Physics 704/804 Electromagnetic Theory II

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Polarization Variables

Formula

$$\vec{k} \cdot \vec{E}_0 = 0 \rightarrow \vec{k} \cdot e^{i\theta} \vec{E}_0 = 0, \qquad \forall\, \theta$$

Allows for potential phase shift between transverse "vector" components of the wave. As there are two distinct directions normal to $\vec{k}$ and two potential phases, a complete description of the radiation (including the overall wave phase) involves 4 quantities, the Stokes parameters. They can be thought of as the amplitudes and phases in the two directions. They depend on the polarization basis chosen

Let the two (real) unit vectors normal to $\vec{k}$ be $\vec{e}_1$ and $\vec{e}_2$, such that

$$\vec{k} = \vec{e}_1 \times \vec{e}_2$$

Linear Polarization Basis

No phase difference between two components

$$\vec{E}\left(\vec{x},t\right)=\left[\vec{e}_1E_1+\vec{e}_2E_2\right]e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

If E_1 and E_2 purely real

$$\left|\vec{E}\right|=\sqrt{E_1^2+E_2^2}$$

E-field direction of a snapshot is constant in space. Polarization direction angle is

$$\tan^{-1}\frac{E_2}{E_1}$$

with respect to the coordinates specified by the basis vectors.

Circular Polarization Basis

Phase difference of $\pm\pi/2 = \pm i$ between two components

$$\vec{E}\left(\vec{x},t\right) = \left[\vec{e}_+ E_+ + \vec{e}_- E_-\right] e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}, \qquad \vec{e}_\pm \equiv \frac{1}{\sqrt{2}}\left(\vec{e}_1 \pm i\vec{e}_2\right)$$

If E_+ and E_- purely real

$$\left|\vec{E}\right| = \sqrt{E_+^2 + E_-^2}$$

but E-field direction of a snapshot rotates in space. Polarization is elliptical and field rotates with a handedness depending on whether

$$\left|E_1\right| < \left|E_2\right| \qquad \text{or} \qquad \left|E_1\right| > \left|E_2\right|$$

with respect to the coordinates specified by the basis vectors.

General Wave Field

$$\vec{E}\left(\vec{x},t\right)=\left[\vec{e}_1 a_1 e^{i\delta_1}+\vec{e}_2 a_2 e^{i\delta_2}\right]e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

The Stokes parameters are

$$s_0=\left|\vec{e}_1\cdot\vec{E}\right|^2+\left|\vec{e}_2\cdot\vec{E}\right|^2=a_1^2+a_2^2$$

$$s_1=\left|\vec{e}_1\cdot\vec{E}\right|^2-\left|\vec{e}_2\cdot\vec{E}\right|^2=a_1^2-a_2^2$$

$$s_2=2\,\mathrm{Re}\left[\left(\vec{e}_1\cdot\vec{E}\right)^*\left(\vec{e}_2\cdot\vec{E}\right)\right]=2a_1a_2\cos\left(\delta_2-\delta_1\right)$$

$$s_3=2\,\mathrm{Im}\left[\left(\vec{e}_1\cdot\vec{E}\right)^*\left(\vec{e}_2\cdot\vec{E}\right)\right]=2a_1a_2\sin\left(\delta_2-\delta_1\right)$$

In Circular Basis

$$\vec{E}\left(\vec{x},t\right)=\left[\vec{e}_{+}a_{+}e^{i\delta_{+}}+\vec{e}_{-}a_{-}e^{i\delta_{-}}\right]e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

