Physics 704/804 Electromagnetic Theory II

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Radiating Systems; Multipoles

$$\rho\left(\vec{x},t\right)=\rho\left(\vec{x}\right)e^{-i\omega t}$$

$$\vec{J}\left(\vec{x},t\right)=\vec{J}\left(\vec{x}\right)e^{-i\omega t}$$

following our standard practice

quantities are real parts

$$\vec{A}\left(\vec{x},t\right)=\frac{\mu_0}{4\pi}\int d^3x'\int dt'\,\frac{\vec{J}\left(\vec{x}',t'\right)}{\left|\vec{x}-\vec{x}'\right|}\delta\left(t'+\left|\vec{x}-\vec{x}'\right|/c-t\right)$$

$$\vec{A}\left(\vec{x}\right)=\frac{\mu_0}{4\pi}\int d^3x'\vec{J}\left(\vec{x}'\right)\frac{e^{ik\left|\vec{x}-\vec{x}'\right|}}{\left|\vec{x}-\vec{x}'\right|}\qquad\qquad k=\omega/c$$

Fields

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A}$$

$$\vec{E} = \frac{iZ_0}{k}\vec{\nabla}\times\vec{H} \qquad Z_0 = \sqrt{\mu_0/\varepsilon_0}$$

near field: $d \ll r \ll \lambda = 2\pi c/\omega$

far field: $d \ll \lambda \ll r$

Far Field

exponential replaced with

$$\left|\vec{x}-\vec{x}'\right| \approx r-\hat{n}\cdot\vec{x}' \qquad\qquad \hat{n}=\vec{x}/\left|\vec{x}\right|$$

$$\lim_{kr\to\infty}\vec{A}\left(\vec{x}\right)=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int\vec{J}\left(\vec{x}'\right)e^{-ik\hat{n}\cdot\vec{x}'}d^3\vec{x}'$$

$$=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_{n=0}\frac{\left(-ik\right)^n}{n!}\int\vec{J}\left(\vec{x}'\right)\left(\hat{n}\cdot\vec{x}'\right)^n d^3\vec{x}'$$

radiation pattern dominated by emission from first non-vanishing n

Electric Dipole Radiation

$n = 0$ term

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}')\, d^3\vec{x}'$$

$$\int \vec{J}(\vec{x}')\, d^3\vec{x}' = -\int \vec{x}' \left(\vec{\nabla}' \cdot \vec{J}\right) d^3\vec{x}' = -i\omega \int \vec{x}' \rho(\vec{x}')\, d^3\vec{x}'$$

because

$$i\omega\rho = \vec{\nabla} \cdot \vec{J}$$

$$\vec{A}(\vec{x}) = -\frac{i\mu_0\omega}{4\pi} \vec{p} \frac{e^{ikr}}{r} \qquad \vec{p} = \int \vec{x}' \rho(\vec{x}')\, d^3\vec{x}'$$

Fields

$$\vec{H} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left\{k^2(\hat{n}\times\vec{p})\times\hat{n}\frac{e^{ikr}}{r}+\left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}$$

radiation zone

$$\vec{H} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}$$

$$\vec{E} = Z_0\vec{H}\times\hat{n}$$

near zone

$$\vec{H} = \frac{i\omega}{4\pi}(\hat{n}\times\vec{p})\frac{1}{r^2}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\frac{1}{r^3}$$

Power

$$\frac{dP}{d\Omega}=\frac{1}{2}\mathrm{Re}\left[r^2\hat{n}\cdot\vec{E}\times\vec{H}^*\right]$$

$$\frac{dP}{d\Omega}=\frac{c^2Z_0}{32\pi^2}k^4\left|\left(\hat{n}\times\vec{p}\right)\times\hat{n}\right|^2$$

if all components of p have the same phase

$$\frac{dP}{d\Omega}=\frac{c^2Z_0}{32\pi^2}k^4\left|\vec{p}\right|^2\sin^2\theta$$

Integrated Over Angle

$$P = \frac{c^2 Z_0}{32\pi^2} k^4 \left|\vec{p}\right|^2 2\pi \int_{-1}^{1} \sin^2\theta d\cos\theta = \frac{c^2 Z_0}{12\pi} k^4 \left|\vec{p}\right|^2$$

center-fed antenna

$$I(z) = I_0 \left(1 - 2|z|/d\right)$$

$$\rho'(z) = \pm \frac{2iI_0}{\omega d}$$

$$p = \int_{-d/2}^{d/2} z\rho'(z) dz = \frac{iI_0 d}{2\omega}$$

$$\frac{dP}{d\Omega} = \frac{Z_0 I_0^2}{128\pi} (kd)^2 \sin^2\theta$$

$$P = \frac{Z_0 I_0^2}{48\pi} (kd)^2 \qquad R_{rad} \approx 5\Omega (kd)^2$$