Physics 704/804 Electromagnetic Theory II

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Perpendicular Polarization

$$0 = \left(\left|\vec{E}_0\right| + \left|\vec{E}_0''\right| - \left|\vec{E}_0'\right|\right) \qquad \text{Tangential } \vec{E}$$

$$0 = \sqrt{\frac{\varepsilon}{\mu}}\left(\left|\vec{E}_0\right| - \left|\vec{E}_0''\right|\right)\cos i - \sqrt{\frac{\varepsilon'}{\mu'}}\left|\vec{E}_0'\right|\cos r \qquad \text{Tangential } \vec{H}$$

$$\frac{\left|\vec{E}_0'\right|}{\left|\vec{E}_0\right|} = \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}$$

$$\frac{\left|\vec{E}_0''\right|}{\left|\vec{E}_0\right|} = \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}$$

Wave Guides: Surface Absorbtion

Homogeneous Maxwell Equations imply

$$\vec{n} \cdot \left(\vec{B} - \vec{B}_c \right) = 0$$

$$\vec{n} \times \left(\vec{E} - \vec{E}_c \right) = 0$$

Curl *H* Maxwell Equation and large but finite conductivity implies

$$\vec{n} \times \left(\vec{H} - \vec{H}_c \right) = 0$$

$$\vec{E}_c = \frac{1}{\sigma} \vec{\nabla} \times \vec{H}_c$$

$$\vec{H}_c = -\frac{i}{\mu_c \omega} \vec{\nabla} \times \vec{E}_c$$

$$\left[\frac{\partial^2}{\partial\xi^2}+\frac{2i}{\delta^2}\right]\left(\vec{n}\times\vec{H}_c\right)=0$$

$$\vec{n}\cdot\vec{H}_c=0$$

$$\delta=\left(\frac{2}{\mu_c\omega\sigma}\right)^{1/2}$$

$$\vec{H}_c=\vec{H}_{par}e^{-\xi/\delta}e^{i\xi/\delta}$$

$\vec{H}_{par}$ is the field at the surface

$$\vec{E}_c\left(\xi=0\right)=\sqrt{\frac{\omega\mu_c}{2\sigma}}\left(1-i\right)\left(\vec{n}\times\vec{H}_{par}\right)$$

$$\frac{dP}{da}=\frac{1}{2}\mathrm{Re}\left[\vec{n}\cdot\vec{E}\times\vec{H}^*\right]=\frac{\mu_c\omega\delta}{4}\left|\vec{H}_{par}\right|^2$$

Alternative Calculation

$$\vec{J} = \sigma \vec{E}_c = \frac{1}{\delta}(1-i)\left(\hat{n} \times \vec{H}_{par}\right) e^{-\xi(1-i)/\delta}$$

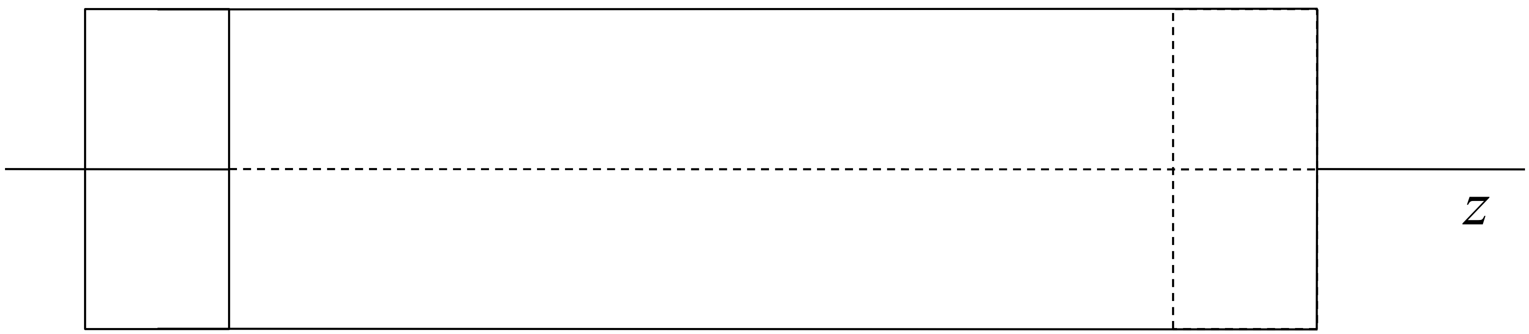
$$\text{power/vol} = \frac{1}{2} \vec{J} \cdot \vec{E}^* = \frac{1}{2\sigma}\left|\vec{J}\right|^2$$

$$K_{eff} = \int_0^\infty J d\xi = \hat{n} \times \vec{H}_{par}$$

surface current for perfect conductor

$$\frac{dP}{da} = \frac{1}{2\sigma\delta}\left|K_{eff}\right|^2 = \frac{\mu_c \omega \delta}{4}\left|K_{eff}\right|^2$$

