Physics 704/804 Electromagnetic Theory II

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Radiation From Relativistic Electrons

From discussion early in the course, in the Lorenz gauge the equation for the potentials is

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi=-\frac{\rho}{\varepsilon_0}$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A}=-\mu_0\vec{J}$$

The solution, using the retarded Green Function is

$$\phi\left(\vec{x},t\right)=\frac{1}{4\pi\varepsilon_0}\int d^3x'dt'\frac{\rho\left(\vec{x}',t'\right)}{\left|\vec{x}-\vec{x}'\right|}\delta\left(\left|\vec{x}-\vec{x}'\right|/c-t+t'\right)$$

$$\vec{A}\left(\vec{x},t\right)=\frac{\mu_0}{4\pi}\int d^3x'dt'\frac{\vec{J}\left(\vec{x}',t'\right)}{\left|\vec{x}-\vec{x}'\right|}\delta\left(\left|\vec{x}-\vec{x}'\right|/c-t+t'\right)$$

Delta Function Representation

$$\phi\left(\vec{x},t\right)=\frac{1}{8\pi^{2}\varepsilon_{0}}\int d^{3}x'dt'd\omega\,\frac{\rho\left(\vec{x}',t'\right)}{\left|\vec{x}-\vec{x}'\right|}e^{i\omega\left[\left|\vec{x}-\vec{x}'\right|/c-\left(t-t'\right)\right]}$$

$$\vec{A}\left(\vec{x},t\right)=\frac{\mu_{0}}{8\pi^{2}}\int d^{3}x'dt'd\omega\,\frac{\vec{J}\left(\vec{x}',t'\right)}{\left|\vec{x}-\vec{x}'\right|}e^{i\omega\left[\left|\vec{x}-\vec{x}'\right|/c-\left(t-t'\right)\right]}$$

$$\rho\left(\vec{x},t\right)=q\delta^{3}\left(\vec{x}-\vec{r}\left(t\right)\right)\qquad\vec{J}\left(\vec{x},t\right)=q\vec{\mathrm{v}}\left(t\right)\delta^{3}\left(\vec{x}-\vec{r}\left(t\right)\right)$$

$$\phi\left(\vec{x},t\right)=\frac{q}{8\pi^{2}\varepsilon_{0}}\int dt'd\omega\,\frac{1}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}$$

$$\vec{A}\left(\vec{x},t\right)=\frac{q\mu_{0}}{8\pi^{2}}\int dt'd\omega\,\frac{\vec{\mathrm{v}}\left(t'\right)}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}$$

Lienard-Weichert Potentials

$$\phi\left(\vec{x},t\right)=\frac{q}{4\pi\varepsilon_0}\int dt'\,\frac{\delta\left(\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right)}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}$$

$$\vec{A}\left(\vec{x},t\right)=\frac{q\mu_0}{4\pi}\int dt'\,\frac{\vec{\mathrm{v}}\left(t'\right)\delta\left(\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right)}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}$$

$$\phi\left(\vec{x},t\right)=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\left|\vec{x}-\vec{r}\left(t'\right)\right|\left(1-\hat{n}\cdot\vec{\beta}\left(t'\right)\right)}\right)_{ret}$$

$$\vec{A}\left(\vec{x},t\right)=\frac{q\mu_0}{4\pi}\left(\frac{\vec{\mathrm{v}}\left(t'\right)}{\left|\vec{x}-\vec{r}\left(t'\right)\right|\left(1-\hat{n}\cdot\vec{\beta}\left(t'\right)\right)}\right)_{ret}$$

EM Field Radiated

$$\phi\left(\vec{x},t\right)=\frac{q}{8\pi^{2}\varepsilon_{0}}\int dt'd\omega\,\frac{1}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}$$

$$\vec{A}\left(\vec{x},t\right)=\frac{q\mu_{0}}{8\pi^{2}}\int dt'd\omega\,\frac{\vec{\mathrm{v}}\left(t'\right)}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}$$

$$\vec{E}=-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}\qquad\qquad\vec{B}=\vec{\nabla}\times\vec{A}$$

$$\vec{E}=\frac{q}{4\pi\varepsilon_{0}}\left[\frac{\hat{n}-\vec{\beta}}{\gamma^{2}\left(1-\hat{n}\cdot\vec{\beta}\right)^{3}R^{2}}\right]_{ret}+\frac{q}{4\pi\varepsilon_{0}c}\left[\frac{\hat{n}\times\left\{\left(\hat{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}}{\left(1-\hat{n}\cdot\vec{\beta}\right)^{3}R}\right]_{ret}$$

$$\vec{B}=\hat{n}\times\vec{E}\,/\,c$$

$$\vec{\nabla}\frac{1}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}=-\frac{\hat{n}}{\left|\vec{x}-\vec{r}\left(t'\right)\right|^2}\qquad \vec{\nabla}\left|\vec{x}-\vec{r}\left(t'\right)\right|=\hat{n}$$

