Physics 704/804
Problem Set 1
Due Jan 28, 2010

1. The totally anti-symmetric 3-tensor, whose components in the standard basis for R^3 are

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{for } \; i=1, j=2, k=3 \text{ and other even permutations of 123} \\ -1 & \text{for } \; i=1, j=3, k=2 \text{ and other odd permutations of 123} \\ 0 & \text{for all other index combinations} \end{cases}$$

is called the Levi-Civita tensor or oriented volume tensor. Show by direct computation in the standard basis that

$$V\left(\vec{\mathrm{v}}_1,\vec{\mathrm{v}}_2,\vec{\mathrm{v}}_3\right) \equiv \varepsilon_{ijk} dx^i \otimes dx^j \otimes dx^k \left(\vec{\mathrm{v}}_1,\vec{\mathrm{v}}_2,\vec{\mathrm{v}}_3\right) = dx^1 \wedge dx^2 \wedge dx^3 \left(\vec{\mathrm{v}}_1,\vec{\mathrm{v}}_2,\vec{\mathrm{v}}_3\right)$$

(summation convention followed) yields the oriented volume of the parallelepiped whose edges are given by the input vectors. The volume calculates positive if $\left(\vec{\mathrm{v}}_1,\vec{\mathrm{v}}_2,\vec{\mathrm{v}}_3\right)$ form a right-handed set of vectors, and negative if they are a left-handed set. Show also that

$$\sum_{i=1}^{3} \varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

where δ_{ij} is the Kronecker delta, equal to one if $i = j$ and equal to zero if $i \neq j$.

2. Show that under a change of basis

$$\begin{pmatrix} \hat{e}'_1 \\ \hat{e}'_2 \\ \hat{e}'_3 \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

the Levi-Civita tensor components transform as

$$\varepsilon'_{ijk} = \det\left(E\right)\varepsilon_{ijk}$$

What then is $V'\left(a\vec{e}'_1, b\vec{e}'_2, c\vec{e}'_3\right)$? Why does your answer expression make sense?

3. Consider the magnetic induction form

$$B = B_x dy \wedge dz + B_z dx \wedge dy \, ,$$

where the magnetic induction components B_x and B_z are constant functions of position. What is dB? What is the magnetic flux ϕ_B through a circle of radius one m in the *x*-*y* plane? Calculate explicitly the flux through the hemispherical surface (positive square root) $z = \sqrt{1 - x^2 - y^2}$. Does your result make sense?

Inside a solenoid magnet (e.g. a high-energy physics detector magnet) the field is very uniform and along the axis of the solenoid $\left(B_x = 0\right)$. Similarly to above, the flux is calculated through a circle of radius a in the *x*-*y* plane, small enough to be inside the uniform field. What is the flux through the rest of the surface of the *x*-*y* plane with $r > a$ extending to ∞, assuming that no flux escapes to ∞? To answer this final question think, don't try to calculate!

4. Consider the electric displacement 2-form

$$D = q\left[\frac{x}{r^3} dy \wedge dz + \frac{y}{r^3} dz \wedge dx + \frac{z}{r^3} dx \wedge dy\right]$$

where $r = \sqrt{x^2 + y^2 + z^2}$. What is ϕ_D evaluated over a sphere of radius a? What is dD for $r > 0$? Interpret. What is ϕ_D evaluated over any closed surface not enclosing the origin of the coordinates? To see the power of the form representation of EM fields, and for practice in performing exterior derivatives and anti-symmetric products, write D in the standard spherical coordinate set

$$\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= r \cos\theta \end{aligned}$$

The myriad terms generated by the exterior derivatives will eventually collapse into a single "obviously right" term. Don't give up too soon.