Cylindrical Systems



z-axis along the cylinder direction

$$\vec{\nabla}\times\vec{E} = i\omega\vec{B} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = -i\mu\varepsilon\vec{E} \qquad \vec{\nabla}\cdot\vec{E} = 0$$

$$\left(\nabla^2 + \mu\varepsilon\omega^2\right)\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix} = 0$$

$$\vec{B} = \vec{B}\left(x, y\right)e^{\pm ikz-\omega t} \qquad \vec{E} = \vec{E}\left(x, y\right)e^{\pm ikz-\omega t}$$

“Transverse” Separation

$$\vec{E} = E_z \hat{z} + \vec{E}_t \qquad \vec{E}_t = \vec{E} - \left(\vec{E}\cdot\hat{z}\right)\hat{z}$$

$$\vec{B} = B_z \hat{z} + \vec{B}_t \qquad \vec{B}_t = \vec{B} - \left(\vec{B}\cdot\hat{z}\right)\hat{z}$$

$$\left[\nabla_t^2 + \left(\mu\varepsilon\omega^2\right) - k^2\right]\begin{cases}\vec{E} \\ \vec{B}\end{cases} = 0$$

$$\frac{\partial \vec{E}_t}{\partial z} + i\omega\hat{z}\times\vec{B}_t = \vec{\nabla}_t E_z \qquad \hat{z}\cdot\left(\vec{\nabla}_t \times \vec{E}_t\right) = i\omega B_z$$

$$\frac{\partial \vec{B}_t}{\partial z} - i\mu\varepsilon\omega\hat{z}\times\vec{E}_t = \vec{\nabla}_t B_z \qquad \hat{z}\cdot\left(\vec{\nabla}_t \times \vec{B}_t\right) = -i\mu\varepsilon\omega E_z$$

$$\vec{\nabla}_t\cdot\vec{E}_t = -\frac{\partial E_z}{\partial z} \qquad \vec{\nabla}_t\cdot\vec{B}_t = -\frac{\partial B_z}{\partial z}$$

TEM Modes

Solutions with transverse field only

$$\vec{E}_{tem} = \vec{E}_{t,tem} \qquad \vec{B}_{tem} = \vec{B}_{t,tem}$$

$$\vec{\nabla}_t \times \vec{E}_{t,tem} = 0 \qquad \vec{\nabla}_t \cdot \vec{E}_{t,tem} = 0$$

$\vec{E}_{tem}$ must solve 2-D *electrostatic* problem

$$k = k_0 = \sqrt{\mu\varepsilon}\,\omega$$

$$\vec{B}_{tem} = \pm\sqrt{\mu\varepsilon}\,\hat{z} \times \vec{E}_{tem}$$

Can there be a solution inside a single closed waveguide?
No, need at least two conductors. No cutoff frequency

More General Case

Transverse field expressible in terms of *z*-field only!

$$\vec{E}_t = \frac{i}{\mu\varepsilon\omega^2 - k^2}\left[k\vec{\nabla}_t E_z - \omega\hat{z}\times\vec{\nabla}_t B_z\right]$$

$$\vec{B}_t = \frac{i}{\mu\varepsilon\omega^2 - k^2}\left[k\vec{\nabla}_t B_z + \mu\varepsilon\omega\hat{z}\times\vec{\nabla}_t E_z\right]$$

Transverse Magnetic (TM)

$$B_z = 0; \qquad \text{Boundary Condition } \left.E_z\right|_S = 0$$

Transverse Electric (TE)

$$E_z = 0; \qquad \text{Boundary Condition } \left.\frac{\partial B_z}{\partial n}\right|_S = 0$$

