Physics 704/804 Electromagnetic Theory II

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Lorenz Gauge

Lorenz Gauge condition is

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$$

Applying this condition yields the following, very symmetrical version of the potential equations

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \phi = -\frac{\rho}{\varepsilon_0}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \vec{A} = -\mu_0 \vec{J}$$

D'Alembertian or Wave Operator

$$\Box \equiv \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right]$$

Solved by

$$\Box f \equiv \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right] f = 0 \rightarrow$$

$$f = \sum_{\vec{k}} h_{\vec{k}} \left(\vec{k} \cdot \vec{x} - \omega t \right) \qquad \omega = \left| \vec{k} \right| c$$

Restricted Lorenz gauge transformation: clearly the time derivative of any such f may be added directly to ϕ, and it's exterior derivative added to ω_A^1, without violating the Lorenz gauge condition.

Conventions on Fourier Transforms

For the time dimensions

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} d\omega$$

Results on Dirac delta functions

$$\tilde{\delta}(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{i\omega t} dt = 1$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega$$

For the three spatial dimensions

$$\tilde{f}\left(\vec{k}\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\vec{x}\right)e^{-i\vec{k}\cdot\vec{x}}d^{3}\vec{x}$$

$$f\left(\vec{x}\right)=\frac{1}{\left(2\pi\right)^{3}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}\left(\vec{k}\right)e^{+i\vec{k}\cdot\vec{x}}d^{3}\vec{k}$$

$$\delta^{3}\left(\vec{x}\right)=\delta\left(\vec{x}\right)=\frac{1}{\left(2\pi\right)^{3}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{+i\vec{k}\cdot\vec{x}}d^{3}\vec{k}$$

Green Function for Wave Equation

Solution to inhomogeneous wave equation

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]G\left(\vec{x},t;\vec{x}',t'\right)$$
$$=-4\pi\delta\left(\vec{x}-\vec{x}'\right)\delta\left(t-t'\right)$$

Will pick out the solution with causal boundary conditions

$$G\left(\vec{x},t;\vec{x}',t'\right)=0 \qquad\qquad t<t'$$

This choice leads automatically to the so-called *Retarded* Green Function

In general

$$G\left(\vec{x},t;\vec{x}',t'\right)=0 \qquad t<t'$$

$$G\left(\vec{x},t;\vec{x}',t'\right)=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[A\left(\vec{k}\right)e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}+B\left(\vec{k}\right)e^{i\left(\vec{k}\cdot\vec{x}+\omega t\right)}\right]\, d^3\vec{k} \qquad t>t'$$

because there are two possible signs of the frequency for each value of the wave vector. To solve the homogeneous wave equation it is necessary that

$$\omega\left(\vec{k}\right)=\left|\vec{k}\right|c$$

i.e., there is no dispersion in free space.

Continuity of G implies

$$A\left(\vec{k}\right)e^{-i\omega t'} = -B\left(\vec{k}\right)e^{i\omega t'}$$

Integrate the inhomogeneous equation between $t = t' + \varepsilon$ and $t = t' - \varepsilon$

$$-\frac{1}{c^2}\frac{\partial G\left(\vec{x},t;\vec{x}',t'\right)}{\partial t}\bigg|_{t'+\varepsilon} = -4\pi\delta\left(\vec{x}-\vec{x}'\right)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-i\omega A\left(\vec{k}\right)e^{i\left(\vec{k}\cdot\vec{x}-\omega t'\right)} + i\omega B\left(\vec{k}\right)e^{i\left(\vec{k}\cdot\vec{x}+\omega t'\right)}\right] d^3\vec{k}$$

$$= 4\pi c^2\delta\left(\vec{x}-\vec{x}'\right)$$

$$A\left(\vec{k}\right) = -\frac{c^2}{\left(2\pi\right)^2 i\omega}e^{-i\vec{k}\cdot\vec{x}'}e^{i\omega t'}$$

$$G\left(\vec{x},t;\vec{x}',t'\right)=$$

$$-\frac{c^2}{(2\pi)^2 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\omega}\left[e^{i\left(\vec{k}\cdot(\vec{x}-\vec{x}')-\omega(t-t')\right)}-e^{i\left(\vec{k}\cdot(\vec{x}-\vec{x}')+\omega(t-t')\right)}\right]d^3\vec{k}$$

$$t>t'$$

$$=\frac{c}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik\left|\vec{x}-\vec{x}'\right|}}{\left|\vec{x}-\vec{x}'\right|}e^{-i\omega(t-t')}dk-\frac{c}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik\left|\vec{x}-\vec{x}'\right|}}{\left|\vec{x}-\vec{x}'\right|}e^{+i\omega(t-t')}dk \quad t>t'$$

$$=\frac{\delta\left(\left|\vec{x}-\vec{x}'\right|/c-t+t'\right)}{\left|\vec{x}-\vec{x}'\right|}+0$$

