Physics 704/804 Electromagnetic Theory II

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Energy

$$\vec{H}_l = \sum_m a_E(l,m)\,\vec{X}_{lm} h_l^1(kr)\,e^{-i\omega t}$$

$$\vec{E}_l = \frac{i}{k} Z_0 \vec{\nabla} \times \vec{H}_l$$

$$u = \frac{\varepsilon_0}{4}\left(\vec{E}\cdot\vec{E}^* + Z_0^2 \vec{H}\cdot\vec{H}^*\right) \approx \frac{\mu_0}{2}\vec{H}\cdot\vec{H}^*$$

in radiation zone

$$dU = \frac{\mu_0 dr}{2k^2}\sum_{mm'} a_E^*(l,m)\,a_E(l,m')\int \vec{X}^*_{lm'}\vec{X}_{lm}\,d\Omega$$

$$\frac{dU}{dr} = \frac{\mu_0}{2k^2}\sum_{mm'}\left|a_E(l,m)\right|^2$$

incoherent sum over multipoles

Angular Momentum

$$\vec{m} = \frac{1}{2c^2}\mathrm{Re}\left[\vec{r}\times\left(\vec{E}\times\vec{H}^*\right)\right]$$

$$\vec{m} = \frac{\mu_0}{2\omega}\mathrm{Re}\left[\vec{H}^*\left(\vec{L}\cdot\vec{H}\right)\right]$$

$$\frac{d\vec{M}}{dr} = \frac{\mu_0}{2\omega k^2}\mathrm{Re}\sum_{mm'} a_E^*\left(l,m\right)a_E\left(l,m'\right)\int Y_{lm'}^*\vec{L}Y_{lm}\,d\Omega$$

$$\frac{dM_z}{dr} = \frac{\mu_0}{2\omega k^2}\sum_m m\left|a_E\left(l,m\right)\right|^2$$

$$\frac{dM_z}{dr} = \frac{m}{\omega}\frac{dU}{dr}$$

photon interpretation: photon takes away

$m\hbar$ units of angular momentum

Angular Distribution

$$\vec{H} \rightarrow \frac{e^{ikr}}{kr}\frac{1}{2c^2}\sum_{lm}(-i)^{l+1}\left[a_E(l,m)\vec{X}_{lm} + a_M(l,m)\hat{n}\times\vec{X}_{lm}\right]$$

$$\vec{E} \rightarrow Z_0\vec{H}\times\hat{n}$$

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|\sum_{lm}(-i)^{l+1}\left[a_E(l,m)\vec{X}_{lm}\times\hat{n} + a_M(l,m)\vec{X}_{lm}\right]\right|^2$$

pure multipole

$$\frac{dP(l,m)}{d\Omega} = \frac{Z_0}{2k^2}\left|a(l,m)\right|^2\left|\vec{X}_{lm}\right|^2$$

$$= \frac{Z_0}{2k^2 l(l+1)}\left|a(l,m)\right|^2\left\{\begin{array}{c}\frac{1}{2}(l-m)(l+m+1)\left|Y_{l,m+1}\right|^2 \\ +\frac{1}{2}(l+m)(l-m+1)\left|Y_{l,m-1}\right|^2 + m^2\left|Y_{l,m}\right|^2\end{array}\right\}$$

Sum Rule

$$\sum_{m=-l}^{l} \left| \vec{X}_{lm}(\theta, \phi) \right|^2 = \frac{2l+1}{4\pi}$$

$$P(l,m) = \frac{Z_0}{2k^2} \left| a(l,m) \right|^2$$

$$P = \frac{Z_0}{2k^2} \sum_{l,m} \left[\left| a_E(l,m) \right|^2 + \left| a_M(l,m) \right|^2 \right]$$

Multipole Sources

$$\vec{\nabla} \cdot \frac{\vec{B}}{\mu} = 0 \qquad \vec{\nabla} \times \vec{E} - ikZ_0 \frac{\vec{B}}{\mu} = 0$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} \times \frac{\vec{B}}{\mu_0} + ik\vec{E} / Z_0 = \vec{J} + \vec{\nabla} \times \vec{M}$$

$$i\omega\rho = \vec{\nabla} \cdot \vec{J} \qquad \vec{H}' = \frac{\vec{B}}{\mu_0} \qquad \vec{E}' = \vec{E} + \frac{i}{\omega\varepsilon_0} \vec{J}$$

$$\vec{\nabla} \cdot \vec{H}' = 0 \qquad \vec{\nabla} \times \vec{E} - ikZ_0 \frac{\vec{B}}{\mu} = \frac{i}{\omega\varepsilon_0} \vec{\nabla} \times \vec{J}$$

equations are now like before

Inhomogeneous Helmholtz Equation

$$\left(\nabla^2 + k^2\right)\vec{r}\cdot\vec{H}' = -i\vec{L}\cdot\left(\vec{J} + \vec{\nabla}\times\vec{M}\right)$$

$$\left(\nabla^2 + k^2\right)\vec{r}\cdot\vec{E}' = Z_0 k\vec{L}\cdot\left(\vec{M} + \frac{1}{k^2}\vec{\nabla}\times\vec{J}\right)$$