Waveguides

$$\vec{H}_t = \frac{\pm 1}{Z}\hat{z}\times\vec{E}_t$$

Wave Impedance

$$Z = \begin{cases} \dfrac{k}{\varepsilon\omega} = \dfrac{k}{k_0}\sqrt{\dfrac{\mu}{\varepsilon}} & \text{(TM)} \\ \dfrac{\mu\omega}{k} = \dfrac{k_0}{k}\sqrt{\dfrac{\mu}{\varepsilon}} & \text{(TE)} \end{cases}$$

Eigenvalue Problem

TM Waves

$$\vec{E}_t = \pm \frac{ik}{\gamma^2} \vec{\nabla}_t \psi$$

TE Waves

$$\vec{H}_t = \pm \frac{ik}{\gamma^2} \vec{\nabla}_t \psi$$

Transverse Helmholz Equation

$$\left(\nabla_t^2 + \gamma^2 \right) \psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2 \right) \psi = 0$$

Boundary Conditions

$$\psi\Big|_S = 0 \qquad (\mathrm{TM}) \qquad \frac{\partial\psi}{\partial n}\Big|_S = 0 \qquad (\mathrm{TE})$$

Spectrum of eigenvalues and eigenfunctions

$$\gamma_\lambda \qquad \psi_\lambda(x,y)$$

Wavelength in mode λ

$$k_\lambda^2 = \mu\varepsilon\omega^2 - \gamma_\lambda^2$$

Cutoff Frequency

$$\omega_\lambda^2 = \frac{\gamma_\lambda^2}{\mu\varepsilon}$$

No propagation at frequencies below cutoff

$$k_\lambda^2 = \mu\varepsilon\left(\omega^2 - \omega_\lambda^2\right)$$

For single-mode propagation choose frequency to be above cutoff for lowest mode and below cutoff for all other modes. Phase velocity infinite at cutoff!

Rectangular Waveguide

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\gamma^2\right)\psi=0$$

$$\left.\frac{\partial\psi}{\partial n}\right|_S=0\ (\mathrm{TE})\rightarrow\psi_{mn}(x,y)=A\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi x}{a}\right)$$

$$\gamma_{mn}^2=\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

$$\omega_{mn}=\frac{\pi}{\sqrt{\mu\varepsilon}}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{1/2}$$

Lowest Mode

$$\omega_{10} = \frac{\pi}{\sqrt{\mu\varepsilon}}\frac{m}{a}$$

$$H_z = A\cos\left(\frac{\pi x}{a}\right)e^{ikz - i\omega t}$$

$$H_z = -\frac{ika}{\pi}A\sin\left(\frac{\pi x}{a}\right)e^{ikz - i\omega t}$$

$$E_y = i\frac{\omega a \mu}{\pi}A\sin\left(\frac{\pi x}{a}\right)e^{ikz - i\omega t}$$

Lowest TM mode has *m* and *n* one with a sin solution. Why? Its cutoff for the lowest mode is at a frequency higher by

$$\left(1 + a^2 / b^2\right)^{1/2}$$

Energy Flow

$$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^*$$

$$\vec{S} = \frac{\omega k}{2\gamma^4}\begin{cases} \varepsilon\left[\hat{z}\left|\vec{\nabla}_t\psi\right|^2 + i\dfrac{\gamma^2}{k}\psi\vec{\nabla}_t\psi^*\right] \\ \mu\left[\hat{z}\left|\vec{\nabla}_t\psi\right|^2 - i\dfrac{\gamma^2}{k}\psi^*\vec{\nabla}_t\psi\right] \end{cases}$$

$$P = \int_A \vec{S}\cdot\hat{z}\,da = \frac{\omega k}{2\gamma^4}\begin{Bmatrix}\varepsilon\\ \mu\end{Bmatrix}\int_A \vec{\nabla}_t\psi^*\cdot\vec{\nabla}_t\psi\,da$$

$$P = \frac{\omega k}{2\gamma^4}\begin{Bmatrix}\varepsilon\\ \mu\end{Bmatrix}\left[\oint_C \psi^*\frac{\partial\psi}{\partial n}dl + \int_A \psi^*\cdot\nabla_t^2\psi\,da\right]$$

