Physics 704/804 Electromagnetic Theory II

G. A. Krafft
Jefferson Lab
Jefferson Lab Professor of Physics
Old Dominion University

Plane Electromagnetic Waves

If there are no sources in the Maxwell equations we must solve

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{D} = 0 \qquad \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = 0$$

For a harmonic time dependence $\vec{B}, \vec{E} \propto e^{-i\omega t}$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} - i\omega \vec{B} = 0$$

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{B} + i\omega\mu\varepsilon \vec{E} = 0$$

Get

$$\left[\nabla^2 + \mu\varepsilon\omega^2\right]\vec{E} = 0 \rightarrow \vec{E} = \vec{E}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

$$\left[\nabla^2 + \mu\varepsilon\omega^2\right]\vec{B} = 0 \rightarrow \vec{B} = \vec{B}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t} \qquad \left|\vec{k}\right| = \sqrt{\mu\varepsilon}\,\omega$$

Divergence equations imply

$$\vec{k}\cdot\vec{E}_0 = \vec{k}\cdot\vec{B}_0 = 0$$

Faraday's Law implies

$$\vec{B}_0 = \frac{\vec{k}\times\vec{E}_0}{\omega} = \frac{n\hat{n}\times\vec{E}_0}{c} \qquad \hat{n} = \frac{\vec{k}}{\left|\vec{k}\right|}$$

where n is the index of refraction

Energy and Energy Flux

Poynting vector for the solution is

$$\vec{S} = \frac{1}{2}\vec{E} \times \vec{H}^* = \frac{n}{2\mu c}\left|\vec{E}_0\right|^2 \hat{n} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left|\vec{E}_0\right|^2 \hat{n}$$

Energy Density

$$u = \frac{1}{4}\left[\varepsilon \vec{E} \cdot \vec{E}^* + \frac{1}{\mu}\vec{B} \cdot \vec{B}^*\right] = \frac{\varepsilon}{2}\left|\vec{E}_0\right|^2$$

Consistency

$$\left|\vec{S}\right| = cu / n$$

Note that the units are correct

Stress Tensor for Plane Wave

Assume free-space conditions and align *z*-axis with $\vec{k}$

$$\varepsilon_0 \begin{pmatrix} E_xE_x + c^2B_xB_x & E_xE_y + c^2B_xB_y & E_xE_z + c^2B_xB_z \\ E_yE_x + c^2B_yB_x & E_yE_y + c^2B_yB_y & E_yE_z + c^2B_yB_z \\ E_zE_x + c^2B_zB_x & E_zE_y + c^2B_zB_y & E_zE_z + c^2B_zB_z \end{pmatrix} - \frac{\varepsilon_0}{2}\begin{pmatrix} \vec{E}\cdot\vec{E} + \\ c^2\vec{B}\cdot\vec{B} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{\varepsilon_0}{2}\left|\vec{E}_0\right|^2 \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = -\frac{\varepsilon_0}{2}\left|\vec{E}_0\right|^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Does this answer make sense?

Radiation Pressure in free space (force directed in the *z*-direction!)

$$\frac{\varepsilon_0}{2}\left|\vec{E}_0\right|^2$$

Polarization Variables

Formula

$$\vec{k} \cdot \vec{E}_0 = 0 \rightarrow \vec{k} \cdot e^{i\theta} \vec{E}_0 = 0, \quad \forall\, \theta$$

Allows for potential phase shift between transverse "vector" components of the wave. As there are two distinct directions normal to $\vec{k}$ and two potential phases, a complete description of the radiation (including the overall wave phase) involves 4 quantities, the Stokes parameters. They can be thought of as the amplitudes and phases in the two directions. They depend on the polarization basis chosen

Let the two (real) unit vectors normal to $\vec{k}$ be $\vec{e}_1$ and $\vec{e}_2$, such that

$$\vec{k} = \vec{e}_1 \times \vec{e}_2$$

Linear Polarization Basis

No phase difference between two components

$$\vec{E}\left(\vec{x},t\right)=\left[\vec{e}_1 E_1+\vec{e}_2 E_2\right]e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

If E_1 and E_2 purely real

$$\left|\vec{E}\right|=\sqrt{E_1^2+E_2^2}$$

E-field direction of a snapshot is constant in space. Polarization direction angle is

$$\tan^{-1}\frac{E_2}{E_1}$$

with respect to the coordinates specified by the basis vectors.

