Physics 704/804 Electromagnetic Theory II

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Perpendicular Polarization

$$0=\left(\left|\vec{E}_0\right|+\left|\vec{E}_0''\right|-\left|\vec{E}_0'\right|\right) \qquad \text{Tangential } \vec{E}$$

$$0=\sqrt{\frac{\varepsilon}{\mu}}\left(\left|\vec{E}_0\right|-\left|\vec{E}_0''\right|\right)\cos i-\sqrt{\frac{\varepsilon'}{\mu'}}\left|\vec{E}_0'\right|\cos r \qquad \text{Tangential } \vec{H}$$

$$\frac{\left|\vec{E}_0'\right|}{\left|\vec{E}_0\right|}=\frac{2n\cos i}{n\cos i+\frac{\mu}{\mu'}\sqrt{n'^2-n^2\sin^2 i}}$$

$$\frac{\left|\vec{E}_0''\right|}{\left|\vec{E}_0\right|}=\frac{n\cos i-\frac{\mu}{\mu'}\sqrt{n'^2-n^2\sin^2 i}}{n\cos i+\frac{\mu}{\mu'}\sqrt{n'^2-n^2\sin^2 i}}$$

Parallel Polarization

$$0 = \cos i\left(\left|\vec{E}_0\right| - \left|\vec{E}_0''\right|\right) - \cos r\left|\vec{E}_0'\right| \qquad \text{Tangential } \vec{E}$$

$$0 = \sqrt{\frac{\varepsilon}{\mu}}\left(\left|\vec{E}_0\right| + \left|\vec{E}_0''\right|\right) - \sqrt{\frac{\varepsilon'}{\mu'}}\left|\vec{E}_0'\right| \qquad \text{Tangential } \vec{H}$$

$$\frac{\left|\vec{E}_0'\right|}{\left|\vec{E}_0\right|} = \frac{2nn'\cos i}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}}$$

$$\frac{\left|\vec{E}_0''\right|}{\left|\vec{E}_0\right|} = \frac{\frac{\mu}{\mu'}n'^2\cos i - n\sqrt{n'^2 - n^2\sin^2 i}}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}}$$

Normal Incidence

$$\frac{\left|\vec{E}_0'\right|}{\left|\vec{E}_0\right|} = \frac{2}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1} = \frac{2n}{n'+n}$$

$$\frac{\left|\vec{E}_0''\right|}{\left|\vec{E}_0\right|} = \frac{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}-1}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1} = \frac{n'-n}{n'+n}$$

Energy Conservation?

$$\sqrt{\frac{\varepsilon}{\mu}}\frac{E_0^2}{2} = \sqrt{\frac{\varepsilon}{\mu}}\frac{E_0''^2}{2} + \sqrt{\frac{\varepsilon'}{\mu'}}\frac{E_0'^2}{2}$$

Normal Incidence

$$\frac{\left|\vec{E}_0'\right|}{\left|\vec{E}_0\right|} = \frac{2}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1} \rightarrow \frac{2n}{n'+n} \quad \text{when } \mu = \mu' = 1$$

$$\frac{\left|\vec{E}_0''\right|}{\left|\vec{E}_0\right|} = \frac{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}-1}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1} \rightarrow \frac{n'-n}{n'+n} \quad \text{when } \mu = \mu' = 1$$

Energy Conservation?

$$\sqrt{\frac{\varepsilon}{\mu}}\frac{E_0^2}{2} = \sqrt{\frac{\varepsilon}{\mu}}\frac{E_0''^2}{2} + \sqrt{\frac{\varepsilon'}{\mu'}}\frac{E_0'^2}{2} \qquad 1 = \frac{\left(\sqrt{}-1\right)^2}{\left(\sqrt{}+1\right)^2} + \sqrt{}\,\frac{2}{\left(\sqrt{}+1\right)^2}$$

