right)$$

$$\frac{d\vec{p}}{d\tau} = \frac{q}{c}\left(U^{0}\vec{E} + \vec{U} \times c\vec{B}\right)$$

$$\frac{dp^{0}}{d\tau} = \frac{q}{c}\vec{U} \cdot \vec{E}$$

$\vec{U} \cdot \vec{E}$ time component of a 4-vector $\rightarrow$

$\vec{E}$ the time-space components of a 4-two tensor

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

can be written in the form

$$\partial_a J^\alpha = 0$$

where the current density 4-vector is

$$J^\mu = \left(c\rho, \vec{J} \right)$$

The continuity equation may indeed be frame covariant, because the 4-density has zero 4-divergence! So assume 4-density is a Lorentz 4-vector as above.

4-potential

Recall

$$\Box \vec{A} = \mu_0 \vec{J} \qquad \Box \phi = \frac{\rho}{\varepsilon_0}$$

but the wave operator $\Box$ is a Lorentz invariant operator, the same in any frame.

$$\therefore\ A^{\mu} = \left(\phi, c\vec{A}\right)$$

must be a 4-vector, called the 4-potential. Have chosen the units so the gradiant gives electric field units

$$\Box A^{\mu} = \mu_0 c J^{\mu}$$

is the inhomogenous Maxwell Equation

Antisymetric Field Strength

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \phi \quad \vec{B} = \nabla \times \vec{A}$$

$$F^{\alpha\beta} = \partial^{\alpha} A^{\beta} - \partial^{\beta} A^{a}$$

$$\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}$$

Lorentz Force Equation becomes

$$\frac{dp^{\alpha}}{d\tau} = qF^{\alpha\beta}U_{\beta}$$

$F_{\alpha\beta}$ is above matrix with $\vec{E} \rightarrow -\vec{E}$

Maxwell's Equations

Dual Tensor

$$\mathrm{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\delta\gamma} F_{\delta\gamma}$$

ε^{0123} and even permutations is 1, is -1 for odd permutations of 0123

$$\mathrm{F}^{\alpha\beta} = \begin{pmatrix} 0 & -cB_x & -cB_y & -cB_z \\ cB_x & 0 & E_z & -E_y \\ cB_y & -E_z & 0 & E_x \\ cB_z & E_y & E_x & 0 \end{pmatrix}$$

Homogeneous Maxwell Equations

$$\partial_\alpha \mathrm{F}^{\alpha\beta} = 0$$

Inhomogeneous Maxwell Equations

$$\partial_\alpha F^{\alpha\beta} = \mu_0 c J^\beta$$

Transformation of EM Field

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}} \frac{\partial x'^{\beta}}{\partial x^{\gamma}} F^{\delta\gamma}$$

For specific case of an x-direction boost

$$E_1' = E_1 \qquad B_1' = B_1$$

$$E_2' = \gamma\left(E_2 - \beta c B_3\right) \qquad B_2' = \gamma\left(B_2 + \beta E_3 / c\right)$$

$$E_3' = \gamma\left(E_3 + \beta c B_2\right) \qquad B_3' = \gamma\left(B_3 - \beta E_2 / c\right)$$

$$\vec{B} = \vec{\mathrm{v}} \times \vec{E}$$

in other frames with relative velocity $\vec{\mathrm{v}}$

Relativistic Spin Equation

Nonrelativistic spin precession equation

$$\frac{d\vec{s}}{dt'} = \frac{ge}{2mc}\vec{s} \times \vec{B}'$$

(prime indicates particle rest frame) should be generalized to include relativistic motion. Create spin 4-vector

$$S^{\alpha} = \left(S^{0}, \vec{S}\right)$$

whose components in the rest frame of the particle are

$$S'^{\alpha} = \left(0, \vec{s}\right)$$

Suppose the particle moves with velocity $\vec{\beta} = \vec{v} / c$

General Lorentz transformation (Eqn. 11.19) gives

$$S'^0 = \gamma\left(S^0 - \vec{\beta}\cdot\vec{S}\right) = \frac{1}{c}U_\alpha S^\alpha = 0 \rightarrow S^0 = \vec{\beta}\cdot\vec{S}$$

$$\vec{s} = \vec{S}' = \vec{S} + \frac{(\gamma-1)}{\beta^2}\left(\vec{\beta}\cdot\vec{S}\right)\vec{\beta} - \gamma\vec{\beta}\left(\vec{\beta}\cdot\vec{S}\right) = \vec{S} - \frac{\gamma}{\gamma+1}\left(\vec{\beta}\cdot\vec{S}\right)\vec{\beta}$$

From this equation it follows

$$\vec{\beta}\cdot\vec{s} = \frac{\vec{\beta}\cdot\vec{S}}{\gamma}$$

so the inverse transformation is

$$S^0 = \gamma\vec{\beta}\cdot\vec{s} \qquad \vec{S} = \vec{s} + \frac{\gamma^2}{\gamma+1}\left(\vec{\beta}\cdot\vec{s}\right)\vec{\beta}$$

Spin Evolution

Suppose the evolution linear in the EM field, spin variable, and only involves 1st derivative in time (does not depend on acceleration), $dS^{\alpha}/d\tau$ can only involve

$$F^{\alpha\beta}S_{\beta} \qquad \left(S_{\lambda}F^{\lambda\mu}U_{\mu}\right)U^{\alpha} \qquad \left(S_{\beta}\frac{dU^{\beta}}{d\tau}\right)U^{\alpha}$$

note

$$F^{\alpha\beta}U_{\beta}\left(S_{\lambda}U^{\lambda}\right) = \left(U_{\lambda}F^{\lambda\mu}U_{\mu}\right)S^{\alpha} = 0, \text{ no others possible}$$

$$\frac{dS^{\alpha}}{d\tau} = A_1F^{\alpha\beta}S_{\beta} + \frac{A_2}{c^2}\left(S_{\lambda}F^{\lambda\mu}U_{\mu}\right)U^{\alpha} + \frac{A_3}{c^2}\left(S_{\beta}\frac{dU^{\beta}}{d\tau}\right)U^{\alpha}$$

$$\frac{d}{d\tau}S^{\alpha}U_{\alpha} = 0 \rightarrow S^{\alpha}\frac{dU_{\alpha}}{d\tau} + U^{\alpha}\frac{dS_{\alpha}}{d\tau} = 0$$

BMT Equation

$$\therefore \left(A_1 - A_2\right) U_\alpha F^{\alpha\beta} S_\beta + \left(1 + A_3\right) S_\beta \frac{dU_\beta}{d\tau} = 0$$

$$A_1 = A_2 \quad A_3 = -1$$

$$\frac{dS^\alpha}{d\tau} = \frac{ge}{2mc}\left[F^{\alpha\beta} S_\beta + \frac{U^\alpha}{c^2}\left(S_\lambda F^{\lambda\mu} U_\mu\right)\right] - \frac{U^\alpha}{c^2}\left(S_\beta \frac{dU^\beta}{d\tau}\right)$$

$$\frac{dU^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha\beta} U_\beta \rightarrow$$

$$\frac{dS^\alpha}{d\tau} = \frac{e}{mc}\left[\frac{g}{2} F^{\alpha\beta} S_\beta + \frac{U^\alpha}{c^2}\left(\frac{g}{2} - 1\right)\left(S_\lambda F^{\lambda\mu} U_\mu\right)\right]$$

contains Thomas precession