Physics 704/804 Electromagnetic Theory II

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Lorentz Invariant Operators

4-divergence

$$\partial_\alpha A^\alpha = \partial^\alpha A_\alpha = \frac{\partial A^0}{\partial x^0} + \vec{\nabla} \cdot \vec{A}$$

wave operator

$$\Box \equiv \partial_\alpha \partial^\alpha = \frac{\partial^2}{\left(\partial x^0\right)^2} - \nabla^2$$

define column vector consisting of contravariant components

$$X = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad \text{as a matrix equation} \quad gX = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$$

Components of the Lorentz Group

Now

$$\mathrm{Det}\left(L^t g L\right) = \mathrm{Det}\left(L^t\right)\mathrm{Det}(g)\,\mathrm{Det}(L) = \mathrm{Det}(L)^2\,\mathrm{Det}(g)$$

$$\therefore\ \mathrm{Det}(L) = \pm 1$$

Lorentz Group has 4 components

$+$Det	$+$ parity	usual component with identity
$+$Det	$-$ parity	time reverse $+$ parity
$-$Det	$-$ parity	usual parity
$-$Det	$+$ parity	time reverse

Component containing identity may be constructed from infinitesimal generators (like all Lie groups)

Infinitesimal Generators

$$\left(1+E\right)^t g\left(1+E\right) \approx g + E^t g + gE + HOT = g$$

$$\therefore\ E^t g + gE = 0 \rightarrow \left(gE\right)^t = -gE \rightarrow gE \quad \text{antisymmetric}$$

6 Possibilities

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Generate Transformations

Rotations generates by the Ss

$$L_{s_3} = \exp\left(-\theta S_3\right) = I - S_3^2 + \cos\theta S_3^2 - \sin\theta S_3 =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Lorentz Boosts generated by the Ks

$$L_{K_1} = \exp\left(-\xi K_1\right) = I - K_1^2 + \cosh\xi K_1^2 - \sinh\xi K_1 =$$

$$\begin{pmatrix} \cosh\xi & -\sinh\xi & 0 & 0 \\ -\sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Commutation Relations

Rapidity

$$\xi = \tanh^{-1}\beta$$

adds for Lorentz boosts in same direction

$$\left[S_i, S_j\right] = \varepsilon_{ijk} S_k$$

$$\left[S_i, K_j\right] = \varepsilon_{ijk} K_k$$

$$\left[K_i, K_j\right] = -\varepsilon_{ijk} S_k$$

Covariance of Electrodynamics

Electric charge a Lorentz invariant

Non-relativistic Lorentz Force

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)$$

$$p^{\mu} = m\left(U^{0}, \vec{U}\right) = m\left(\gamma c, \gamma \vec{\mathrm{v}}\right)$$

$$\frac{d\vec{p}}{d\tau} = \frac{q}{c}\left(U^{0}\vec{E} + \vec{U} \times c\vec{B}\right)$$

$$\frac{dp^{0}}{d\tau} = \frac{q}{c}\vec{U} \cdot \vec{E}$$

$\vec{U} \cdot \vec{E}$ time component of a 4-vector $\rightarrow$

$\vec{E}$ the time-space components of a 4-two tensor

Transformation of EM Field

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}} \frac{\partial x'^{\beta}}{\partial x^{\gamma}} F^{\delta\gamma}$$

For specific case of an x-direction boost

$$E_1' = E_1 \qquad B_1' = B_1$$

$$E_2' = \gamma\left(E_2 - \beta c B_3\right) \qquad B_2' = \gamma\left(B_2 + \beta E_3 / c\right)$$

$$E_3' = \gamma\left(E_3 + \beta c B_2\right) \qquad B_3' = \gamma\left(B_3 - \beta E_2 / c\right)$$

$$\vec{B} = \vec{\mathrm{v}} \times \vec{E}$$

in other frames with relative velocity $\vec{\mathrm{v}}$

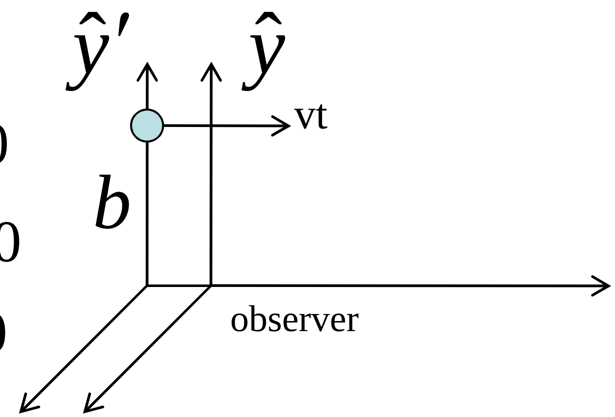
Point Charge Passing Observer

Prime frame rest frame of charge, charge passes with minimum distance b

$$E_1' = -qvt'/4\pi\varepsilon_0 r'^3 \qquad B_1' = 0$$

$$E_2' = -qb/4\pi\varepsilon_0 r'^3 \qquad B_2' = 0$$

$$E_3' = 0 \qquad B_3' = 0$$



rest frame of observer

$$E_1' = E_1 = -\frac{\gamma q v t}{4\pi\varepsilon_0\left(\sqrt{b^2 + (\gamma v t)^2}\right)^3} \qquad B_1' = 0$$

$$E_2' = -\frac{\gamma q b}{4\pi\varepsilon_0\left(\sqrt{b^2 + (\gamma v t)^2}\right)^3} \qquad B_2' = 0$$

$$E_3' = 0 \qquad B_3' = \gamma\beta E_2 / c$$