$$\frac{d}{dt'}\frac{1}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}=\frac{\hat{n}\cdot\dot{\vec{\beta}}}{\left|\vec{x}-\vec{r}\left(t'\right)\right|^2}\qquad \frac{d\hat{n}}{dt'}=\frac{-d\vec{r}\,/\,dt'+\hat{n}\left(\hat{n}\cdot d\vec{r}\,/\,dt'\right)}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}\qquad\cdots$$

$$-\nabla\phi\left(\vec{x},t\right)=\frac{q}{8\pi^2\varepsilon_0}\int dt'd\omega\,\frac{\hat{n}\left(1-i\omega\left|\vec{x}-\vec{r}\left(t'\right)\right|/c\right)}{\left|\vec{x}-\vec{r}\left(t'\right)\right|^2}e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}$$

$$-\frac{\partial}{\partial t}\vec{A}\left(\vec{x},t\right)=\frac{q\mu_0}{8\pi^2}\int dt'd\omega\,\frac{\vec{\mathrm{v}}\left(t'\right)i\omega}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}$$

$$\frac{d}{dt'}e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}=i\omega\left(1-\vec{\beta}\left(t'\right)\cdot\hat{n}\left(t'\right)\right)e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}$$

$$\vec{E}\left(\vec{x},t\right)=\frac{q}{8\pi^2\varepsilon_0}\int dt'd\omega e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}\left[\frac{\hat{n}}{\left|\vec{x}-\vec{r}\left(t'\right)\right|^2}+\frac{i\omega\left(\vec{\beta}-\hat{n}\right)}{c\left|\vec{x}-\vec{r}\left(t'\right)\right|}\right]$$

integrate by parts to get final result

$$\vec{E}\left(\vec{x},t\right)_{vel}=\frac{q}{8\pi^2\varepsilon_0}\int dt'd\omega\frac{e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}}{\left(1-\vec{\beta}\cdot\hat{n}\right)^2\left|\vec{x}-\vec{r}\left(t'\right)\right|^2}$$

$$\times\left[\begin{array}{l}\hat{n}\left(\begin{array}{l}1-2\vec{\beta}\cdot\hat{n}+\left(\vec{\beta}\cdot\hat{n}\right)^2+\vec{\beta}\cdot\hat{n}-\left(\vec{\beta}\cdot\hat{n}\right)^2+\vec{\beta}\cdot\hat{n}-\left(\vec{\beta}\cdot\hat{n}\right)^2\\-\beta^2+\left(\vec{\beta}\cdot\hat{n}\right)^2\end{array}\right)\\-\vec{\beta}\left(1-\vec{\beta}\cdot\hat{n}+\vec{\beta}\cdot\hat{n}-\left(\vec{\beta}\cdot\hat{n}\right)^2-\beta^2+\left(\vec{\beta}\cdot\hat{n}\right)^2\right)\end{array}\right]$$

$$\vec{E}\left(\vec{x},t\right)_{acc} = \frac{q}{8\pi^2\varepsilon_0 c}\int dt' d\omega \frac{e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}}{\left(1-\vec{\beta}\cdot\hat{n}\right)^2\left|\vec{x}-\vec{r}\left(t'\right)\right|}$$

$$\times\begin{bmatrix}\hat{n}\left(\dot{\vec{\beta}}\cdot\hat{n}\right)\\ -\dot{\vec{\beta}}\left(1-\vec{\beta}\cdot\hat{n}\right)-\vec{\beta}\left(\dot{\vec{\beta}}\cdot\hat{n}\right)\end{bmatrix}$$

$$= \frac{q}{8\pi^2\varepsilon_0 c}\int dt' d\omega \frac{e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}}{\left(1-\vec{\beta}\cdot\hat{n}\right)^2\left|\vec{x}-\vec{r}\left(t'\right)\right|}\left[\hat{n}\times\left\{\left(\hat{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}\right]$$

Larmor's Formula

For small velocities can neglect retardation

$$\vec{E}\left(\vec{x},t\right)_{acc} = \frac{q}{4\pi\varepsilon_0 c}\left[\hat{n}\times\left\{\hat{n}\times\dot{\vec{\beta}}\right\}\right]/R$$

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0\mu_0 c^3}\left|\left[\hat{n}\times\left\{\hat{n}\times\dot{\vec{\beta}}\right\}\right]\right|^2$$

$$= \frac{q^2}{16\pi^2\varepsilon_0 c^3}\left|\dot{\vec{\mathrm{v}}}\right|^2\sin^2\theta$$

$$P = \frac{q^2}{6\pi\varepsilon_0 c^3}\left|\dot{\vec{\mathrm{v}}}\right|^2$$