Mid-term Examination
Physics 704/804
Electromagnetic Theory

1. (20 points) A function $f(z) = u(x,y) + iv(x,y)$ of a complex variable $z = x + iy$ is called *analytic* if it can be differentiated. Such functions must satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

   a. Resolve $f(z)dz$ into $\omega_{real} + i\omega_{imaginary}$ where ω_{real} and $\omega_{imaginary}$ are two dimensional 1-forms in the variables x and y involving u and v. Give expressions for ω_{real} and $\omega_{imaginary}$.

   b. Compute $d\omega_{real}$ and $d\omega_{imaginary}$ for an analytic $f(z)$.

   c. Show Cauchy's Theorem, $\oint f(z)dz = 0$, applies to analytic $f(z)$. Hint: Generalized Stoke's Theorem.

2. (30 points) Suppose a single charge q at $\vec{x} = (0,0,d)$ is held next to an infinite metallic plate with surface at $\vec{x} = (x,y,0)$ and $-\infty < x, y < \infty$ in vacuum.

   a. Using the method of images or otherwise, show the total electric field next to the surface is

$$\vec{E}(x,y,0) = \frac{-2qd}{4\pi\varepsilon_0 \left(x^2 + y^2 + d^2\right)^{3/2}}\hat{z}$$

   b. What is the stress tensor $\overleftrightarrow{T}$ next to the metallic surface?

   c. What is the total force on the plate? You may assume the following elementary integral

$$\int_0^\infty \frac{rdr}{\left(r^2 + d^2\right)^3} = \frac{1}{4d^4}.$$

   d. Given what you know about image forces, does your answer in c. make sense?

3. (20 points) Four plane electromagnetic waves in free space are represented by the following expressions

$$\vec{E}_1 = \mathrm{Re}\left[E_1\hat{x}e^{i(kz-\omega t)}\right] \qquad \vec{B}_1 = \mathrm{Re}\left[\left(E_1/c\right)\hat{y}e^{i(kz-\omega t)}\right]$$

$$\vec{E}_2 = \mathrm{Re}\left[E_2\hat{y}e^{i(kz-\omega t+\alpha)}\right] \qquad \vec{B}_2 = -\mathrm{Re}\left[\left(E_2/c\right)\hat{x}e^{i(kz-\omega t+\alpha)}\right]$$

$$\vec{E}_3 = \mathrm{Re}\left[E_3\hat{x}e^{i(kz-\omega t+\alpha)}\right] \qquad \vec{B}_3 = \mathrm{Re}\left[\left(E_3/c\right)\hat{y}e^{i(kz-\omega t+\alpha)}\right]$$

$$\vec{E}_4 = \mathrm{Re}\left[E_1\hat{x}e^{i(-kz-\omega t)}\right] \qquad \vec{B}_4 = -\mathrm{Re}\left[\left(E_1/c\right)\hat{y}e^{i(-kz-\omega t)}\right]$$

a. Compute the instantaneous Poynting vector $\vec{S}$ and the time-averaged Poynting vector $\left\langle \vec{S} \right\rangle$ for the superposition of waves one and two. Show that these quantities are just the sums of the corresponding quantities for the waves taken separately. Why?

b. Compute $\vec{S}$ and $\left\langle \vec{S} \right\rangle$ for the superposition of waves one and three. Compare with the results in a) and explain the difference.

c. Compute $\vec{S}$ for the superposition of waves one and four. Show that the energy oscillates, being purely electric then purely magnetic. Show that $\left\langle \vec{S} \right\rangle$ vanishes. Explain.

4. (30 points) A resonant cavity is constructed from a half-circle metallic box of inner radius R and inner side dimension d (See Figure (1)).

a. What are the field patterns ψ for the TM and TE modes?

b. What is the lowest resonant mode frequency when $R = d$?