Brewster's Angle Calculation

$$n'^2 \cos i_B = n\sqrt{n'^2 - n^2 \sin^2 i_B}$$

$$n'^4 \cos^2 i_B = n^2\left(n'^2 - n^2 \sin^2 i_B\right)$$

$$n'^2 = \frac{+n^2 \pm \sqrt{n^4 - 4n^4 \cos^2 i_B \sin^2 i_B}}{2\cos^2 i_B} = \frac{+n^2 \pm n^2\left(1 - 2\cos^2 i_B\right)}{2\cos^2 i_B}$$

$$i_{Brewster} = \tan^{-1}\frac{n'}{n}$$

Zero reflection in perpendicular polarization implies $n' = n$

Brewster's Angle

If $\mu = \mu'$ reflected parallel polarization amplitude vanishes when incident at the Brewster angle

$$i_{Brewster} = \tan^{-1}\frac{n'}{n}$$

Reflected wave completely plane-polarized (polarization perpendicular to plane of incidence) if mixed-polarization beam incident at Brewster angle.

$$i_{Brewster} = 56^{\circ} \quad \text{for} \quad \frac{n'}{n} = 1.5$$

Total Internal Reflection

Examine Snell's Law in case $n > n'$

$$i_0 = \sin^{-1} \frac{n'}{n}$$

For angles of incidence greater, there is no transmitted wave solution to attach to, only an exponentially damped solution. This implies total reflection, also called *total internal reflection*. Optical communication systems are based on this phenomenon!

Group Velocity

Until now, we have assumed that the relative permitivity and permeability are independent of frequency. This may be far from the case. Relaxing the requirement of constant phase velocity as a function of frequency leads to more general wave phenomena. Allow the frequency to depend on wavelength in 1 dimension:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(k) e^{ikx - \omega(k)t} dk$$

The function $\omega(k)$ is known as the "dispersion function". A strictly linear dispersion function, as we've had up to now, does not lead to pulse spreading, or dispersion.

$$A(k)=\int_{-\infty}^{\infty}u(x,0)\,e^{-ikx}\,dx$$

$$\omega(k)=\omega_0+\left.\frac{d\omega}{dk}\right|_0(k-k_0)+\cdots \qquad \omega_0=\omega(k_0)$$

$$u(x,t)\approx\frac{e^{i\left[k_0(d\omega/dk)|_0-\omega_0\right]t}}{2\pi}\int_{-\infty}^{\infty}A(k)\,e^{i\left[x-(d\omega/dk)|_0 t\right]k}\,dk$$

$$=e^{i\left[k_0(d\omega/dk)|_0-\omega_0\right]t}\,u\left(x-(d\omega/dk)|_0\,t,0\right)$$

The pulse shape travels at the group velocity

$$\mathrm{v}_g=\frac{d\omega}{dk}$$

Dispersion

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k) e^{ikx-\omega(k)t}\,dk$$

Have exact calculation for modulated Gaussian function

$$u(x,0) = \exp\left(-x^2/2L^2\right)e^{ik_0x}$$

$$A(k) = \int_{-\infty}^{\infty} u(x,0)e^{-ikx}\,dx$$

$$= \sqrt{2\pi}L\exp\left(-L^2/2\right)\left(k-k_0\right)^2$$

$$\omega(k) = \nu\left(1+\frac{a^2k^2}{2}\right)$$

Pulse Spreading, or Dispersion

$$\mathrm{v}_g = \frac{d\omega}{dk} = \nu a^2 k_0$$

$$u(x,t) = \frac{L}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left(-L^2/2\right)\left(k-k_0\right)^2} e^{ikx - i\nu t\left[1+\left(a^2k^2/2\right)\right]} dk$$

$$= \frac{\exp\left[-\dfrac{\left(x-\nu a^2 k_0 t\right)^2}{2L^2\left(1+\dfrac{ia^2\nu t}{L^2}\right)}\right]}{\left(1+\dfrac{ia^2\nu t}{L^2}\right)^{1/2}} \exp\left[ik_0 x - i\nu\left(1+a^2k^2/2\right)t\right]$$

$$L(t) = \frac{d\omega}{dk} = \sqrt{L^2 + \left(\nu a^2 t / L\right)^2}$$

$$\Delta \mathrm{v}_g = \frac{d^2\omega}{dk^2}\Delta k = \frac{\nu a^2}{L}$$

$$\Delta x(t) = \sqrt{\left(\Delta x\right)^2 + \left(\Delta \mathrm{v}_g t\right)^2}$$