Energy and Group Velocity

$$P = \frac{1}{2\sqrt{\mu\varepsilon}} \left(\frac{\omega}{\omega_\lambda} \right)^2 \left(1 - \frac{\omega_\lambda^2}{\omega^2} \right)^{1/2} \begin{Bmatrix} \varepsilon \\ \mu \end{Bmatrix} \int_A \psi^* \psi \, da$$

$$U = \frac{1}{2} \left(\frac{\omega}{\omega_\lambda} \right)^2 \begin{Bmatrix} \varepsilon \\ \mu \end{Bmatrix} \int_A \psi^* \psi \, da$$

$$\frac{P}{U} = \frac{k}{\omega} \frac{1}{\mu\varepsilon} = \frac{1}{\sqrt{\mu\varepsilon}} \sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}} = \mathrm{v}_g$$

$$\mathrm{v}_p \mathrm{v}_g = \frac{1}{\mu\varepsilon}$$

Attenuation

Field attenuation constant given by β_λ

$$P(z) = P_0 e^{-2\beta_\lambda z}$$

$$\beta_\lambda = -\frac{1}{2P}\frac{dP}{dz}$$

$$\frac{dP}{dz} = -\frac{1}{2\sigma\delta}\oint_C \left|\hat{n}\times\vec{H}\right|^2 dl$$

$$= -\frac{1}{2\sigma\delta}\oint_C \left\{ \begin{array}{c} \dfrac{1}{\mu^2\omega_\lambda^2}\left|\dfrac{\partial\psi}{\partial n}\right|^2 \\ \dfrac{1}{\mu^2\omega_\lambda^2}\left(1-\dfrac{\omega_\lambda^2}{\omega^2}\right)\left|\hat{n}\times\vec{\nabla}_t\psi\right|^2 + \dfrac{\omega_\lambda^2}{\omega^2}\left|\psi\right|^2 \end{array} \right\} dl$$

$$\left\langle \left| \frac{\partial \psi}{\partial n} \right|^2 \right\rangle \approx \left\langle \left| \hat{n} \times \vec{\nabla}_t \psi \right|^2 \right\rangle \approx \mu\varepsilon\omega_\lambda^2 \left\langle \left| \psi \right|^2 \right\rangle$$

$$\oint_C \frac{1}{\omega_\lambda^2} \left| \frac{\partial \psi}{\partial n} \right|^2 dl = \varsigma_\lambda \mu\varepsilon \frac{C}{A} \int_A \left| \psi \right|^2 da$$

$$\beta_\lambda = \sqrt{\frac{\varepsilon}{\mu}} \frac{1}{\sigma\delta_\lambda} \left(\frac{C}{2A} \right) \frac{\left(\omega / \omega_\lambda \right)^{1/2}}{\left(1 - \omega^2 / \omega_\lambda^2 \right)^{1/2}} \left[\varsigma_\lambda + \eta_\lambda \left(\frac{\omega_\lambda}{\omega} \right)^2 \right]$$

Resonant Cavities

Put end conductors on a cylindrical waveguide. Example: Cylindrical cavity of length *d* and radius *R.* In general, *z* dependence is

$$A\sin(kz) + B\cos(kz)$$

$$BCs \rightarrow k = p\frac{\pi}{d}, \quad p = 0, 1, 2, \ldots$$

TM

$$E_z = \psi(x, y)\cos\left(\frac{p\pi}{d}\right), \quad p = 0, 1, 2, \ldots$$

TE

$$H_z = \psi(x, y)\sin\left(\frac{p\pi}{d}\right), \quad p = 1, 2, \ldots$$

Field Patterns

TM

$$\vec{E}_t = -\frac{p\pi}{d\gamma^2}\sin\left(\frac{p\pi}{d}\right)\vec{\nabla}_t\psi$$

$$\vec{H}_t = \frac{i\varepsilon\omega}{\gamma^2}\cos\left(\frac{p\pi}{d}\right)\hat{z}\times\vec{\nabla}_t\psi$$

TE

$$\vec{E}_t = -\frac{i\omega\mu}{\gamma^2}\sin\left(\frac{p\pi}{d}\right)\hat{z}\times\vec{\nabla}_t\psi$$

$$\vec{H}_t = \frac{p\pi}{d\gamma^2}\cos\left(\frac{p\pi}{d}\right)\vec{\nabla}_t\psi$$

$$\gamma^2 = \mu\varepsilon\omega^2 - \left(p\pi/d\right)^2$$

Eigenvalue Equation

$$\omega_{\lambda p}^{2} = \frac{1}{\mu\varepsilon}\left(\gamma_{\lambda}^{2} + \left(p\pi / d\right)^{2}\right)$$