The Stokes parameters are

$$s_0=\left|\vec{e}_{+}^{\,*}\cdot\vec{E}\right|^2+\left|\vec{e}_{-}^{\,*}\cdot\vec{E}\right|^2=a_{+}^2+a_{-}^2$$

$$s_1=2\,\mathrm{Re}\left[\left(\vec{e}_{+}^{\,*}\cdot\vec{E}\right)^*\left(\vec{e}_{-}^{\,*}\cdot\vec{E}\right)\right]=2a_{+}a_{-}\cos\left(\delta_{-}-\delta_{+}\right)$$

$$s_2=2\,\mathrm{Im}\left[\left(\vec{e}_{+}^{\,*}\cdot\vec{E}\right)^*\left(\vec{e}_{-}^{\,*}\cdot\vec{E}\right)\right]=2a_{+}a_{-}\sin\left(\delta_{-}-\delta_{+}\right)$$

$$s_3=\left|\vec{e}_{+}^{\,*}\cdot\vec{E}\right|^2-\left|\vec{e}_{-}^{\,*}\cdot\vec{E}\right|^2=a_{+}^2-a_{-}^2$$

Generally

$$s_0^2=s_1^2+s_2^2+s_3^2$$

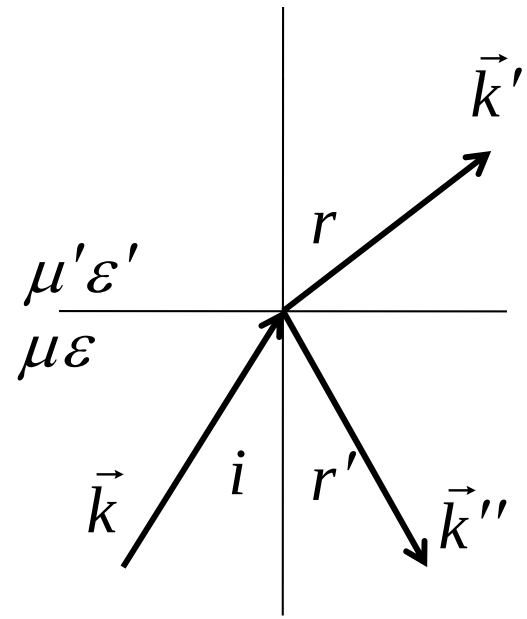
Reflection/Refraction

At the boundary between two media of index of refraction n and n' must have continuity of field:

Incident:

$$\vec{E}\left(\vec{x},t\right)=\vec{E}_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

$$\vec{B}\left(\vec{x},t\right)=\sqrt{\mu\varepsilon}\,\frac{\vec{k}\times\vec{E}}{\left|\vec{k}\right|}$$



Refracted:

$$\vec{E}'\left(\vec{x},t\right)=\vec{E}'_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

$$\vec{B}'\left(\vec{x},t\right)=\sqrt{\mu'\varepsilon'}\,\frac{\vec{k}'\times\vec{E}'}{\left|\vec{k}'\right|}$$

Reflected:

$$\vec{E}''(\vec{x},t) = \vec{E}_0'' e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

$$\vec{B}(\vec{x},t) = \sqrt{\mu\varepsilon}\,\frac{\vec{k}''\times\vec{E}''}{\left|\vec{k}''\right|}$$

Magnitudes:

$$\left|\vec{k}\right| = \left|\vec{k}''\right| = k = \omega\sqrt{\mu\varepsilon}$$

$$\left|\vec{k}'\right| = k' = \omega\sqrt{\mu'\varepsilon'}$$

Must have:

$$\left(\vec{k}\cdot\vec{x}\right)_{z=0} = \left(\vec{k}'\cdot\vec{x}\right)_{z=0} = \left(\vec{k}''\cdot\vec{x}\right)_{z=0}$$

Snell's Law

$$k \sin i = k' \sin r = k'' \sin r'$$

And so

$$i = r'$$

$$\frac{\sin i}{\sin r} = \frac{k'}{k} = \sqrt{\frac{\mu' \varepsilon'}{\mu \varepsilon}} = \frac{n'}{n}$$

