

# Physics 704/804 Electromagnetic Theory II

G. A. Krafft  
Jefferson Lab  
Jefferson Lab Professor of Physics  
Old Dominion University

# 4-Vectors and Proper Time



Any set of four quantities that transform as the coordinate displacements is called a (contravariant) 4-vector

$$x'^0 = \gamma(x^0 - \beta x^1)$$

$$x'^1 = \gamma(x^1 - \beta x^0)$$

$$x'^2 = x^2$$

$$x'^3 = x^3$$

proper (invariant!) time

$$d\tau = \sqrt{1 - \vec{v} \cdot \vec{v} / c^2} dt$$

# Other 4-vectors

velocity 4-vector

$$U = \left( \frac{dt}{d\tau} c, \frac{d\vec{x}}{d\tau} \right) = c(\gamma, \gamma\vec{\beta})$$

$$U \cdot U = c^2$$

Energy-momentum 4-vector

$$p = mU$$

$$p \cdot p = m^2 c^2 = \frac{E^2}{c^2} - m^2 \vec{v} \cdot \vec{v}$$

$$E^2 = |\vec{p}|^2 c^2 + m^2 c^4$$

# Invariance of wave phase



$$\omega't' - \vec{k}' \cdot \vec{x}' = \omega t - \vec{k} \cdot \vec{x}$$

$(\omega, \vec{k}c)$  for photons is a 4-vector

$$\omega' = \gamma(\omega - \beta k_z c)$$

$$k_z = \gamma(k_z - \beta \omega / c)$$

$$k'_x = k_x$$

$$k'_y = k_y$$

relativistic Doppler Shift

$$\omega' = \gamma \omega (1 - \beta \cos \theta)$$

$$\tan \theta' = \frac{\sin \theta'}{\cos \theta'} = \frac{\sin \theta}{\gamma(\cos \theta - \beta)}$$

# Tensor Analysis

Lorentz Transformations leave invariant the space-time interval

$$ds^2(x, y) = (x^0 - y^0)^2 - (x^1 - y^1)^2 - (x^2 - y^2)^2 - (x^3 - y^3)^2$$

and more generally the Lorentz scalar products of 4-vectors

$$v \cdot w = v^0 w^0 - v^1 w^1 - v^2 w^2 - v^3 w^3$$

Goal in Special Relativity; write laws of physics in terms of quantities with specific Lorentz transformation properties, for example invariants, 4-vectors, 4-tensors, etc.

$$x'^\alpha = x'^\alpha(x^0, x^1, x^2, x^3)$$

$$dx'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} dx^\beta \quad \text{Einstein summation convention on } \beta$$

Contravariant (4)-vectors: Transform like coordinate differentials

$$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta}$$

Examples: 4-velocity, energy-momentum 4-vector,  $k$  for photons  
 Not only possibility. Consider the gradient operator

$$\frac{\partial}{\partial x'^{\alpha}} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial}{\partial x^{\beta}}$$

uses inverse matrix to contravariant vectors. Quantities transforming similarly to gradient called covariant vectors

$$B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta}$$

# Second Rank 4-Tensors

Contravariant rank two tensors transform like the outer product of two contravariant (4)-vectors:

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}} \frac{\partial x'^{\beta}}{\partial x^{\gamma}} F^{\delta\gamma}$$

Covariant rank two tensors transform like the outer product of two covariant (4)-vectors:

$$H'_{\alpha\beta} = \frac{\partial x^{\delta}}{\partial x'^{\alpha}} \frac{\partial x^{\gamma}}{\partial x'^{\beta}} H_{\delta\gamma}$$

Mixed rank two tensors transform like:

$$H'^{\beta}_{\alpha} = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} H_{\gamma}^{\delta}$$