TM

$$\psi\left(\rho,\phi\right) = AJ_{m}\left(\gamma_{mn}\rho\right)e^{\pm im\phi}$$

$$\gamma_{mn} = \frac{x_{mn}}{R} \qquad J_{m}\left(x_{mn}\right) = 0$$

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\frac{x_{mn}^{2}}{R^{2}} + \frac{p^{2}\pi^{2}}{d^{2}}} \qquad \omega_{010} = \frac{2.405}{\sqrt{\mu\varepsilon}R}$$

$$E_{z} = AJ_{0}\left(\frac{2.405\rho}{R}\right)e^{-i\omega t} \qquad H_{\phi} = -i\sqrt{\frac{\varepsilon}{\mu}}AJ_{1}\left(\frac{2.405\rho}{R}\right)e^{-i\omega t}$$

TE

$$\psi(\rho,\phi) = AJ_m(\gamma_{mn}\rho)e^{\pm im\phi}$$

$$\gamma_{mn} = \frac{x'_{mn}}{R} \qquad J'_m(x'_{mn}) = 0$$

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\frac{x'^2_{mn}}{R^2} + \frac{p^2\pi^2}{d^2}}$$

$$\omega_{111} = \frac{1.841}{\sqrt{\mu\varepsilon}R}\left(1 + 2.912\frac{R^2}{d^2}\right)$$

$$H_z = AJ_1\left(\frac{1.841\rho}{R}\right)\cos\phi\sin\left(\frac{\pi z}{d}\right)e^{-i\omega t}$$

Cavity Losses

$$Q = \omega \frac{\text{Stored energy}}{\text{Power loss}} \quad \text{both proportional to } |\psi|^2$$

$$\frac{dU}{dt} = -\frac{\omega_0}{Q} U$$

$$U(t) = U_0 e^{-\omega_0 t/Q}$$

$$E(t) = E_0 e^{-\omega_0 t/2Q} e^{-i(\omega_0 + \Delta\omega)t}$$

Parallel Polarization

$$0 = \cos i\left(\left|\vec{E}_0\right| - \left|\vec{E}_0''\right|\right) - \cos r\left|\vec{E}_0'\right| \qquad \text{Tangential } \vec{E}$$

$$0 = \sqrt{\frac{\varepsilon}{\mu}}\left(\left|\vec{E}_0\right| + \left|\vec{E}_0''\right|\right) - \sqrt{\frac{\varepsilon'}{\mu'}}\left|\vec{E}_0'\right| \qquad \text{Tangential } \vec{H}$$

$$\frac{\left|\vec{E}_0'\right|}{\left|\vec{E}_0\right|} = \frac{2nn'\cos i}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}}$$

$$\frac{\left|\vec{E}_0''\right|}{\left|\vec{E}_0\right|} = \frac{\frac{\mu}{\mu'}n'^2\cos i - n\sqrt{n'^2 - n^2\sin^2 i}}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}}$$

Normal Incidence

$$\frac{\left|\vec{E}_0'\right|}{\left|\vec{E}_0\right|} = \frac{2}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1} = \frac{2n}{n'+n}$$

$$\frac{\left|\vec{E}_0''\right|}{\left|\vec{E}_0\right|} = \frac{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}-1}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1} = \frac{n'-n}{n'+n}$$

Energy Conservation?

$$\sqrt{\frac{\varepsilon}{\mu}}\frac{E_0^2}{2} = \sqrt{\frac{\varepsilon}{\mu}}\frac{E_0''^2}{2} + \sqrt{\frac{\varepsilon'}{\mu'}}\frac{E_0'^2}{2}$$

Mistakes from Last Time

Energy Conservation at Normal Incidence

$$\sqrt{\frac{\varepsilon}{\mu}}\frac{E_0^2}{2} = \sqrt{\frac{\varepsilon}{\mu}}\frac{E_0''^2}{2} + \sqrt{\frac{\varepsilon'}{\mu'}}\frac{E_0'^2}{2}$$

$$\frac{\left|\vec{E}_0'\right|}{\left|\vec{E}_0\right|} = \frac{2}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1} = \frac{2n}{n'+n}$$

$$\frac{\left|\vec{E}_0''\right|}{\left|\vec{E}_0\right|} = \frac{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}-1}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1} = \frac{n'-n}{n'+n}$$

