Physics 704/804 Electromagnetic Theory II

G. A. Krafft

Jefferson Lab

Jefferson Lab Professor of Physics

Old Dominion University

Lienard's Relativistic Generalization

For photon emission, the emitted energy increment dE and duration increment dt are time components of (different) 4-vectors. Therefore their ratio, the power emitted

$$P = \frac{dE}{dt}$$

is a Lorentz invariant! Must be of form

$$P = -\frac{q^2}{6\pi\varepsilon_0 m^2 c^3}\left(\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau}\right)$$

$$E = \gamma m c^2 \qquad \vec{p} = \gamma m \vec{v} \qquad d\tau = dt/\gamma$$

$$P = -\frac{q^2}{6\pi\varepsilon_0 m^2 c^3}\left(\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau}\right) = \frac{q^2}{6\pi\varepsilon_0 c}\gamma^6\left(\left(\dot{\vec{\beta}}\right)^2 - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right)$$

Very strong energy dependence!

For Circular Motion

$$\vec{x}(t) = \rho\left[\cos\omega t\hat{x} + \sin\omega t\hat{y}\right]$$

$$\vec{\mathrm{v}}(t) = \rho\omega\left[-\sin\omega t\hat{x} + \cos\omega t\hat{y}\right] \quad \dot{\vec{\mathrm{v}}}(t) = -\omega^2\vec{x}$$

$$\dot{\vec{\beta}}^2 = \left(\frac{\rho\omega^2}{c}\right)^2 \qquad \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2 = \left(\frac{\rho\omega}{c}\frac{\rho\omega^2}{c}\right)^2$$

$$P = \frac{q^2}{6\pi\varepsilon_0 c}\gamma^4\left(\frac{\rho\omega^2}{c}\right)^2 = \frac{q^2}{6\pi\varepsilon_0 c}\gamma^4\beta^4\frac{c^2}{\rho^2} = \frac{q^2 c}{6\pi\varepsilon_0\rho^2}\gamma^4\beta^4$$

Energy loss per turn

$$\delta E = \frac{2\pi\rho}{\beta c}\frac{q^2 c}{6\pi\varepsilon_0\rho^2}\gamma^4\beta^4 = \frac{q^2}{3\varepsilon_0\rho}\gamma^4\beta^3 = 8.85\times 0.01\frac{\left[E(\mathrm{GeV})\right]^4}{\rho(\mathrm{m})}$$

Relativistic Peaking

In far field after short acceleration

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c} \frac{\left|\hat{n} \times \left\{\left(\hat{n} - \vec{\beta}\right) \times \dot{\vec{\beta}}\right\}\right|^2}{\left(1 - \hat{n} \cdot \vec{\beta}\right)^5}$$

$$\frac{dP(t')}{d\Omega} = \frac{q^2 \dot{\beta}^2}{16\pi^2\varepsilon_0 c} \frac{\sin^2\theta}{\left(1 - \beta\cos\theta\right)^5}$$

$$\theta_{\max} \to \frac{1}{2\gamma}$$

For circular motions

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c} \frac{\dot{\beta}^2}{\left(1 - \beta\cos\theta\right)^3} \left[1 - \frac{\sin^2\theta\cos^2\varphi}{\gamma^2\left(1 - \beta\cos\theta\right)^2}\right]$$

Spectrum Radiated by Motion

$$\frac{dE}{d\Omega} = \int_{-\infty}^{\infty} \frac{dP}{d\Omega} dt = \int_{-\infty}^{\infty} \vec{E} \times \vec{H} \cdot \hat{n} R^2 dt = \frac{1}{c\mu_0} \int_{-\infty}^{\infty} \left(\vec{E} \cdot \vec{E} \right) R^2 dt =$$

$$\frac{1}{c\mu_0} \left(\frac{q}{8\pi^2 \varepsilon_0 c} \right)^2 \int_{-\infty}^{\infty} \int\int\int\int \left[\frac{\hat{n} \times \left\{ \left(\hat{n} - \vec{\beta} \right) \times \dot{\vec{\beta}} \right\}}{\left(1 - \hat{n} \cdot \vec{\beta} \right)^2} (t') \right] \cdot \left[\frac{\hat{n} \times \left\{ \left(\hat{n} - \vec{\beta} \right) \times \dot{\vec{\beta}} \right\}}{\left(1 - \hat{n} \cdot \vec{\beta} \right)^2} (t'') \right]$$

$$\times e^{i\omega\left[-\hat{n} \cdot \vec{r}(t')/c - t + t' \right]} e^{i\omega'\left[-\hat{n} \cdot \vec{r}(t'')/c - t + t'' \right]} dt' d\omega dt'' d\omega' dt =$$

clearing the unprimed time integral and omega prime integral with delta represntation

$$\frac{2\pi}{c\mu_0} \left(\frac{q}{8\pi^2 \varepsilon_0 c} \right)^2 \int_{-\infty}^{\infty} \int\int \left[\frac{\hat{n} \times \left\{ \left(\hat{n} - \vec{\beta} \right) \times \dot{\vec{\beta}} \right\}}{\left(1 - \hat{n} \cdot \vec{\beta} \right)^2} (t') \right] \cdot \left[\frac{\hat{n} \times \left\{ \left(\hat{n} - \vec{\beta} \right) \times \dot{\vec{\beta}} \right\}}{\left(1 - \hat{n} \cdot \vec{\beta} \right)^2} (t'') \right]$$