$$\vec{r}\cdot\vec{H}' = \frac{i}{4\pi}\int\frac{e^{ik\left|\vec{x}-\vec{x}'\right|}}{\left|\vec{x}-\vec{x}'\right|}\vec{L}'\cdot\left[\vec{J} + \vec{\nabla}'\times\vec{M}\right]d^3\vec{x}'$$

$$\vec{r}\cdot\vec{E}' = -\frac{Z_0 k}{4\pi}\int\frac{e^{ik\left|\vec{x}-\vec{x}'\right|}}{\left|\vec{x}-\vec{x}'\right|}\vec{L}'\cdot\left[\vec{M} + \frac{1}{k^2}\vec{\nabla}'\times\vec{J}\right]d^3\vec{x}'$$

$$\frac{1}{4\pi}\int d\Omega\, Y_{lm}^*\left(\theta,\phi\right)\frac{e^{ik\left|\vec{x}-\vec{x}'\right|}}{\left|\vec{x}-\vec{x}'\right|} = ikh_l^1\left(kr\right) j_l\left(kr'\right)Y_{lm}^*\left(\theta,\phi\right)$$

$$a_E(l,m) = \frac{ik^3}{\sqrt{l(l+1)}} \int j_l(kr) Y_{lm}^* \vec{L} \cdot \left(\vec{M} + \frac{1}{k^2} \vec{\nabla} \times \vec{J} \right) d^3\vec{x}$$

$$a_M(l,m) = -\frac{k^2}{\sqrt{l(l+1)}} \int j_l(kr) Y_{lm}^* \vec{L} \cdot \left(\vec{J} + \vec{\nabla} \times \vec{M} \right) d^3\vec{x}$$

$$a_E(l,m) = \frac{k^2}{i\sqrt{l(l+1)}} \int Y_{lm}^* \left\{ \begin{matrix} c\rho \frac{\partial}{\partial r}\left[r j_l(kr) \right] + ik\left(\vec{r} \cdot \vec{J} \right) j_l(kr) \\ -ik\vec{\nabla} \cdot \left(\vec{r} \times \vec{M} \right) j_l(kr) \end{matrix} \right\} d^3\vec{x}$$

$$a_M(l,m) = \frac{k^2}{i\sqrt{l(l+1)}} \int Y_{lm}^* \left\{ \begin{matrix} \vec{\nabla} \cdot \left(\vec{r} \times \vec{J} \right) j_l(kr) \\ +\left(\vec{r} \cdot \vec{M} \right) \frac{\partial}{\partial r}\left[r j_l(kr) \right] \\ -k^2 \left(\vec{r} \cdot \vec{M} \right) j_l(kr) \end{matrix} \right\} d^3\vec{x}$$

Small Source Moments

for small sources

$$a_E(l,m) = \frac{ck^{l+2}}{i(2l+1)!!}\sqrt{\frac{l+1}{l}}\left(Q_{lm} + Q'_{lm}\right)$$

$$a_M(l,m) = \frac{ick^{l+2}}{(2l+1)!!}\sqrt{\frac{l+1}{l}}\left(M_{lm} + M'_{lm}\right)$$

$$Q_{lm} = \int r^l Y^*_{lm}\rho\, d^3x \qquad Q'_{lm} = \frac{-ik}{(l+1)c}\int r^l Y^*_{lm}\vec{\nabla}\cdot\left(\vec{r}\times\vec{M}\right)d^3x$$

$$M'_{lm} = -\int r^l Y^*_{lm}\vec{\nabla}\cdot\vec{M}\, d^3x \qquad M_{lm} = -\frac{1}{(l+1)}\int r^l Y^*_{lm}\vec{\nabla}\cdot\left(\vec{r}\times\vec{J}\right)d^3x$$

Scattering and Diffraction

$$\vec{E}_{inc} = \varepsilon_0 E_0 e^{ik\hat{n}_0\cdot\vec{x}}$$

$$\vec{H}_{inc} = \hat{n}_0 \times \vec{E}_{inc} / Z_0$$

$$\vec{E}_{sc} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r}\left[\left(\hat{n}\times\vec{p}\right)\times\hat{n} - \hat{n}\times\vec{m}/c\right] \quad \vec{H}_{sc} = \hat{n}\times\vec{E}_{sc}/Z_0$$

$$\frac{d\sigma}{d\Omega}\left(\hat{n},\hat{\varepsilon};\hat{n}_0,\hat{\varepsilon}_0\right) = \frac{r^2 \frac{1}{2Z_0}\left|\hat{\varepsilon}^*\cdot\vec{E}_{sc}\right|^2}{\frac{1}{2Z_0}\left|\hat{\varepsilon}_0^*\cdot\vec{E}_{inc}\right|^2}$$

$$Rayleigh's\ Law$$

$$\frac{d\sigma}{d\Omega}\left(\hat{n},\hat{\varepsilon};\hat{n}_0,\hat{\varepsilon}_0\right) = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2}\left|\hat{\varepsilon}^*\cdot\vec{p} - \left(\hat{n}\times\hat{\varepsilon}^*\right)\cdot\vec{m}/c\right|^2$$