Brewster's Angle Calculation

$$n'^2 \cos i_B = n\sqrt{n'^2 - n^2 \sin^2 i_B}$$

$$n'^4 \cos^2 i_B = n^2 \left(n'^2 - n^2 \sin^2 i_B\right)$$

$$n'^2 = \frac{+n^2 \pm \sqrt{n^4 - 4n^4 \cos^2 i_B \sin^2 i_B}}{2\cos^2 i_B} = \frac{+n^2 \pm n^2 \left(1 - 2\cos^2 i_B\right)}{2\cos^2 i_B}$$

$$i_{Brewster} = \tan^{-1} \frac{n'}{n}$$

Zero reflection in perpendicular polarization implies $n' = n$

Brewster's Angle

If $\mu = \mu'$ reflected parallel polarization amplitude vanishes when incident at the Brewster angle

$$i_{Brewster} = \tan^{-1}\frac{n'}{n}$$

Reflected wave completely plane-polarized (polarization perpendicular to plane of incidence) if mixed-polarization beam incident at Brewster angle.

$$i_{Brewster} = 56^{\circ} \quad \text{for} \quad \frac{n'}{n} = 1.5$$

Total Internal Reflection

Examine Snell's Law in case $n > n'$

$$i_0 = \sin^{-1} \frac{n'}{n}$$

For angles of incidence greater, there is no transmitted wave solution to attach to, only an exponentially damped solution. This implies total reflection, also called *total internal reflection*. Optical communication systems are based on this phenomenon!

Group Velocity

Until now, we have assumed that the relative permitivity and permeability are independent of frequency. This may be far from the case. Relaxing the requirement of constant phase velocity as a function of frequency leads to more general wave phenomena. Allow the frequency to depend on wavelength in 1 dimension:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k) e^{ikx-\omega(k)t}\,dk$$

The function $\omega(k)$ is known as the "dispersion function". A strictly linear dispersion function, as we've had up to now, does not lead to pulse spreading, or dispersion.

$$A(k) = \int_{-\infty}^{\infty} u(x,0)\, e^{-ikx}\, dx$$

$$\omega(k) = \omega_0 + \left.\frac{d\omega}{dk}\right|_0 (k - k_0) + \cdots \qquad \omega_0 = \omega(k_0)$$

$$u(x,t) \approx \frac{e^{i\left[k_0 (d\omega/dk)|_0 - \omega_0\right]t}}{2\pi} \int_{-\infty}^{\infty} A(k)\, e^{i\left[x-(d\omega/dk)|_0 t\right]k}\, dk$$

$$= e^{i\left[k_0 (d\omega/dk)|_0 - \omega_0\right]t}\, u\left(x - (d\omega/dk)|_0\, t, 0\right)$$

The pulse shape travels at the group velocity

$$\mathrm{v}_g = \frac{d\omega}{dk}$$

Dispersion

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k) e^{ikx-\omega(k)t} dk$$

Have exact calculation for modulated Gaussian function

$$u(x,0) = \exp\left(-x^2 / 2L^2\right) e^{ik_0 x}$$

$$A(k) = \int_{-\infty}^{\infty} u(x,0) e^{-ikx} dx$$

$$= \sqrt{2\pi} L \exp\left(-L^2 / 2\right)\left(k - k_0\right)^2$$

$$\omega(k) = \nu\left(1 + \frac{a^2 k^2}{2}\right)$$

Pulse Spreading, or Dispersion

$$\mathrm{v}_g = \frac{d\omega}{dk} = \nu a^2 k_0$$

$$u(x,t) = \frac{L}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left(-L^2/2\right)\left(k-k_0\right)^2} e^{ikx - i\nu t\left[1+\left(a^2k^2/2\right)\right]} dk$$

$$= \frac{\exp\left[-\dfrac{\left(x-\nu a^2 k_0 t\right)^2}{2L^2\left(1+\dfrac{ia^2\nu t}{L^2}\right)}\right]}{\left(1+\dfrac{ia^2\nu t}{L^2}\right)^{1/2}} \exp\left[ik_0 x - i\nu\left(1+a^2k^2/2\right)t\right]$$

$$L(t) = \frac{d\omega}{dk} = \sqrt{L^2 + \left(\nu a^2 t / L\right)^2}$$

$$\Delta \mathrm{v}_g = \frac{d^2\omega}{dk^2}\Delta k = \frac{\nu a^2}{L}$$

$$\Delta x(t) = \sqrt{\left(\Delta x\right)^2 + \left(\Delta \mathrm{v}_g t\right)^2}$$

