# Physics 704/804 Electromagnetic Theory II

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# Pulse Spreading, or Dispersion

$$\mathrm{v}_g = \frac{d\omega}{dk} = \nu a^2 k_0$$

$$u(x,t) = \frac{L}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left(-L^2/2\right)\left(k-k_0\right)^2} e^{ikx - i\nu t\left[1+\left(a^2k^2/2\right)\right]} dk$$

$$= \frac{\exp\left[-\dfrac{\left(x-\nu a^2 k_0 t\right)^2}{2L^2\left(1+\dfrac{ia^2\nu t}{L^2}\right)}\right]}{\left(1+\dfrac{ia^2\nu t}{L^2}\right)^{1/2}} \exp\left[ik_0 x - i\nu\left(1+a^2k^2/2\right)t\right]$$

$$L(t) = \frac{d\omega}{dk} = \sqrt{L^2 + \left(\nu a^2 t / L\right)^2}$$

$$\Delta \mathrm{v}_g = \frac{d^2\omega}{dk^2}\Delta k = \frac{\nu a^2}{L}$$

$$\Delta x(t) = \sqrt{\left(\Delta x\right)^2 + \left(\Delta \mathrm{v}_g t\right)^2}$$

# Causality

$$\vec{D}\left(\vec{x},\omega\right)=\varepsilon\left(\omega\right)\vec{E}\left(\vec{x},\omega\right)$$

Convolution Theorem (Fahltung Theorem) implies non-locality in time.

$$\vec{D}\left(\vec{x},t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\vec{D}\left(\vec{x},\omega\right)e^{-i\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon\left(\omega\right)\vec{E}\left(\vec{x},\omega\right)e^{-i\omega t}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon\left(\omega\right)e^{-i\omega t}\int_{\infty}^{\infty}dt'e^{+i\omega t'}\vec{E}\left(\vec{x},t'\right)d\omega$$

$$=\varepsilon_0\left\{\vec{E}\left(\vec{x},t\right)+\int_{-\infty}^{\infty}G\left(\tau\right)\vec{E}\left(\vec{x},t-\tau\right)d\tau\right\}$$

Green function for connection

$$G(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \varepsilon(\omega) / \varepsilon_0 - 1 \right] e^{-i\omega\tau} d\omega$$

Damped oscillator connection

$$\varepsilon(\omega) / \varepsilon_0 - 1 = \omega_p^2 \left( \omega_0^2 - \omega^2 - i\gamma\omega \right)^{-1}$$

$$G(\tau) = \omega_p^2 e^{-\gamma t/2} \frac{\sin \nu_0 \tau}{\nu_0} \Theta(\tau)$$

$$\nu_0 = \sqrt{\omega_0^2 - \gamma^2 / 4}$$

Vanishes for negative $\tau$, cause cannot precede effect. Causal Green's functions must be analytic in upper ½ of complex plane.

# Kramers-Kronig Relations

$$\varepsilon(\omega)/\varepsilon_0 - 1 = \int_0^{\infty} G(\tau) e^{i\omega\tau} d\tau$$

Is automatically causal for a wide variety of choices for *G*. Analyticity in UH-$\omega$P implies a relationship between real and imaginary part of the permitivity. Cauchy's theorem for *z* inside a closed curve *C*

$$\varepsilon(z)/\varepsilon_0 = 1 + \frac{1}{2\pi i} \oint_C \frac{\varepsilon(\omega')/\varepsilon_0 - 1}{\omega' - z} d\omega'$$

$$= 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varepsilon(\omega')/\varepsilon_0 - 1}{\omega' - z} d\omega'$$

where the integral is now along the real axis

$$\frac{1}{\omega-\omega'-i\delta}=P\left(\frac{1}{\omega-\omega'}\right)+\pi i\delta\left(\omega'-\omega\right)$$

$$\mathrm{Re}\,\varepsilon\left(\omega\right)/\varepsilon_0=1+\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,\varepsilon\left(\omega'\right)/\varepsilon_0}{\omega'-z}d\omega'$$

$$\mathrm{Im}\,\varepsilon\left(\omega\right)/\varepsilon_0=-\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Re}\,\varepsilon\left(\omega'\right)/\varepsilon_0-1}{\omega'-z}d\omega'$$

$$\mathrm{Re}\,\varepsilon\left(\omega\right)/\varepsilon_0=1+\frac{2}{\pi}P\int_{0}^{\infty}\frac{\omega'\,\mathrm{Im}\,\varepsilon\left(\omega'\right)/\varepsilon_0}{\omega'^2-\omega^2}d\omega'$$

$$\mathrm{Im}\,\varepsilon\left(\omega\right)/\varepsilon_0=-\frac{2\omega}{\pi}P\int_{0}^{\infty}\frac{\mathrm{Re}\,\varepsilon\left(\omega'\right)/\varepsilon_0-1}{\omega'^2-\omega^2}d\omega'$$

$$\varepsilon\left(-\omega\right)=\varepsilon^*\left(\omega^*\right)$$

# Sum Rules

Sum Rules for oscillator strengths

$$\omega_p^2 = 1 + \frac{2}{\pi} P \int_0^\infty \omega \,\mathrm{Im}\, \varepsilon(\omega) / \varepsilon_0 \, d\omega$$

Second Sum Rule

$$\frac{1}{N} \int_0^N \mathrm{Re}\, \varepsilon(\omega) / \varepsilon_0 \, d\omega = 1 + \frac{\omega_p^2}{N^2}$$

# Wave Guides: Surface Absorbtion

$$\vec{n}\cdot\left(\vec{B}-\vec{B}_c\right)=0$$

$$\vec{n}\times\left(\vec{E}-\vec{E}_c\right)=0$$

$$\vec{n}\times\left(\vec{H}-\vec{H}_c\right)=0$$

$$\vec{E}_c=\frac{1}{\sigma}\vec{\nabla}\times\vec{H}_c$$

$$\vec{H}_c=-\frac{i}{\mu_c\omega}\vec{\nabla}\times\vec{E}_c$$

$$\left[\frac{\partial^2}{\partial \xi^2}+\frac{2i}{\delta^2}\right]\left(\vec{n}\times\vec{H}_c\right)=0$$

$$\vec{n}\cdot\vec{H}_c=0$$

$$\delta=\left(\frac{2}{\mu_c\omega\sigma}\right)^{1/2}$$

$$\vec{H}_c=\vec{H}_{par}e^{-\xi/\delta}e^{i\xi/\delta}$$

$$\vec{E}_c=\sqrt{\frac{\omega\mu_c}{2\sigma}}(1-i)\left(\vec{n}\times\vec{H}_{par}\right)$$

$$\frac{dP}{da}=\frac{\mu_c\omega\delta}{2}\left|\vec{H}_{par}\right|^2$$