$$\times e^{i\omega\left[-\hat{n} \cdot \vec{r}(t')/c - t + t' \right]} e^{-i\omega\left[-\hat{n} \cdot \vec{r}(t'')/c - t + t'' \right]} dt' dt'' d\omega$$

$$\frac{d^2E}{d\omega d\Omega} = \frac{q^2}{32\pi^3\varepsilon_0 c}\left|\int_{-\infty}^{\infty} \frac{\hat{n}\times\left\{\left(\hat{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}}{\left(1-\hat{n}\cdot\vec{\beta}\right)^2} e^{i\omega\left[-\hat{n}\cdot\vec{r}(t')/c - t + t'\right]} dt'\right|^2$$

$$\frac{d^2E}{d\omega d\Omega} = \frac{q^2\omega^2}{32\pi^3\varepsilon_0 c}\left|\int_{-\infty}^{\infty} \hat{n}\times\left(\hat{n}\times\vec{\beta}\right) e^{i\omega\left[t' - \hat{n}\cdot\vec{r}(t')/c\right]} dt'\right|^2$$

Factor of two difference from Jackson because he combines positive frequency and negative frequency contributions in one positive frequency integral. I don't like because Parseval's formula, etc. don't work! I've written papers about performing this calculation in new regimes of high intensity pulsed lasers.

Snychrotron Radiation

Case of instantaneous circular motion

$$\xi = \frac{\omega\rho}{3c}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}$$

$$\frac{d^2E}{d\omega d\Omega} = \frac{q^2}{24\pi^3\varepsilon_0 c}\left(\frac{\omega\rho}{c}\right)^2\left(\frac{1}{\gamma^2}+\theta^2\right)^2\left[K_{2/3}^2(\xi) + \frac{\theta^2}{(1/\gamma)^2+\theta^2}K_{1/3}^2(\xi)\right]$$

$$\frac{dE}{d\omega} = \frac{\sqrt{3}q^2}{8\pi\varepsilon_0 c}\gamma\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{\infty} K_{5/3}(x)dx$$

critical frequecy

$$\omega_c = \frac{3}{2}\gamma^3\left(\frac{c}{\rho}\right)$$

Photon Interpretation

Photon Energy Distribution

$$\frac{dN}{dy} = \frac{I}{\hbar\omega_c}\frac{9\sqrt{3}}{64\pi^2\varepsilon_0}\int_y^\infty K_{5/3}(x)dx$$

Number emitted per revolution

$$N = \frac{5\pi}{\sqrt{3}}\gamma\alpha$$

α fine structure constant $= \dfrac{e^2}{4\pi\varepsilon_0\hbar c}$

Average energy

$$\langle\hbar\omega\rangle = \frac{8}{15\sqrt{3}}\hbar\omega_c$$

Undulator emission

Wiggle electrons at a certain frequency with alternating pole magnets

$$K = \frac{eB\lambda}{2\pi mc}$$

emission frequency

$$\omega' = \gamma\left(1 - \beta\cos\theta\right)\omega \approx \omega\left(1 + \gamma^2\theta^2\right)/\,2\bar{\gamma}^2$$

$$\bar{\gamma} = \frac{\gamma}{\sqrt{1 + K^2/2}}$$

By Larmor, in undulator

$$P = \frac{q^2 c \overline{\gamma}^2 K^2 k_0^2}{12\pi\varepsilon_0}$$

Number emitted per pass of undulator

$$N_\gamma = \frac{2\pi}{3}\alpha N K^2$$

By uncertainty relation, spectral width of emission is $1/N$

Thomson Scattering

Nonrelativistic Compton Scattering

$$\frac{dP}{d\Omega} = \frac{e^2}{16\pi^2\varepsilon_0 c^3}\left|\vec{\varepsilon}^{\,*}\cdot\dot{\vec{\mathrm{v}}}\right|^2$$

Incident plane wave

$$\vec{E} = \vec{\varepsilon}_0 E_0 e^{i\left(\vec{k}_0\cdot\vec{x}-\omega t\right)}$$

$$\dot{\vec{\mathrm{v}}} = \vec{\varepsilon}_0 \frac{e}{m} E_0 e^{i\left(\vec{k}_0\cdot\vec{x}-\omega t\right)}$$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^2}{32\pi^2\varepsilon_0 c^3}\left|E_0\right|^2\left(\frac{e}{m}\right)^2\left|\vec{\varepsilon}^{\,*}\cdot\vec{\varepsilon}_0\right|^2$$

Scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{dP_{scattered} / d\Omega}{I_{incident}} = \frac{\frac{e^2}{32\pi^2\varepsilon_0 c^3}\left|E_0\right|^2\left(\frac{e}{m}\right)^2\left|\vec{\varepsilon}^* \cdot \vec{\varepsilon}_0\right|^2}{\frac{\varepsilon_0 c}{2}\left|E_0\right|^2}$$

$$= \left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 \left|\vec{\varepsilon}^* \cdot \vec{\varepsilon}_0\right|^2 = r_e^2 \left|\vec{\varepsilon}^* \cdot \vec{\varepsilon}_0\right|^2$$

r_e classical electron radius

$$\frac{d\sigma}{d\Omega} = \frac{r_e^2}{2}\left(1 + \cos^2\theta\right)$$

$$\sigma_T = \frac{8\pi}{3} r_e^2$$