Circular Polarization Basis

Phase difference of ±π/2 = ±*i* between two components

$$\vec{E}\left(\vec{x},t\right)=\left[\vec{e}_+E_+ + \vec{e}_-E_-\right]e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)},\qquad \vec{e}_\pm \equiv \frac{1}{\sqrt{2}}\left(\vec{e}_1 \pm i\vec{e}_2\right)$$

If E_+ and E_- purely real

$$\left|\vec{E}\right|=\sqrt{E_+^2+E_-^2}$$

but *E*-field direction of a snapshot rotates in space. Polarization is elliptical and field rotates with a handedness depending on whether

$$\left|E_1\right|<\left|E_2\right| \qquad \text{or} \qquad \left|E_1\right|>\left|E_2\right|$$

with respect to the coordinates specified by the basis vectors.

General Wave Field

$$\vec{E}(\vec{x},t) = \left[\vec{e}_1 a_1 e^{i\delta_1} + \vec{e}_2 a_2 e^{i\delta_2}\right] e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

The Stokes parameters are

$$s_0 = \left|\vec{e}_1 \cdot \vec{E}\right|^2 + \left|\vec{e}_2 \cdot \vec{E}\right|^2 = a_1^2 + a_2^2$$

$$s_1 = \left|\vec{e}_1 \cdot \vec{E}\right|^2 - \left|\vec{e}_2 \cdot \vec{E}\right|^2 = a_1^2 - a_2^2$$

$$s_2 = 2\,\mathrm{Re}\left[\left(\vec{e}_1 \cdot \vec{E}\right)^* \left(\vec{e}_2 \cdot \vec{E}\right)\right] = 2a_1 a_2 \cos\left(\delta_2 - \delta_1\right)$$

$$s_3 = 2\,\mathrm{Im}\left[\left(\vec{e}_1 \cdot \vec{E}\right)^* \left(\vec{e}_2 \cdot \vec{E}\right)\right] = 2a_1 a_2 \sin\left(\delta_2 - \delta_1\right)$$

In Circular Basis

$$\vec{E}(\vec{x},t) = \left[\vec{e}_+ a_+ e^{i\delta_+} + \vec{e}_- a_- e^{i\delta_-}\right] e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

The Stokes parameters are

$$s_0 = \left|\vec{e}_+^{\,*}\cdot\vec{E}\right|^2 + \left|\vec{e}_-^{\,*}\cdot\vec{E}\right|^2 = a_+^2 + a_-^2$$

$$s_1 = 2\,\mathrm{Re}\left[\left(\vec{e}_+^{\,*}\cdot\vec{E}\right)^*\left(\vec{e}_-^{\,*}\cdot\vec{E}\right)\right] = 2a_+a_-\cos\left(\delta_- - \delta_+\right)$$

$$s_2 = 2\,\mathrm{Im}\left[\left(\vec{e}_+^{\,*}\cdot\vec{E}\right)^*\left(\vec{e}_-^{\,*}\cdot\vec{E}\right)\right] = 2a_+a_-\sin\left(\delta_- - \delta_+\right)$$

$$s_3 = \left|\vec{e}_+^{\,*}\cdot\vec{E}\right|^2 - \left|\vec{e}_-^{\,*}\cdot\vec{E}\right|^2 = a_+^2 - a_-^2$$

Generally

$$s_0^2 = s_1^2 + s_2^2 + s_3^2$$