Physics 704/804 Electromagnetic Theory II

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Tensor Analysis

Lorentz Transformations leave invariant the space-time interval

$$ds^2(x,y) = \left(x^0 - y^0\right)^2 - \left(x^1 - y^1\right)^2 - \left(x^2 - y^2\right)^2 - \left(x^3 - y^3\right)^2$$

and more generally the Lorentz scaler products of 4-vectors

$$v \cdot w = v^0 w^0 - v^1 w^1 - v^2 w^2 - v^3 w^3$$

Goal in Special Relativity; write laws of physics in terms of quantities with specific Lorentz transformation properties, for example invariants, 4-vectors, 4-tensors, etc.

$$x'^{\alpha} = x'^{\alpha}\left(x^0, x^1, x^2, x^3\right)$$

$$dx'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} dx^{\beta}$$ Einstein summation convention on β

Contravariant (4)-vectors: Transform like coordinate differentials

$$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta}$$

Examples: 4-velocity, energy-momentum 4-vector, k for photons

Not only possibility. Consider the gradient operator

$$\frac{\partial}{\partial x'^{\alpha}} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial}{\partial x^{\beta}}$$

uses inverse matrix to contravariant vectors. Quantities transforming similarly to gradiant called covariant vectors

$$B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta}$$

Second Rank 4-Tensors

Contravariant rank two tensors transform like the outer product of two contravariant (4)-vectors:

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}} \frac{\partial x'^{\beta}}{\partial x^{\gamma}} F^{\delta\gamma}$$

Covariant rank two tensors transform like the outer product of two covariant (4)-vectors:

$$H'_{\alpha\beta} = \frac{\partial x^{\delta}}{\partial x'^{\alpha}} \frac{\partial x^{\gamma}}{\partial x'^{\beta}} H_{\delta\gamma}$$

Mixed rank two tensors transform like:

$$H'^{\ \beta}_{\alpha} = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} H_{\gamma}^{\ \delta}$$

Not necessarily true that $H_{\alpha}^{\ \beta} = H^{\beta}_{\ \alpha}$

Tensor Contraction

Summing upper and lower indices leaves tensor with rank two lower. For mixed 2 tensor yields an invariant

$$H'^{\alpha}_{\alpha} = \frac{\partial x^{\delta}}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} H_{\delta}{}^{\gamma} = \frac{\partial x^{\delta}}{\partial x^{\gamma}} H_{\delta}{}^{\gamma} = H_{\delta}{}^{\delta}$$

$\leftrightarrow$ to the linear algebra theorem that the trace of a matrix is invariant under similarity transformations

$$\mathrm{Tr}\left(SMS^{-1}\right) = \mathrm{Tr}\left(M\right)$$

Similar example: inner product is outer product of covariant and contravariant components followed by a contraction

$$v' \cdot w' = v'_{\alpha} w'^{\alpha} = \frac{\partial x^{\delta}}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} v_{\delta} w^{\gamma} = \frac{\partial x^{\delta}}{\partial x^{\gamma}} v_{\delta} w^{\gamma} = v_{\delta} w^{\delta} = v \cdot w$$

Metric Tensor

Write invariant interval

$$(ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$$

$g_{\mu\nu}$ gives the scalar product by $v \cdot w = g_{\mu\nu} v^\mu w^\nu$.

Because $v \cdot w = w \cdot v$, $g_{\mu\nu}$ is symmetric. We'll use Minkowski metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

inverse matrix

$$g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha{}_\gamma, \; g_{\gamma\beta} g^{\beta\alpha} = \delta_\gamma{}^\alpha \quad \delta^\alpha{}_\gamma = \delta_\gamma{}^\alpha = \begin{cases} 1 & \alpha = \gamma \\ 0 & \alpha \neq \gamma \end{cases}$$

Inverse Matrix

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

metric may be used to provide an association between contravariant and covariant components

$$x_\alpha = g_{\alpha\beta} x^\beta = \left(x^0, -\vec{x} \right)$$

$$F^{\alpha\cdot\cdot}_{\dots} = g^{\alpha\beta} F^{\cdot\cdot}_{\beta\dots}$$

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right) \qquad \partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla} \right)$$

Lorentz Invariant Operators

4-divergence

$$\partial_\alpha A^\alpha = \partial^\alpha A_\alpha = \frac{\partial A^0}{\partial x^0} + \vec{\nabla} \cdot \vec{A}$$

wave operator

$$\Box \equiv \partial_\alpha \partial^\alpha = \frac{\partial^2}{\left(\partial x^0\right)^2} - \nabla^2$$

define column vector consisting of contravariant components

$$X = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad \text{as a matrix equation} \quad gX = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$$

Lorentz Group

Note

$$ggX = g^2 X = IX = X$$

If

$$\left(X, Y\right) \equiv X^t Y$$

The Lorentz scalar product is

$$a \cdot b = \left(a, gb\right) = \left(ga, b\right) = a^t gb$$

Elements of the Lorentz group of matrices leave the Lorentz scalar product invariant

$$X' = LX \qquad Y' = LY$$

$X' \cdot Y' = \left(LX\right)^t gLY = X^t L^t gLY = X \cdot Y$ for all X, Y if

$L^t gL = g$ as a matrix equation

Components of the Lorentz Group

Now

$$\mathrm{Det}\left(L^t g L\right) = \mathrm{Det}\left(L^t\right)\mathrm{Det}(g)\,\mathrm{Det}(L) = \mathrm{Det}(L)^2\,\mathrm{Det}(g)$$

$$\therefore\ \mathrm{Det}(L) = \pm 1$$

Lorentz Group has 4 components

$+$Det	$+$ parity	usual component with identity
$+$Det	$-$ parity	time reverse $+$ parity
$-$Det	$-$ parity	usual parity
$-$Det	$+$ parity	time reverse

Component containing identity may be constructed from infinitesimal generators (like all Lie groups)

Infinitesimal Generators

$$(1+E)^t g(1+E) \approx g + E^t g + gE + HOT = g$$

$$\therefore E^t g + gE = 0 \to (gE)^t = -gE \to gE \quad \text{antisymmetric}$$

6 Possibilities

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Generate Transformations

Rotations generates by the Ss

$$L_{s_3} = \exp\left(-\theta S_3\right) = I - S_3^2 + \cos\theta S_3^2 - \sin\theta S_3 =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Lorentz Boosts generated by the Ks

$$L_{K_1} = \exp\left(-\xi K_1\right) = I - K_1^2 + \cosh\xi K_1^2 - \sinh\xi K_1 =$$

$$\begin{pmatrix} \cosh\xi & -\sinh\xi & 0 & 0 \\ -\sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Commutation Relations

Rapidity

$\xi = \tanh^{-1}\beta$

adds for Lorentz boosts in same direction

$$\left[S_i, S_j\right] = \varepsilon_{ijk} S_k$$

$$\left[S_i, K_j\right] = \varepsilon_{ijk} K_k$$

$$\left[K_i, K_j\right] = -\varepsilon_{ijk} S_k$$