Dynamic Conditions

at $z = 0$, boundary conditions imply

$$0 = \left[\varepsilon \left(\vec{E}_0 + \vec{E}_0'' \right) - \varepsilon' \vec{E}_0' \right] \cdot \vec{n} \qquad \text{Normal } \vec{D}$$

$$0 = \left[\vec{k} \times \vec{E}_0 + \vec{k}'' \times \vec{E}_0'' - \vec{k}' \times \vec{E}_0' \right] \cdot \vec{n} \qquad \text{Normal } \vec{B}$$

$$0 = \left(\vec{E}_0 + \vec{E}_0'' - \vec{E}_0' \right) \times \vec{n} \qquad \text{Tangential } \vec{E}$$

$$0 = \left[\frac{1}{\mu} \left(\vec{k} \times \vec{E}_0 + \vec{k}'' \times \vec{E}_0'' \right) - \frac{1}{\mu'} \vec{k}' \times \vec{E}_0' \right] \times \vec{n} \qquad \text{Tangential } \vec{H}$$

Perpendicular Polarization

$$0 = \left(\left|\vec{E}_0\right| + \left|\vec{E}_0''\right| - \left|\vec{E}_0'\right|\right) \qquad \text{Tangential } \vec{E}$$

$$0 = \sqrt{\frac{\varepsilon}{\mu}}\left(\left|\vec{E}_0\right| - \left|\vec{E}_0''\right|\right)\cos i - \sqrt{\frac{\varepsilon'}{\mu'}}\left|\vec{E}_0'\right|\cos r \qquad \text{Tangential } \vec{H}$$

$$\frac{\left|\vec{E}_0'\right|}{\left|\vec{E}_0\right|} = \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}$$

$$\frac{\left|\vec{E}_0''\right|}{\left|\vec{E}_0\right|} = \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}$$

Parallel Polarization

$$0 = \cos i \left(\left| \vec{E}_0 \right| - \left| \vec{E}_0'' \right| \right) - \cos r \left| \vec{E}_0' \right| \qquad \text{Tangential } \vec{E}$$

$$0 = \sqrt{\frac{\varepsilon}{\mu}} \left(\left| \vec{E}_0 \right| + \left| \vec{E}_0'' \right| \right) - \sqrt{\frac{\varepsilon'}{\mu'}} \left| \vec{E}_0' \right| \qquad \text{Tangential } \vec{H}$$

$$\frac{\left| \vec{E}_0' \right|}{\left| \vec{E}_0 \right|} = \frac{2nn' \cos i}{\frac{\mu}{\mu'} n'^2 \cos i + n\sqrt{n'^2 - n^2 \sin^2 i}}$$

$$\frac{\left| \vec{E}_0'' \right|}{\left| \vec{E}_0 \right|} = \frac{\frac{\mu}{\mu'} n'^2 \cos i - n\sqrt{n'^2 - n^2 \sin^2 i}}{\frac{\mu}{\mu'} n'^2 \cos i + n\sqrt{n'^2 - n^2 \sin^2 i}}$$

Normal Incidence

$$\frac{\left|\vec{E}_0'\right|}{\left|\vec{E}_0\right|} = \frac{2}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1} \rightarrow \frac{2n}{n'+n} \quad \text{when } \mu = \mu' = 1$$

$$\frac{\left|\vec{E}_0''\right|}{\left|\vec{E}_0\right|} = \frac{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}-1}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1} \rightarrow \frac{n'-n}{n'+n} \quad \text{when } \mu = \mu' = 1$$

Energy Conservation?

$$\sqrt{\frac{\varepsilon}{\mu}}\frac{E_0^2}{2} = \sqrt{\frac{\varepsilon}{\mu}}\frac{E_0''^2}{2} + \sqrt{\frac{\varepsilon'}{\mu'}}\frac{E_0'^2}{2} \qquad 1 = \frac{\left(\sqrt{\ }-1\right)^2}{\left(\sqrt{\ }+1\right)^2} + \sqrt{\ }\frac{2}{\left(\sqrt{\ }+1\right)^2}$$