Not necessarily true that  $H_{\alpha}^{\beta} = H^{\beta}_{\alpha}$

# Tensor Contraction

Summing upper and lower indices leaves tensor with rank two lower. For mixed 2 tensor yields an invariant

$$H'^{\alpha}_{\alpha} = \frac{\partial x^{\delta}}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} H_{\delta}^{\gamma} = \frac{\partial x^{\delta}}{\partial x^{\gamma}} H_{\delta}^{\gamma} = H_{\delta}^{\delta}$$

↔ to the linear algebra theorem that the trace of a matrix is invariant under similarity transformations

$$\text{Tr}(S M S^{-1}) = \text{Tr}(M)$$

Similar example: inner product is outer product of covariant and contravariant components followed by a contraction

$$v' \cdot w' = v'_{\alpha} w'^{\alpha} = \frac{\partial x^{\delta}}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} v_{\delta} w^{\gamma} = \frac{\partial x^{\delta}}{\partial x^{\gamma}} v_{\delta} w^{\gamma} = v_{\delta} w^{\delta} = v \cdot w$$

# Metric Tensor

---

Write invariant interval

$$(ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$$

$g_{\mu\nu}$  gives the scalar product by  $v \cdot w = g_{\mu\nu} v^\mu w^\nu$ .

Because  $v \cdot w = w \cdot v$ ,  $g_{\mu\nu}$  is symmetric. We'll use Minkowski metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

inverse matrix

$$g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma, \quad g_{\gamma\beta} g^{\beta\alpha} = \delta^\alpha_\gamma \quad \delta^\alpha_\gamma = \delta_\gamma^\alpha = \begin{cases} 1 & \alpha = \gamma \\ 0 & \alpha \neq \gamma \end{cases}$$

# Inverse Matrix

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

metric may be used to provide an association between contravariant and covariant components

$$x_\alpha = g_{\alpha\beta} x^\beta = (x^0, -\vec{x})$$

$$F^{\alpha ..} = g^{\alpha\beta} F_{\beta ..}$$

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left( \frac{\partial}{\partial x^0}, \vec{\nabla} \right) \quad \partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} = \left( \frac{\partial}{\partial x^0}, -\vec{\nabla} \right)$$

# Lorentz Invariant Operators



4-divergence

$$\partial_\alpha A^\alpha = \partial^\alpha A_\alpha = \frac{\partial A^0}{\partial x^0} + \vec{\nabla} \cdot \vec{A}$$

wave operator

$$\square \equiv \partial_\alpha \partial^\alpha = \frac{\partial^2}{(\partial x^0)^2} - \nabla^2$$

define column vector consisting of contravariant components

$$X = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\text{as a matrix equation } gX = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$$

# Lorentz Group



Note

$$ggX = g^2 X = IX = X$$

If

$$(X, Y) \equiv X^t Y$$

The Lorentz scalar product is

$$a \cdot b = (a, gb) = (ga, b) = a^t gb$$

Elements of the Lorentz group of matrices leave the Lorentz scalar product invariant

$$X' = LX \quad Y' = LY$$

$$X' \cdot Y' = (LX)^t g LY = X^t L^t g LY = X \cdot Y \text{ for all } X, Y \text{ if}$$

$$L^t g L = g \quad \text{as a matrix equation}$$

# Components of the Lorentz Group



Now

$$\text{Det}(L^t g L) = \text{Det}(L^t) \text{Det}(g) \text{Det}(L) = \text{Det}(L)^2 \text{Det}(g)$$

$$\therefore \text{Det}(L) = \pm 1$$

Lorentz Group has 4 components

|      |          |                               |
|------|----------|-------------------------------|
| +Det | + parity | usual component with identity |
| +Det | - parity | time reverse + parity         |
| -Det | - parity | usual parity                  |
| -Det | + parity | time reverse                  |

Component containing identity may be constructed from infinitesimal generators (like all Lie groups)

# Infinitesimal Generators



$$(1 + E)^t g (1 + E) \approx g + E^t g + gE + HOT = g$$

$$\therefore E^t g + gE = 0 \rightarrow (gE)^t = -gE \rightarrow gE \text{ antisymmetric}$$

6 Possibilities

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$