Causality

$$\vec{D}\left(\vec{x},\omega\right) = \varepsilon\left(\omega\right)\vec{E}\left(\vec{x},\omega\right)$$

Convolution Theorem (Fahltung Theorem) implies non-locality in time.

$$\vec{D}\left(\vec{x},t\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\vec{D}\left(\vec{x},\omega\right)e^{-i\omega t}d\omega = \int_{-\infty}^{\infty}\varepsilon\left(\omega\right)\vec{E}\left(\vec{x},\omega\right)e^{-i\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon\left(\omega\right)e^{-i\omega t}\int_{\infty}^{\infty}dt' e^{+i\omega t'}\vec{E}\left(\vec{x},t'\right)d\omega$$

$$= \varepsilon_0\left\{\vec{E}\left(\vec{x},t\right) + \int_{-\infty}^{\infty}G\left(\tau\right)\vec{E}\left(\vec{x},t-\tau\right)d\tau\right\}$$

Green function for connection

$$G(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\varepsilon(\omega)/\varepsilon_0 - 1\right]e^{-i\omega\tau}d\omega$$

Damped oscillator connection

$$\varepsilon(\omega)/\varepsilon_0 - 1 = \omega_p^2\left(\omega_0^2 - \omega^2 - i\gamma\omega\right)^{-1}$$

$$G(\tau) = \omega_p^2 e^{-\gamma t/2}\frac{\sin\nu_0\tau}{\nu_0}\Theta(\tau)$$

$$\nu_0 = \sqrt{\omega_0^2 - \gamma^2/4}$$

Vanishes for negative τ, cause cannot precede effect. Causal Green's functions must be analytic in upper ½ of complex plane.

Kramers-Kronig Relations

$$\varepsilon(\omega)/\varepsilon_0 - 1 = \int_0^\infty G(\tau) e^{i\omega\tau} d\tau$$

Is automatically causal for a wide variety of choices for G. Analyticity in UH-ωP implies a relationship between real and imaginary part of the permitivity. Cauchy's theorem for z inside a closed curve C

$$\varepsilon(z)/\varepsilon_0 = 1 + \frac{1}{2\pi i} \oint_C \frac{\varepsilon(\omega')/\varepsilon_0 - 1}{\omega' - z} d\omega'$$

$$= 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varepsilon(\omega')/\varepsilon_0 - 1}{\omega' - z} d\omega'$$

where the integral is now along the real axis

$$\frac{1}{\omega-\omega'-i\delta}=P\left(\frac{1}{\omega-\omega'}\right)+\pi i\delta\left(\omega'-\omega\right)$$

$$\mathrm{Re}\,\varepsilon\left(\omega\right)/\varepsilon_0=1+\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,\varepsilon\left(\omega'\right)/\varepsilon_0}{\omega'-z}d\omega'$$

$$\mathrm{Im}\,\varepsilon\left(\omega\right)/\varepsilon_0=-\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Re}\,\varepsilon\left(\omega'\right)/\varepsilon_0-1}{\omega'-z}d\omega'$$

$$\mathrm{Re}\,\varepsilon\left(\omega\right)/\varepsilon_0=1+\frac{2}{\pi}P\int_{0}^{\infty}\frac{\omega'\,\mathrm{Im}\,\varepsilon\left(\omega'\right)/\varepsilon_0}{\omega'^2-\omega^2}d\omega'$$

$$\mathrm{Im}\,\varepsilon\left(\omega\right)/\varepsilon_0=-\frac{2\omega}{\pi}P\int_{0}^{\infty}\frac{\mathrm{Re}\,\varepsilon\left(\omega'\right)/\varepsilon_0-1}{\omega'^2-\omega^2}d\omega'$$

$$\varepsilon\left(-\omega\right)=\varepsilon^*\left(\omega^*\right)$$

Sum Rules

Sum Rules for oscillator strengths

$$\omega_p^2 = 1 + \frac{2}{\pi} P \int_0^\infty \omega \operatorname{Im} \varepsilon(\omega) / \varepsilon_0 d\omega$$

Second Sum Rule

$$\frac{1}{N} \int_0^N \operatorname{Re} \varepsilon(\omega) / \varepsilon_0 d\omega = 1 + \frac{\omega_p^2}{N^2}$$