Called retarded because the influence at time t is due to the source evaluated at the retarded time

$$t'=t-\left|\vec{x}-\vec{x}'\right|/c$$

Advanced Green Function

If work with the boundary condition that

$$G_{adv}\left(\vec{x},t;\vec{x}',t'\right)=0 \qquad\qquad t>t'$$

one notices that all that happens in the analysis is the sign of all terms reverses, leaving

$$G_{adv}\left(\vec{x},t;\vec{x}',t'\right)=$$

$$=\frac{c}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik\left|\vec{x}-\vec{x}'\right|}}{\left|\vec{x}-\vec{x}'\right|}e^{+i\omega\left(t-t'\right)}dk-\frac{c}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik\left|\vec{x}-\vec{x}'\right|}}{\left|\vec{x}-\vec{x}'\right|}e^{-i\omega\left(t-t'\right)}dk \quad t<t'$$

$$=\frac{\delta\left(\left|\vec{x}-\vec{x}'\right|/c+t-t'\right)}{\left|\vec{x}-\vec{x}'\right|}+0$$

Because not causal, not generally used

Retarded Solutions for Fields

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi=-\frac{\rho}{\varepsilon_0}$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A}=-\mu_0\vec{J}$$

$$\phi\left(\vec{x},t\right)=\frac{1}{4\pi\varepsilon_0}\int d^3x'dt'\,\frac{\rho\left(\vec{x}',t'\right)}{\left|\vec{x}-\vec{x}'\right|}\delta\left(\left|\vec{x}-\vec{x}'\right|/c-t+t'\right)$$

$$\vec{A}\left(\vec{x},t\right)=\frac{\mu_0}{4\pi}\int d^3x'dt'\,\frac{\vec{J}\left(\vec{x}',t'\right)}{\left|\vec{x}-\vec{x}'\right|}\delta\left(\left|\vec{x}-\vec{x}'\right|/c-t+t'\right)$$

Tip: Leave the delta function in it's integral form to do derivations. Don't have to remember complicated delta-function rules

Retarded Solutions for Fields

$$\phi(\vec{x},t) = \frac{1}{8\pi^2\varepsilon_0}\int d^3x'dt'd\omega \frac{\rho(\vec{x}',t')}{|\vec{x}-\vec{x}'|} e^{i\omega[|\vec{x}-\vec{x}'|/c-(t-t')]}$$

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{8\pi^2}\int d^3x'dt'd\omega \frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|} e^{i\omega[|\vec{x}-\vec{x}'|/c-(t-t')]}$$

Evaluation can be expedited by noting and using the symmetry of the Green function and using relations such as

$$\frac{\partial}{\partial t} f(t-t') = -\frac{\partial}{\partial t'} f(t-t')$$

$$\frac{\partial}{\partial \vec{x}} f(|\vec{x}-\vec{x}'|) = -\frac{\partial}{\partial \vec{x}'} f(|\vec{x}-\vec{x}'|)$$

Yields *Jackson* 6.51-6.52

$$\vec{E}\left(\vec{x},t\right) = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$$

$$= \frac{1}{8\pi^2\varepsilon_0}\int d^3x'dt'd\omega\,\frac{-\vec{\nabla}'\rho\left(\vec{x}',t'\right)}{\left|\vec{x}-\vec{x}'\right|}e^{i\omega\left[\left|\vec{x}-\vec{x}'\right|/c-\left(t-t'\right)\right]}$$

$$-\frac{1}{8\pi^2\varepsilon_0}\int d^3x'dt'd\omega\,\frac{\partial\vec{J}\left(\vec{x}',t'\right)/\partial t'}{c^2\left|\vec{x}-\vec{x}'\right|}e^{i\omega\left[\left|\vec{x}-\vec{x}'\right|/c-\left(t-t'\right)\right]}$$

$$\vec{B}\left(\vec{x},t\right) = \vec{\nabla}\times\vec{A}$$

$$= \frac{\mu_0}{8\pi^2}\int d^3x'dt'd\omega\,\frac{\vec{\nabla}'\times\vec{J}\left(\vec{x}',t'\right)}{\left|\vec{x}-\vec{x}'\right|}e^{i\omega\left[\left|\vec{x}-\vec{x}'\right|/c-\left(t-t'\right)\right]}$$

Direct computation, using the unprimed variables, gives

$$\frac{\partial}{\partial \vec{x}} \frac{e^{i\omega|\vec{x}-\vec{x}'|/c}}{|\vec{x}-\vec{x}'|} = -\frac{e^{i\omega|\vec{x}-\vec{x}'|/c}}{|\vec{x}-\vec{x}'|^2}\hat{R} + \frac{(i\omega/c)\,e^{i\omega|\vec{x}-\vec{x}'|/c}}{|\vec{x}-\vec{x}'|}\hat{R}$$

$$\hat{R} = \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{|\vec{x}-\vec{x}'|}$$

Jefimenko Expressions (Jackson 6.55 and 6.56)

$$\vec{E}(\vec{x},t) = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} = \frac{1}{8\pi^2\varepsilon_0}\int d^3x'dt'd\omega\,\frac{\hat{R}\rho(\vec{x}',t')}{R^2}e^{i\omega\left[|\vec{x}-\vec{x}'|/c-(t-t')\right]}$$

$$+\frac{1}{8\pi^2\varepsilon_0}\int d^3x'dt'd\omega\left[\frac{\hat{R}}{cR}\frac{\partial\rho}{\partial t'}(\vec{x}',t') - \frac{1}{c^2R}\frac{\partial \vec{J}}{\partial t'}(\vec{x}',t')\right]e^{i\omega\left[|\vec{x}-\vec{x}'|/c-(t-t')\right]}$$